Analysis of minimal representations

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§1 What are minimal representations?

§2 Conformal model of minimal representations

§3 Geometric quantization of minimal nilpotent orbits and $L^2$ model

§4 Deformation of Fourier transform
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What are minimal reps?

Minimal representations of a reductive group $G$
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Minimal representations of a reductive group $G$

Loosely, minimal representations are

- ‘smallest’ infinite dimensional unitary rep. of $G$
What are minimal reps?

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Algebraically, minimal reps are infinite dim’l reps whose annihilators are the Joseph ideals in $U(g)$

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- ‘smallest’ infinite dimensional unitary rep. of $G$
- one of ‘building blocks’ of unitary reps.
- ‘isolated’ among the unitary dual (finitely many) (continuously many)
- ‘attached to’ minimal nilpotent orbits (orbit method)
- matrix coefficients are of bad decay
Building blocks of unitary reps

unitary reps of Lie groups
Building blocks of unitary reps

unitary reps of Lie groups
  ↑ direct integral (Mautner)
irred. unitary reps of Lie groups
Building blocks of unitary reps

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  ↑ direct integral (Mautner)
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  ↑ construction (Mackey, Kirillov, Duflo)
irred. unitary reps of reductive groups
Building blocks of unitary reps

unitary reps of Lie groups
  ↑ direct integral (Mautner)
irred. unitary reps of Lie groups
  ↑ construction (Mackey, Kirillov, Duflo)
irred. unitary reps of reductive groups
  ↑ “induction”, etc.
finitely many “very small” irred. unitary reps.
of reductive groups
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Minimal ⇔ Maximal

My (ambiguous and ambitious) project

collaborated with S.Ben Saïd, J.Hilgert, G.Mano, J.Möllers, B.Ørsted, M.Pevzner:

Use minimal reps to get an inspiration in finding new interactions with other fields of mathematics.
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Observation. \( \varpi \): minimal rep of \( G \)

\[
\text{DIM}(\varpi) \quad (\text{Gelfand–Kirillov dimension})
\]

\[= \frac{1}{2} \text{ dimension of minimal nilpotent orbits}
\]

\[< \text{ dimension of any non-trivial } G\text{-space}\]
Minimal ⇔ Maximal

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My viewpoint:
Minimal representation (⇔ group)
≈ Maximal symmetries (⇔ rep. space)
Geometric analysis on minimal reps

[6] Special functions associated to a fourth order differential equation · · · Ramanujan J. Math (2011)
[8] Generalized Fourier transforms $\mathcal{F}_{k,a}$ · · · C.R.A.S. Paris 2009

Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers, Ørsted and M. Pevzner
Indefinite orthogonal group $O(p + 1, q + 1)$

Throughout this talk, $p, q \geq 1$, $p + q$: even > 2

$$G = O(p + 1, q + 1)$$

$$= \{ g \in GL(p + q + 2, \mathbb{R}) : g \begin{pmatrix} I_{p+1} & 0 \\ 0 & -I_{q+1} \end{pmatrix} g^T \begin{pmatrix} I_{p+1} & 0 \\ 0 & -I_{q+1} \end{pmatrix} \}$$

$\cdots$ real simple Lie group of type D
Minimal representation of $G = O(p + 1, q + 1)$

- $q = 1$
  - highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom
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- $p = q$
  - spherical case
    - $p = q = 3$ case: Kostant (1990)
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- $p, q$: general
  non-highest, non-spherical
  - algebraic construction (e.g. dual pair)
    (Binegar–Zierau, Howe–Tan, Huang–Zhu)
  - construction by conformal geometry (K–Ørsted)
  - $L^2$ construction (K–Ørsted, K–Mano 2011)
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  - $L^2$ construction (K–Ørsted, K–Mano 2011)
    ($L^2$ model of minimal reps of some other groups Sahi, Hilgert–K–Möllers)
Two constructions of minimal reps.

1. Conformal model
   Theorem B

2. $L^2$ model
   (Schrödinger model)
   Theorem D
Two constructions of minimal reps.

Group action  Hilbert structure

1. Conformal model
   Theorem B  Clear
   v.s.

2. $L^2$ model
   (Schrödinger model)  ?  Clear
   Theorem D

Clear Picture ··· advantage of the model
Two constructions of minimal reps.

Group action    Hilbert structure

1. Conformal model
   Theorem B    Clear
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Clear Picture · · · advantage of the model

No single model of minimal representations has clear pictures for both group actions and Hilbert structures
Two constructions of minimal reps.

Group action   Hilbert structure

1. Conformal model
   Theorem B
   v.s.

2. $L^2$ model
   (Schrödinger model)  Theorem D
   (Schweig model)  Theorem E

Clear Picture · · · advantage of the model

No single model of minimal representations has clear pictures for both group actions and Hilbert structures
Two constructions of minimal reps.

Group action Hilbert structure

1. Conformal model
   Theorem B Clear Theorem C
   v.s.

2. $L^2$ model
   (Schrödinger model) Theorem E Clear
   Theorem D

   Clear Picture ⋮ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)
   (interpolation, special functions, Dunkl operators)
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Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

\[ \text{holomorphic} \circ \text{holomorphic} = \text{holomorphic} \]
Conformal construction of minimal reps.

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\[ \downarrow \text{taking real parts} \]

\[ \text{harmonic} \circ \text{conformal} = \text{harmonic} \quad \text{on } \mathbb{C} \simeq \mathbb{R}^2 \]
Conformal construction of minimal reps.

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make sense for general Riemannian manifolds.
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Make sense for general Riemannian manifolds.

But \[ \text{harmonic} \circ \text{conformal} \neq \text{harmonic} \quad \text{in general} \]
Conformal construction of minimal reps.

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make sense for general Riemannian manifolds.

But \[ \text{harmonic} \circ \text{conformal} \neq \text{harmonic} \quad \text{in general} \]

\[ \Rightarrow \text{Try to modify the definition!} \]
\[
\text{Conf}(X, g) \supset \text{Isom}(X, g)
\]

\[ (X, g) \text{ Riemannian manifold} \]

\[ \varphi \in \text{Diffeo}(X) \]
Conf(\(X, g\)) \supset \text{Isom}(X, g)

\((X, g)\) Riemannian manifold
\(\varphi \in \text{Diffeo}(X)\)

\[
\text{Def.} \quad \begin{align*}
\varphi \text{ is isometry} & \iff \varphi^* g = g \\
\varphi \text{ is conformal} & \iff \exists \text{positive function } C_\varphi \in C^\infty(X) \text{ s.t.} \\
& \quad \varphi^* g = C_\varphi^2 g
\end{align*}
\]

\(C_\varphi\) : conformal factor
Conf\((X, g) \supset \text{Isom}(X, g)\)

\((X, g)\) Riemannian manifold
\(\varphi \in \text{Diffeo}(X)\)

**Def.**

\(\varphi\) is isometry \iff \(\varphi^* g = g\)
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\(C_\varphi: \text{conformal factor}\)

\(\text{Diffeo}(X) \supset \text{Conf}(X, g) \supset \text{Isom}(X, g)\)

Conformal group \hspace{1cm} Isometry group
Conf\((X, g) \supset \text{Isom}(X, g)\)

\((X, g)\) pseudo-Riemannian manifold
\(\varphi \in \text{Diffeo}(X)\)

**Def.**

\(\varphi\) is isometry \iff \(\varphi^*g = g\)
\(\varphi\) is conformal \iff \(\exists\) positive function \(C_\varphi \in C^\infty(X)\) s.t.
\(\varphi^*g = C_\varphi^2 g\)

\(C_\varphi\) : conformal factor

\(\text{Diffeo}(X) \supset \text{Conf}(X, g) \supset \text{Isom}(X, g)\)

Conformal group  isometry group
Harmonic ◦ conformal ≠ harmonic

Modification

\( \varphi \in \text{Conf}(X^n, g), \quad \varphi^* g = C^2_\varphi g \)
Harmonic ▪ conformal ≠ harmonic

Modification
φ ∈ Conf(X^n, g), φ^*g = C_{φ}^2g

- pull-back ⇒ twisted pull-back
  f ◦ φ ⇒ C_{φ}^{-\frac{n-2}{2}} f ◦ φ

conformal factor
Harmonic $\circ$ conformal $\neq$ harmonic

Modification
$\varphi \in \text{Conf}(X^n, g), \quad \varphi^* g = C_\varphi^2 g$

- pull-back $\rightsquigarrow$ twisted pull-back
  
  $f \circ \varphi \rightsquigarrow C_\varphi^{-\frac{n-2}{2}} f \circ \varphi$

  conformal factor

- $\text{Sol}(\Delta_X) = \{ f \in C^\infty(X) : \Delta_X f = 0 \}$ (harmonic functions)
  
  $\rightsquigarrow \text{Sol}(\widetilde{\Delta}_X) = \{ f \in C^\infty(X) : \widetilde{\Delta}_X f = 0 \}$

  $\widetilde{\Delta}_X := \Delta_X + \frac{n-2}{4(n-1)} \kappa$

  Yamabe operator  Laplacian  scalar curvature
Distinguished rep. of conformal groups

harmonic ◦ conformal ÷ harmonic

↓ Modification
Distinguished rep. of conformal groups

\[ \text{harmonic} \circ \text{conformal} \triangleq \text{harmonic} \]

\[ \downarrow \text{Modification} \]

\begin{quote}
\textbf{Theorem A} ([K–Ørsted 03]) \quad (X^n, g): \text{Riemannian mfd}

\[ \implies \text{Conf}(X, g) \text{ acts on } Sol(\tilde{\Delta}_X) \text{ by } f \mapsto C_{\varphi}^{-\frac{n-2}{2}} f \circ \varphi \]
\end{quote}
Distinguished rep. of conformal groups

harmonic \circ \text{conformal} \vdash \text{harmonic}

\downarrow \text{Modification}

\textbf{Theorem A ([K–Ørsted 03])} \quad (X^n, g): \text{pseudo-Riemannian mfd}

\implies \text{Conf}(X, g) \text{ acts on } Sol(\Delta_X) \text{ by } f \mapsto C_{\varphi}^{-\frac{n-2}{2}} f \circ \varphi
Distinguished rep. of conformal groups

harmonic ◦ conformal \lneq harmonic

\[\downarrow \text{Modification}\]

**Theorem A ([K–Ørsted 03])** \((X^n, g)\): pseudo-Riemannian mfd

\[\Rightarrow \text{Conf}(X, g) \text{ acts on } \text{Sol}(\widetilde{\Delta}_X) \text{ by } f \mapsto C_{\varphi}^{\frac{n-2}{2}} f \circ \varphi\]

**Point** \(\widetilde{\Delta}_X = \Delta_X + \frac{n-2}{4(n-1)} \kappa\)

\(\Delta_X\) is not invariant by Conf\((X, g)\).

But \(\text{Sol}(\widetilde{\Delta}_X)\) is invariant by Conf\((X, g)\).
Distinguished rep. of conformal groups

harmonic \circ \text{conformal} \div \text{harmonic}

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\textbf{Theorem A ([K–Ørsted 03])} \ (X^n, g): \text{pseudo-Riemannian mfd}

\implies \text{Conf}(X, g) \text{ acts on } \text{Sol}(\widetilde{\Delta}_X) \text{ by } f \mapsto C^{-\frac{n-2}{2}} f \circ \varphi

\begin{align*}
\text{Point} & \quad \widetilde{\Delta}_X = \Delta_X + \frac{n-2}{4(n-1)} \kappa \\
\Delta_X & \text{ is not invariant by } \text{Conf}(X, g). \\
\text{But } \text{Sol}(\widetilde{\Delta}_X) & \text{ is invariant by } \text{Conf}(X, g).
\end{align*}

\text{Diffeo}(X) \supset \text{Conf}(X, g) \supset \text{Isom}(X, g)

Conformal group \quad \text{isometry group}
Application of Theorem A

$$(X, g) := (S^p \times S^q, \overbrace{+ \cdots +}^p \overbrace{- \cdots -}^q)$$
Application of Theorem A

\[(X, g) := (S^p \times S^q, \underbrace{+ \cdots +}_{p} \underbrace{- \cdots -}_{q})\]

Theorem B ([7, Part I]) \(\widetilde{\Delta}_X = \Delta_{S^p} - \Delta_{S^q} + \text{const.}\)

0) \(\text{Conf}(X, g) \approx O(p + 1, q + 1)\)

1) \(\text{Sol}(\widetilde{\Delta}_X) \neq \{0\} \iff p + q \text{ even}\)

2) If \(p + q\) is even and \(> 2\), then
   \(\text{Conf}(X, g) \sim \text{Sol}(\widetilde{\Delta}_X)\) is irreducible,
   and for \(p + q > 6\) it is a minimal rep of \(O(p + 1, q + 1)\).
Application of Theorem A

\[(X, g) := (S^p \times S^q, \underbrace{+ \cdots +}_p \underbrace{- \cdots -}_q)\]

**Theorem B ([7, Part I])** \(\widetilde{\Delta}_X = \Delta_{S^p} - \Delta_{S^q} + \text{const.}\)

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   \(\text{Conf}(X, g) \overset{\sim}{\lhr} \text{Sol}(\widetilde{\Delta}_X)\) is irreducible,
   and for \(p + q > 6\) it is a minimal rep of \(O(p + 1, q + 1)\).

**Remark 1.** \(\text{Sol} \cdots\) in the sense of \(C^\infty\), hyperfunction, etc
   (the result (1) remains true)

**Remark 2.** For \(q = 1\), take a double cover \(X'\) of \(X\).
   Then \(\text{Sol}(\widetilde{\Delta}_{X'}) \neq \{0\}\) for all \(p\).
Theorem B ([7, Part I]) \[ \widetilde{\Lambda}_X = \Delta_{S^p} - \Delta_{S^q} + \text{const.} \]

0) \( \text{Conf}(X, g) \approx O(p + 1, q + 1) \)

1) \( \text{Sol}(\widetilde{\Lambda}_X) \neq \{0\} \iff p + q \text{ even} \)

2) If \( p + q \) is even and \( > 2 \), then
   \( \text{Conf}(X, g) \) is irreducible,
   and for \( p + q > 6 \) it is a minimal rep of \( O(p + 1, q + 1) \).
Two constructions of minimal reps.

1. Conformal construction
   **Theorem B**
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   **Theorem D**

Group action       Hilbert structure

Clear  ?

Clear  ... advantage of the model
Hilbert structure on \( \text{Sol}(\tilde{\Delta}) \)

Three methods:

1. Parseval-type formula [7, Part II]
2. Using Green function [7, Part III]
3. ‘Intrinsic formula’
Hilbert structure on $\text{Sol}(\tilde{\Delta})$

Three methods:

1. Parseval-type formula [7, Part II]
2. Using Green function [7, Part III]
3. ‘Intrinsic formula’
Flat model

Stereographic projection

\[ S^n \to \mathbb{R}^n \cup \{\infty\} \text{ conformal map} \]
Flat model

Stereographic projection

\[ S^n \rightarrow \mathbb{R}^n \cup \{\infty\} \]  conformational map

More generally

\[ S^p \times S^q \leftrightarrow \mathbb{R}^{p+q} \]  conformational embedding

\[ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]
Flat model

Stereographic projection

\[ S^n \rightarrow \mathbb{R}^n \cup \{\infty\} \quad \text{conformal map} \]

More generally

\[ S^p \times S^q \leftrightarrow \mathbb{R}^{p+q} \quad \text{conformal embedding} \]

\[ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

Functoriality of Theorem A

\[
\begin{align*}
\text{Sol}(\tilde{\Delta}_{S^p \times S^q}) & \subset \text{Sol}(\tilde{\Delta}_{\mathbb{R}^{p,q}}) \\
\text{Conf}(S^p \times S^q) & \leftrightarrow \text{Conf}(\mathbb{R}^{p,q})
\end{align*}
\]
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]
Conservative quantity for ultra-hyperbolic eqn.

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**Problem** Find an ‘intrinsic’ inner product on (a ‘large’ subspace of) \( \text{Sol}(\Box_{p,q}) \) if exists.
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

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Unitarization of subrep (representation theory) \iff Conservative quantity (differential eqn)
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \widetilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

\[ q = 1 \quad \text{wave operator} \]
Conservative quantity for ultra-hyperbolic eqn.

\[ R^{p,q} = R^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \Delta_{R^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

\[ q = 1 \quad \text{wave operator} \]

\[ \text{energy} \cdots \text{conservative quantity for wave equations} \]

\[ \text{w.r.t. time translation} \quad R \]
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \square_{p,q} \]

\[ q = 1 \quad \text{wave operator} \]

\[ \downarrow \]

energy \cdots \text{conservative quantity for wave equations w.r.t. time translation} \quad \mathbb{R} \]

\[ ? \cdots \text{conservative quantity for ultra-hyperbolic eqs w.r.t. conformal group} \quad O(p + 1, q + 1) \]
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \text{Sol}(\Box_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \quad \text{(to be defined soon)} \quad \ldots \ldots \text{①}$$
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f \quad \text{(to be defined soon)} \quad \ldots \ldots \text{①}$$


1) ① is independent of hyperplane $\alpha$. 
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \text{Sol}(\Box_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f \quad \text{(to be defined soon)}$$

\[ \text{\textbf{Theorem C } ([7, Part III] + K–2011)} \]

1) $\Box$ is independent of hyperplane $\alpha$.

2) $\Box$ gives the unique inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$. 
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f \quad \text{(to be defined soon)}$$

\[ \ldots \ldots \quad (1) \]

\[ \text{Theorem C} \quad ([7, \text{Part III}] + K-2011) \]

1) $(1)$ is independent of hyperplane $\alpha$.

2) $(1)$ gives the \textit{unique} inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$.

$$O(p, q) \quad \sim \quad \mathbb{R}^{p,q} \quad \text{(linear)}$$
Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \text{Sol}(\square_{p,q})$

$$(f, f) := \int_{\alpha} Q_\alpha f \quad (\text{to be defined soon})$$

\[ \text{Theorem C } ([7, \text{ Part III}] + \text{K–2011}) \]

1) (1) is independent of hyperplane $\alpha$.

2) (1) gives the \textbf{unique} inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$.

\[
O(p, q) \quad \sim \quad \mathbb{R}^{p,q} \quad \text{(linear)}
\]

(Möbius transform)
Construction of $\mathcal{Q}_\alpha f$

$\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2)$

Fix $v \in \mathbb{R}^{p,q}$ s.t. $(v, v)_{\mathbb{R}^{p,q}} = \pm 1$

$c \in \mathbb{R}$

$\mathbb{R}^{p,q} \ni \alpha \equiv \alpha_{v,c} := \{x \in \mathbb{R}^{p+q} : (x, v)_{\mathbb{R}^{p,q}} = c\}$

non-characteristic hyperplane
Construction of $Q_\alpha f$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)
Construction of $Q_{\alpha}f$

For $\alpha = \alpha_{v, c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)
Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)

$f_\pm$ extends holomorphically to the direction $\pm \sqrt{-1}v$
Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)

$$f_\pm(x; v) := \frac{1}{2\pi i} \int_{\mathbb{R}} \mp f(x - tv) \frac{dt}{t \pm i0}$$
Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)

$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. $v$

$\alpha_{v,c}$

$(x, v)_{\mathbb{R}^{p,q}} = c$
Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)

$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. $v$

$$Q_\alpha f := \frac{1}{i} \left( f_+ \overline{f'_+} - f_- \overline{f'_-} \right)$$
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in Sol(\Box_{p,q})$

$$(f, f) := \int_{\alpha} Q_\alpha f$$

\[\cdots \cdots \textcircled{1}\]

\[\textbf{Theorem C}\]

1) \textcircled{1} is independent of hyperplane $\alpha$.

2) \textcircled{1} gives the unique inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$. 
Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in Sol(\square_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f$$

\[ \cdots \cdots \text{(1)} \]

**Theorem C**

1) \text{(1)} is independent of hyperplane $\alpha$.
2) \text{(1)} gives the **unique** inner product (up to scalar) which is invariant under $O(p+1, q+1)$. 

Theorem C is non-trivial even for $q = 1$ (wave equation)

In space-time $\mathbb{R}^{p+1} = \mathbb{R}^p_x \times \mathbb{R}_t$,

average in **space** (i.e. **time** $t$ = constant)

= average in (any hyperplane in **space**) $\times \mathbb{R}_t$ (**time**)
Two constructions of minimal reps.

1. Conformal construction
   
   Theorems A, B
   v.s.

2. Clear

   Clear ··· advantage of the model
Two constructions of minimal reps.

1. Conformal construction
   **Theorems A, B**
   v.s.

2. Group action
   Clear

   **Hilbert structure**
   conservative quantity
   Theorem C

Clear ••• advantage of the model
Two constructions of minimal reps.

1. Conformal construction
   Theorems A, B
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

Clear ••• advantage of the model
§1 What are minimal representations?

§2 Conformal model of minimal representations

§3 Geometric quantization of minimal nilpotent orbits and $L^2$ model

§4 Deformation of Fourier transform
§ 1  What are minimal representations?

§ 2  Conformal model of minimal representations

§ 3  Geometric quantization of minimal nilpotent orbits and $L^2$ model

§ 4  Deformation of Fourier transform
Conformal model $\implies L^2$-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$
Conformal model $\Rightarrow L^2$-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}$$

= $\triangle$ (figure for $(p, q) = (2, 1)$)
Conformal model $\implies L^2$-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}$$

$$\Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi$$

*Fourier trans.*
Conformal model $\implies L^2$-model

$$\square_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$\square_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi$$

Fourier trans.

$$\mathcal{F} : S'(\mathbb{R}^{p,q}) \xrightarrow{\sim} S'(\mathbb{R}^{p,q})$$
Conformal model $\cong L^2$-model

$$\square_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$\square_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi$$

Fourier trans.

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^{p,q}) \sim \mathcal{S}'(\mathbb{R}^{p,q})$$

$$\text{Sol}(\square_{p,q})$$
Conformal model $\implies L^2$-model

$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$

$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$

\[ \Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi \]

$\mathcal{F} : S'(\mathbb{R}^{p,q}) \sim \rightarrow S'(\mathbb{R}^{p,q})$

$\overline{\text{Sol}(\Box_{p,q})} \sim \rightarrow \square$?

\[ \sim \] denotes the Hilbert completion w.r.t. the invariant inner product.
Conformal model \( \Rightarrow L^2\)-model

\[
\square_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}
\]

\(\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}\)

\[
\square_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi
\]

Fourier trans.

\[
\mathcal{F} : S'(\mathbb{R}^{p,q}) \xrightarrow{\sim} S'(\mathbb{R}^{p,q})
\]

Theorem D ([7, Part III]) \(\overline{\text{Sol}(\square_{p,q})} \xrightarrow{\sim} L^2(\Xi)\)
Conformal model $\iff L^2$-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$\Box_{p,q} f = 0 \iff \text{Supp } \mathcal{F} f \subset \Xi$$

Fourier trans.

$$\mathcal{F} : S'(\mathbb{R}^{p,q}) \sim \rightarrow S'(\mathbb{R}^{p,q})$$

Theorem D ([7, Part III])

$$\text{Sol}(\Box_{p,q}) \sim \rightarrow L^2(\Xi)$$

conformal model $L^2$-model
Two constructions of minimal reps.

1. Conformal construction
   Theorems A, B
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

Clear \cdots advantage of the model
$L^2$-model of minimal reps.

**Theorem D** \( p + q > 2 \), even. \( \overline{\text{Sol}(\Box_{p,q})} \xrightarrow{\sim} L^2(\mathbb{E}) \)

conformal model \( L^2 \)-model
$L^2$-model of minimal reps.

**Theorem D** \( p + q > 2, \text{ even.} \)  \( \overline{\text{Sol}(\Box_{p,q})} \xrightarrow{\sim} L^2(\Xi) \)

conformal model  \( L^2 \)-model

minimal rep.

\[ G = \text{O}(p + 1, q + 1) \xrightarrow{\sim} L^2(\Xi) \quad \text{unitary rep.} \]
$L^2$-model of minimal reps.

**Theorem D**  \( p + q > 2 \), even.  \( \overline{\text{Sol}(\Box_{p,q})} \xrightarrow{\sim} L^2(\Xi) \)

conformal model  \( L^2 \)-model

minimal rep.

\[
G = O(p + 1, q + 1) \xrightarrow{\sim} L^2(\Xi) \quad \text{unitary rep.}
\]

\[
\text{dim } \Xi = p + q - 1 \implies \Xi \text{ is too small to be acted by } G.
\]
$L^2$-model of minimal reps.

**Theorem D**  \[ p + q > 2, \text{ even.} \quad \overline{\text{Sol}(\square_{p,q})} \sim L^2(\Xi) \]

**conformal model** \[ L^2\text{-model} \]

**minimal rep.**

\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \text{ unitary rep.} \]

\[ \dim \Xi = p + q - 1 \implies \Xi \text{ is too small to be acted by } G. \]

\[ \Xi \subset \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1} \]
$L^2$-model of minimal reps.

**Theorem D** \( p + q > 2, \text{ even.} \) \( \overline{Sol(\Delta_{p,q})} \sim L^2(\mathbb{E}) \)

conformal model $L^2$-model

**minimal rep.**

\[ G = O(p + 1, q + 1) \sim L^2(\mathbb{E}) \]

unitary rep.

\[ \dim \mathbb{E} = p + q - 1 \implies \mathbb{E} \text{ is too small to be acted by } G. \]

\[ \mathbb{E} \subset \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1} \]

What is $\mathbb{E}$?
Geometric quantization of minimal nilpotent orbit

$G \not\sim \Xi$ but $G \sim L^2(\Xi)$

What is $\Xi$?
Geometric quantization of minimal nilpotent orbit

\[ \mathfrak{g}^* \ni \mathcal{O}_{\text{min}} = \text{Ad}^*(G)\lambda \text{ coadjoint orbit} \]

\[ \downarrow \text{``geometric quantization''} \]

\[ \widehat{G} \ni \pi \text{ irred. unitary rep of } G \]
Geometric quantization of minimal nilpotent orbit

\[ g^* \supset O_{\text{min}} = \text{Ad}^*(G)\lambda \quad \text{coadjoint orbit} \]

\[ \downarrow \quad ? \quad \text{"geometric quantization"} \]

\[ \tilde{G} \ni \pi \quad \text{irred. unitary rep of } G \]

hyperbolic orbit \( \rightsquigarrow \) real polarization \( \rightsquigarrow \) principal series reps
(induced reps from parabolic subgroups)

elliptic orbit \( \rightsquigarrow \) complex polarization \( \rightsquigarrow \) Zuckerman derived functor
modules/Dolbeault cohomologies on non-compact homogeneous
complex manifolds

(Kostant, Schmid, Zuckerman, Vogan, Schmid, Bernstein, ... late
1970s–1990s)
Geometric quantization of minimal nilpotent orbit

Difficult case

\[ \mathfrak{g}^* \ni \mathcal{O}_{\text{min}} = \text{Ad}^*(G) \lambda \quad \text{minimal nilp. orbit} \]

\[ \downarrow \quad ? \quad \quad \text{“geometric quantization”} \]

\[ \widehat{G} \ni \pi \quad \text{minimal rep of } G \]

hyperbolic orbit \( \rightsquigarrow \) real polarization \( \rightsquigarrow \) principal series reps
(induced reps from parabolic subgroups)

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modules/Dolbeault cohomologies on non-compact homogeneous complex manifolds

(Kostant, Schmid, Zuckerman, Vogan, Schmid, Bernstein, ... late 1970s–1990s)
Geometric quantization of minimal nilpotent orbit

Difficult case

\[ \mathfrak{g}^* \supset O_{\text{min}} = \text{Ad}^*(G)\lambda \quad \text{minimal nilp. orbit} \]
\[ \Downarrow ? \quad \text{“geometric quantization”} \]
\[ \widehat{G} \ni \pi \quad \text{minimal rep of } G \]

hyperbolic orbit \(\rightsquigarrow\) real polarization \(\rightsquigarrow\) principal series reps
(induced reps from parabolic subgroups)

elliptic orbit \(\rightsquigarrow\) complex polarization \(\rightsquigarrow\) Zuckerman derived functor modules/Dolbeault cohomologies on non-compact homogeneous complex manifolds

(Kostant, Schmid, Zuckerman, Vogan, Schmid, Bernstein, ... late 1970s–1990s)

In the previous theorem, \(\Xi\) is Lagrangian submanifold of \(O_{\text{min}}\) for \(G = O(p + 1, q + 1)\)!
$L^2$-model of minimal rep.

$V$: simple Jordan algebra

$G = (a$ finite covering of$)$ the conformal group of $V$
$L^2$-model of minimal rep.

$V$: simple Jordan algebra

$G = (a$ finite covering of$) the conformal group of $V$

**Ex 1**  
$V = \text{Symm}(n, \mathbb{R})$

$G = Mp(n, \mathbb{R})$, a double cover of $Sp(n, \mathbb{R})$

**Ex 2**  
$V = \mathbb{R}^{p,q+1}$

$G = O(p + 1, q + 1)$
**$L^2$-model of minimal rep.**

$V$: simple Jordan algebra

$G = (a$ finite covering of) the conformal group of $V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$

$\Xi := O_{\text{min}} \cap V$

**Observation**

$$g \sim g^*$$

$$\cup \quad \cup$$

$$V \quad O_{\text{min}}$$

$$\cup \quad \cup$$

$$\Xi$$
$L^2$-model of minimal rep.

$V$: simple Jordan algebra

$G = (a$ finite covering of) the conformal group of $V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$

$\Xi := O_{\text{min}} \cap V$

Ex 1  \[ g = \text{sp}(n, \mathbb{R}) \]

\[
\begin{align*}
V &= \text{Symm}(n, \mathbb{R}) \\
\cup &\quad \cup \text{Lagrangian} \\
\Xi &= \{X : X = {}^t X, \text{ rank } X \leq 1\}
\end{align*}
\]
$L^2$-model of minimal rep.

$V$: simple Jordan algebra

$G = (a$ finite covering of) the conformal group of $V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$

$\Xi := O_{\text{min}} \cap V$

\begin{align*}
\text{Ex 2} & \quad g = \vartheta(p + 1, q + 1) \\
V & = \mathbb{R}^{p+q+1} \\
& \subseteq O_{\text{min}} \\
& \subseteq \text{Lagrangian} \\
\Xi & = \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}
\end{align*}
$L^2$-model of minimal rep.

$V$: simple Jordan algebra

$G = (a$ finite covering of$) the conformal group of $V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$

$\Xi := O_{\text{min}} \cap V$

Assume that a maximal euclidean Jordan subalgebra of $V$ is simple, and $V \neq \mathbb{R}^{p.q+1}$ with $p + q$: odd.

**Theorem** (with Hilgert, Moellers, arXiv:1106.3621)

1) $\Xi$ is a Lagrangian submanifold of $O_{\text{min}}$. 
$L^2$-model of minimal rep.

$V$: simple Jordan algebra

$G = \text{(a finite covering of) the conformal group of } V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$

$\Xi := O_{\text{min}} \cap V$

Assume that a maximal euclidean Jordan subalgebra of $V$ is simple, and $V \not\cong \mathbb{R}^{p,q+1}$ with $p + q$: odd.

**Theorem** (with Hilgert, Moellers, [arXiv:1106.3621](http://arxiv.org/abs/1106.3621))

1) $\Xi$ is a Lagrangian submanifold of $O_{\text{min}}$.

2) We get an irreducible unitary representation of $G$ on $L^2(\Xi)$. 
$L^2$-model of minimal rep.

$V$: simple Jordan algebra

$G = (a$ finite covering of$) the conformal group of V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$

$\Xi := O_{\text{min}} \cap V$

Assume that a maximal euclidean Jordan subalgebra of $V$ is simple, and $V \neq \mathbb{R}^{p,q+1}$ with $p + q$: odd.


1) $\Xi$ is a Lagrangian submanifold of $O_{\text{min}}$.
2) We get an irreducible unitary representation of $G$ on $L^2(\Xi)$.
3) The Gelfand–Kirillov dimension attains its minimum among all $(\infty\text{-dim}'l)$ irreducible unitary representations of $G$. 
$L^2$-model of minimal rep.

$V$: simple Jordan algebra
$G = \text{(a finite covering of) the conformal group of } V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$
$\Xi := O_{\text{min}} \cap V$

Assume that a maximal euclidean Jordan subalgebra of $V$ is simple, and $V \not\cong \mathbb{R}^{p,q+1}$ with $p + q$: odd.

---

**Theorem** (with Hilgert, Moellers, [arXiv:1106.3621](http://arxiv.org/abs/1106.3621))

1) $\Xi$ is a Lagrangian submanifold of $O_{\text{min}}$.
2) We get an irreducible unitary representation of $G$ on $L^2(\Xi)$.
3) The Gelfand–Kirillov dimension attains its minimum among all ($\infty$-dim’l) irreducible unitary representations of $G$.
4) The annihilator of the differential rep $d\pi$ is the Joseph ideal in $U(\mathfrak{g})$ if $V$ is split and $\mathfrak{g} \not\cong A_n$. 
$L^2$-model of minimal rep.

$V$: simple Jordan algebra
$G = (a$ finite covering of$)$ the conformal group of $V$

$O_{\text{min}}$: minimal nilpotent coadjoint orbit of $G$
$\Xi := O_{\text{min}} \cap V$

**Ex 1**

$V = \text{Symm}(n, \mathbb{R})$
$G = Mp(n, \mathbb{R})$
$\Rightarrow$ Schrödinger model of the Weil representation

$G \sim L^2(\mathbb{R}^n_{\text{even}}) \simeq L^2(\Xi)$

**Ex 2**

$V = \mathbb{R}^{p,q+1}$, $p + q$: even
$G = O(p + 1, q + 1)$
$\Rightarrow G \sim L^2(\Xi)$
(Theorem D)
Inversion element

\[ G = PGL(2, \mathbb{C}) \sim \mathbb{P}^1 \mathbb{C} \simeq \mathbb{C} \cup \{\infty\} \]

Möbius transform
Inversion element

\[ G = PGL(2, \mathbb{C}) \sim \mathbb{P}^1 \mathbb{C} \simeq \mathbb{C} \cup \{\infty\} \]

Möbius transform

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \quad z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \quad \text{(inversion)} \]
Inversion element

\[ G = \text{PGL}(2, \mathbb{C}) \quad \sim \quad \mathbb{P}^1 \mathbb{C} \cong \mathbb{C} \cup \{\infty\} \]

Möbius transform

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \quad \quad z \mapsto az + b \]

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\( G \) is generated by \( P \) and \( w \).
Inversion element

\[ G = PGL(2, \mathbb{C}) \sim \mathbb{P}^1 \mathbb{C} \cong \mathbb{C} \cup \{\infty\} \]

 Möbius transform

\[ \div O(3, 1) \quad \div \mathbb{R}^{2,0} \]

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \quad z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \quad \text{(inversion)} \]

\( G \) is generated by \( P \) and \( w \).
**Inversion element**

\[ G = PGL(2, \mathbb{C}) \quad \sim \quad \mathbb{P}^1 \mathbb{C} \cong \mathbb{C} \cup \{ \infty \} \]

Möbius transform

\[ \div O(3, 1) \quad \div \mathbb{R}^{2,0} \]

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \quad z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \quad \text{(inversion)} \]

\( G \) is generated by \( P \) and \( w \).

\[ G = O(p + 1, q + 1) \quad \sim \quad \mathbb{R}^{p,q} \]

Möbius transform

\[ P = \{(A, b) : A \in O(p, q) \cdot \mathbb{R}^\times, \ b \in \mathbb{R}^{p+q}\} \quad x \mapsto Ax + b \]

\[ w = \begin{pmatrix} I_p & -I_q \end{pmatrix} \quad \text{(inversion)} \]
Inversion element

\[ G = PGL(2, \mathbb{C}) \sim \mathbb{P}^1 \mathbb{C} \simeq \mathbb{C} \cup \{\infty\} \]

\[ \overset{\text{Möbius transform}}{\div} O(3, 1) \overset{\text{div}}{\div} \mathbb{R}^{2,0} \]

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \]

\[ z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ z \mapsto -\frac{1}{z} \quad \text{(inversion)} \]

\( G \) is generated by \( P \) and \( w \).

\[ G = O(p + 1, q + 1) \sim \mathbb{R}^{p,q} \]

\[ \overset{\text{Möbius transform}}{\div} P = \{ (A, b) : A \in O(p, q) \cdot \mathbb{R}^\times, \ b \in \mathbb{R}^{p+q} \} \quad x \mapsto Ax + b \]

\[ w = \begin{pmatrix} I_p & -I_q \end{pmatrix} : (x', x'') \mapsto \frac{4}{|x'|^2 - |x''|^2} (-x', x'') \quad \text{(inversion)} \]
Towards a global formula

\[ p + q: \text{even} > 2 \]

\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{minimal rep.} \]
Towards a global formula

\[ p + q: \text{even} > 2 \]
\[ G = O(p + 1, q + 1) \overset{\sim}{\longrightarrow} L^2(\Xi) \quad \text{minimal rep.} \]

- \( P \)-action \( \cdots \) translation and multiplication
- \( w \)-action \( \cdots \) \( \mathcal{F}_\Xi \) (unitary inversion operator)
Towards a global formula

\[ p + q: \text{even} > 2 \]

\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{minimal rep.} \]

- \text{\textit{P}}\text{-action} \quad \cdots \quad \text{translation and multiplication}
- \text{\textit{w}}\text{-action} \quad \cdots \quad \mathcal{F}_\Xi \quad \text{(unitary inversion operator)}

\textbf{Problem} \quad \text{What is } \mathcal{F}_\Xi?
Towards a global formula

\[ p + q: \text{ even} > 2 \]

\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{minimal rep.} \]

- \( P \)-action \( \cdots \) translation and multiplication
- \( w \)-action \( \cdots \) \( \mathcal{F}_\Xi \) (unitary inversion operator)

**Problem** What is \( \mathcal{F}_\Xi \)?

**Cf.** Analogous operator for the oscillator rep.

\[ Mp(n, \mathbb{R}) \sim L^2(\mathbb{R}^n) \]

unitary inversion operator coincides with

Euclidean Fourier transform \( \mathcal{F}_{\mathbb{R}^n} \) (up to scalar)!
New Fourier transform $\mathcal{F}_\Xi$ on $\Xi$

$$\Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}$$

= \phantom{\Xi}

(figure for $(p, q) = (2, 1)$)
New Fourier transform $\mathcal{F}_\Xi$ on $\Xi$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$\Xi \approx \bigcap \bigcup$ (figure for $(p, q) = (2, 1)$)

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi \approx \bigcap \bigcup$
New Fourier transform $\mathcal{F}_\Xi$ on $\Xi$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$= \quad \text{(figure for} \ (p, q) = (2, 1))$$

Fourier trans. $\mathcal{F}_\mathbb{R}^n$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi = \quad \text{(figure)}$

Problem
1. Algebraic properties of $\mathcal{F}_\Xi$
2. Analytic formula of $\mathcal{F}_\Xi$. 
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi = \cdots$
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}^4 = \text{id}$

$\mathcal{F}_\Xi$ on $\Xi = \ldots$
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}^4 = \text{id}$

$\mathcal{F}_\Xi$ on $\Xi = $ 

$\mathcal{F}_\Xi^2 = \text{id}$
'Fourier transform' $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

\[
\begin{align*}
Q_j & \leftrightarrow -P_j \\
P_j & \leftrightarrow Q_j
\end{align*}
\]

$Q_j = x_j$ \hspace{0.5cm} (multiplication by coordinates function)

$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$

$\mathcal{F}_\Xi$ on $\Xi = \bigcirc$
`Fourier transform` $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

- $Q_j \mapsto -P_j$
- $P_j \mapsto Q_j$

$Q_j = x_j$ \hspace{1cm} (multiplication by coordinates function)

$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$

$R_j = \exists$ second order differential op. on $\Xi$
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

<table>
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<tr>
<th>Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$</th>
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<tr>
<td>$Q_j \mapsto -P_j$</td>
<td>$Q_j \mapsto R_j$</td>
</tr>
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<td>$R_j \mapsto Q_j$</td>
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$Q_j = x_j$ \hspace{1cm} (multiplication by coordinates function)

$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$

$R_j = \exists$ second order differential op. on $\Xi$

Rediscover Bargmann–Todorov’s operators (1977)
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$Q_j \mapsto -P_j$

$P_j \mapsto Q_j$

$Q_j = x_j$ (multiplication by coordinates function)

$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$

$R_j = \Xi$ second order differential op. on $\Xi$

Notice

\[
\begin{align*}
Q_1^2 + \cdots + Q_p^2 - Q_{p+1}^2 - \cdots - Q_{p+q}^2 &= 0 \\
R_1^2 + \cdots + R_p^2 - R_{p+1}^2 - \cdots - R_{p+q}^2 &= 0
\end{align*}
\] on $\Xi$
Unitary inversion operator $\mathcal{F}_\Xi$

$p + q$: even > 2

$G = O(p + 1, q + 1) \underset{\sim}{\sim} L^2(\Xi)$ minimal rep.

$\omega$-action $\cdots \mathcal{F}_\Xi$ (unitary inversion operator)

**Problem** Find the unitary operator $\mathcal{F}_\Xi$ explicitly.
Unitary inversion operator $F_{\Xi}$

$p + q$: even $> 2$

\[ G = O(p + 1, q + 1) \cong L^2(\Xi) \quad \text{minimal rep.} \]

\[ w\text{-action} \quad \cdots \quad F_{\Xi} \text{ (unitary inversion operator)} \]

Problem Find the unitary operator $F_{\Xi}$ explicitly.

Cf. Euclidean case $\varphi(t) = e^{-it}$ (one variable)

\[ F_{\mathbb{R}^N} f(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy \]
Unitary inversion operator $\mathcal{F}_\Xi$

$p + q$: even $> 2$

\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{minimal rep.} \]

$\omega$-action \cdots $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem Find the unitary operator $\mathcal{F}_\Xi$ explicitly.

Cf. Euclidean case \[ \varphi(t) = e^{-it} \quad \text{(one variable)} \]

\[ \mathcal{F}_{\mathbb{R}^N} f(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy \]

Thm E (K–Mano, Memoirs AMS, 2011, vol.1000) \[ (\mathcal{F}_\Xi f)(x) = c \int_{\Xi} \Phi(\langle x, y \rangle) f(y) dy \]
$\mathcal{F}_{\mathbb{R}^N} \text{ v.s. } \mathcal{F}_\Xi$

On $\mathbb{R}^N$

\[(\mathcal{F}_{\mathbb{R}^N} f)(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy \]

$\varphi(t) = e^{-it}$ satisfies

\[\left(\frac{d}{dt} + i\right)\varphi(t) = 0\]
\( \mathcal{F}_{\mathbb{R}^N} \text{ v.s. } \mathcal{F}_\Xi \)

On \( \mathbb{R}^N \)

\[(\mathcal{F}_{\mathbb{R}^N} f)(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy \]

\[\varphi(t) = e^{-it} \text{ satisfies } \left( \frac{d}{dt} + i \right) \varphi(t) = 0 \]

On \( \Xi \quad (\subset \mathbb{R}^{p,q}) \)

\[(\mathcal{F}_\Xi f)(x) = c \int_{\Xi} \Phi(\langle x, y \rangle) f(y) dy \]

\[\Phi(t) \text{ satisfies } \left( \left( t \frac{d}{dt} \right)^2 + \frac{1}{2} (p + q - 4) t \frac{d}{dt} + 2t \right) \Phi(t) = 0 \]
Bessel functions

\[ J_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j \left( \frac{z}{2} \right)^{2j}}{j! \Gamma(j + \nu + 1)} \]

\[ I_\nu(z) := e^{-\frac{\sqrt{-1}\nu\pi}{2}} J_\nu \left( e^{\frac{\sqrt{-1}\pi}{2}} z \right) \]

\[ Y_\nu(z) := \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi} \quad \text{(second kind)} \]

\[ K_\nu(z) := \frac{\pi}{2 \sin \nu\pi} (I_{-\nu}(z) - I_\nu(z)) \quad \text{(third kind)} \]
Bessel distribution

Prop. \( \Phi_{m}^{\varepsilon}(t) \) solves the differential equation
\[
(\theta^2 + m\theta + 2t)u = 0
\]
where \( \theta = t \frac{d}{dt} \).
Bessel distribution

Prop. \( \Phi_m^x(t) \) solves the differential equation
\[
(\theta^2 + m\theta + 2t)u = 0
\]
where \( \theta = t \frac{d}{dt} \).

Explicit forms

\[
\Phi^0_m(t) = 2\pi i (2t)^{\frac{m}{2}} J_m(2 \sqrt{2t})
\]
\[
\Phi^1_m(t) = \Phi^0_m(t) - \pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!} \delta^{(l)}(t)
\]
Bessel distribution

Prop. $\Phi_m^\epsilon(t)$ solves the differential equation

$$(\theta^2 + m\theta + 2t)u = 0$$

where $\theta = t \frac{d}{dt}$.

Explicit forms

$$\Phi_0^0(t) = 2\pi i (2t)_+^{-m/2} J_m(2 \sqrt{2t_+})$$

$$\Phi_1^0(t) = \Phi_0^0(t) - \pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!} \delta^{(l)}(t)$$

$$\Phi_2^0(t) = 2\pi i (2t)_+^{-m/2} Y_m(2 \sqrt{2t_+})$$

$$\quad + 4(-1)^{m+1} i (2t)_-^{-m/2} K_m(2 \sqrt{2t_-})$$
Bessel distribution

**Prop.** \( \Phi_m(t) \) solves the differential equation

\[
(\theta^2 + m\theta + 2t)u = 0
\]

where \( \theta = t \frac{d}{dt} \).


\[
(\mathcal{F}_\Xi f)(x) = c \int_{\Xi} \Phi^{(p,q)}_{\frac{1}{2}+q-4} \langle x, y \rangle f(y) dy
\]
Bessel distribution

**Prop.** $\Phi_m(t)$ solves the differential equation

$$(\theta^2 + m\theta + 2t)u = 0$$

where $\theta = t \frac{d}{dt}$.


$$(\mathcal{F}_\Xi f)(x) = c \int_{\Xi} \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)} (\langle x, y \rangle) f(y) dy$$

Here, $\varepsilon(p, q) = \begin{cases} 
0 & \text{if } \min(p, q) = 1, \\
1 & \text{if } p, q > 1 \text{ are both odd,} \\
2 & \text{if } p, q > 1 \text{ are both even.}
\end{cases}$
Two constructions of minimal reps.

1. Conformal construction
   Theorems A, B
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

3. Deformation of Fourier transforms
   (Theorems F, G, H)

Group action

Hilbert structure

Conservative quantity

‘Fourier transform’ $\mathcal{F}_\Xi$
Two constructions of minimal reps.

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<td>Theorem E</td>
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<td>(Schrödinger model)</td>
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<td>Clear</td>
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§1 What are minimal representations?

§2 Conformal model of minimal representations

§3 Geometric quantization of minimal nilpotent orbits and $L^2$ model

§4 Deformation of Fourier transform
§ 1 What are minimal representations?

§ 2 Conformal model of minimal representations

§ 3 Geometric quantization of minimal nilpotent orbits and $L^2$ model

§ 4 Deformation of Fourier transform
Deformation theory of Fourier transform

- Generalized Fourier transform $\mathcal{F}_{k,a}$ C.R.A.S. Paris (2009)
- Laguerre semigroup and Dunkl operators 74 pp. Compositio Math (to appear), with Ben Saïd and Bent Ørsted
- Inversion and holomorphic extension R. Howe 60th birthday volume (2007), 65 pp. with Mano
Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$(k, a)$-generalized Fourier transform $\mathcal{F}_{k,a}$

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

- $a \rightarrow 2$
- $a \rightarrow 1$

$\mathcal{I}_{k,2}(t)$
- $t \rightarrow \frac{\pi i}{2}$
- $k \rightarrow 0$

Dunkl transform

$\mathcal{I}_{k,1}(t)$
- $t \rightarrow \frac{\pi i}{2}$
- $k \rightarrow 0$

Hermite semigroup

Laguerre semigroup

Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

Hankel-type transform $\mathcal{F}_{\Xi}$

‘unitary inversion operator’

the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of the conformal group $O(N + 1, 2)$
Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_\Xi \cdots \text{‘Fourier transform’ on } \Xi \subset \mathbb{R}^{p,q}$

$\mathcal{F}_{\mathbb{R}^N} \cdots \text{Fourier transform on } \mathbb{R}^N$
Interpolation of Fourier transform $\mathcal{F}_{R^N}$

$\mathcal{F}_\Xi \quad \cdots \quad \text{‘Fourier transform’ on } \Xi \subset \mathbb{R}^{p,q}$

$\mathcal{F}_{R^N} \quad \cdots \quad \text{Fourier transform on } R^N$

Assume $q = 1$. Set $p = N$.

$\mathbb{R}^{N,1} \supset \Xi = \begin{array}{c}
\text{projection} \\
\downarrow
\end{array} \Rightarrow \begin{array}{c}
\square = R^N
\end{array}$
Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_\Xi \cdots$ ‘Fourier transform’ on $\Xi \subset \mathbb{R}^{p,q}$

$\mathcal{F}_{\mathbb{R}^N} \cdots$ Fourier transform on $\mathbb{R}^N$

Assume $q = 1$. Set $p = N$.

$\mathbb{R}^{N,1} \supset \Xi = \begin{array}{c}
\text{projection} \\
\end{array} = \mathbb{R}^N$

$\mathcal{F}_\Xi \quad \quad \quad \quad \quad \mathcal{F}_{\mathbb{R}^N}$

$O(N + 1, 2) \quad \quad \quad \quad Mp(N, \mathbb{R})$
Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_{\Xi} \cdots \text{ ‘Fourier transform’ on } \Xi \subset \mathbb{R}^{p,q}$

$\mathcal{F}_{\mathbb{R}^N} \cdots \text{ Fourier transform on } \mathbb{R}^N$

Assume $q = 1$. Set $p = N$.

$$\mathbb{R}^{N,1} \supset \Xi = \begin{array}{c} \text{projection} \\ \longrightarrow \end{array} \Rightarrow = \mathbb{R}^N$$

$\mathcal{F}_{\Xi}$ interpolate $\mathcal{F}_{\mathbb{R}^N}$

$a = 1$ $a = 2$
\((k, a)\)-deformation of \( \exp \frac{t}{2}(\Delta - |x|^2) \)

Fourier transform

\[
\mathcal{F}_{\mathbb{R}^N} = c \exp \left( \frac{\pi i}{4} (\Delta - |x|^2) \right)
\]
(k, a)-deformation of \( \exp \frac{i}{2}(\Delta - |x|^2) \)

Fourier transform

self-adjoint op. on \( L^2(\mathbb{R}^N) \)

\[
F_{\mathbb{R}^N} = c \exp \left( \frac{\pi i}{4} \left( \Delta - |x|^2 \right) \right)
\]

phase factor \quad Laplacian

\[
= e^{\frac{\pi i N}{4}}
\]
(k, a)-deformation of \( \exp \frac{t}{2}(\Delta - |x|^2) \)

Fourier transform

self-adjoint op. on \( L^2(\mathbb{R}^N) \)

\[
\mathcal{F}_{\mathbb{R}^N} = c \exp \left( \frac{\pi i}{4} \left( \Delta - |x|^2 \right) \right)
\]

phase factor \ Laplacian
\[= e^{\frac{\pi i N}{4}}\]

Hermite semigroup

\[
I(t) := \exp \frac{t}{2}(\Delta - |x|^2)
\]

Mehler kernel using \( \exp(-x^2) \)
\( (k, a) \)-deformation of \( \exp \frac{t}{2}(\Delta - |x|^2) \)

\( (k, a) \)-generalized Fourier transform

self-adjoint op. on \( L^2(\mathbb{R}^N, \theta_{k,a}(x)dx) \)

\[
\mathcal{F}_{k,a} = c \exp \left( \frac{\pi i}{2a} \left( |x|^{2-a} \Delta_k - |x|^a \right) \right)
\]

phase factor \quad Dunkl Laplacian

\[
= e^{i\frac{\pi(N+2<k>+a-2)}{2a}}
\]

\( (k, a) \)-deformation of Hermite semigroup

\[
\mathcal{I}_{k,a}(t) := \exp \frac{t}{a}(|x|^{2-a} \Delta_k - |x|^a)
\]

Mehler kernel using \( \exp(-x^2) \)

\( k: \) multiplicity on root system \( \mathcal{R} \), \( a > 0 \)
(k, a)-deformation of \( \exp \frac{t}{2}(\Delta - |x|^2) \)

Hankel-type transform on \( \Xi \)

self-adjoint op. on \( L^2(\mathbb{R}^N, \frac{dx}{|x|}) \)

\[
\mathcal{F}_\Xi = \quad c \quad \exp \left( \frac{\pi i}{2} \left( |x| \Delta - |x| \right) \right)
\]

phase factor   Laplacian
\[ = e^{\frac{\pi i(N-1)}{2}} \]

“Laguerre semigroup” ([K–Mano], 2007)

\[ \mathcal{I}(t) := \exp \left( t(|x| \Delta - |x|) \right) \quad \text{Re } t > 0 \]

closed formula using Bessel function
(\(k, a\))-deformation of \(\exp \frac{t}{2}(\Delta - |x|^2)\)

(k, a)-generalized Fourier transform

self-adjoint op. on \(L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)\)

\[
F_{k,a} = c \exp\left(\frac{\pi i}{2a} \left( |x|^{2-a} \Delta_k - |x|^a \right) \right)
\]

phase factor Dunkl Laplacian

\[
= e^{i \frac{\pi(N+2(k)+a-2)}{2a}}
\]

(k, a)-deformation of Hermite semigroup ([with Ben Saïd, Ørsted])

\[
I_{k,a}(t) := \exp \frac{t}{a} (|x|^{2-a} \Delta_k - |x|^a) \quad \text{Re } t > 0
\]

\(k: \) multiplicity on root system \(\mathcal{R}, a > 0\)
Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$(k, a)$-generalized Fourier transform $\mathcal{F}_{k,a}$

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$t \rightarrow \frac{i}{2}$

$a \rightarrow 2$

$a \rightarrow 1$

$t \rightarrow \frac{i}{2}$

$k \rightarrow 0$

$t \rightarrow \frac{\pi i}{2}$

Dunkl transform

Hermite semigp

Laguerre semigp

$\mathcal{F}_{k,1}$

Fourier transform

Hankel transform
Special values of holomorphic semigroup $I_{k,a}(t)$

$(k, a)$-generalized Fourier transform $\mathcal{F}_{k,a}$

$\mathcal{F}_{k,1}$

Dunkl transform \quad Hermite semigroup \quad Laguerre semigroup

$t \to \frac{\pi i}{2}$

$k \to 0$

$\mathcal{I}_{k,2}(t)$

$t \to \frac{\pi i}{2}$

$\mathcal{I}_{k,1}(t)$

$t \to \frac{\pi i}{2}$

$a \to 2$

$a \to 1$

$\mathcal{F}_{k,1}$

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$\mathcal{F}_{k,1}$

Fourier transform

Hankel transform

$\Leftrightarrow$ ‘unitary inversion operator’ $\Rightarrow$

the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of the conformal group $O(N + 1, 2)$
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \, I_{k,a}(\frac{\pi i}{2})$$
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \mathcal{I}_{k,a}(\frac{\pi i}{2}) = c \exp\left(\frac{\pi i}{2a}(|x|^{2-a} \Delta_k - |x|^a)\right)$$

**Thm G** (with Ben Saïd, Ørsted, to appear)

1) $\mathcal{F}_{k,a}$ is a unitary operator
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \, i_{k,a} \left( \frac{\pi i}{2} \right) = c \exp \left( \frac{\pi i}{2a} (|x|^{2-a} \Delta_k - |x|^a) \right)$$

**Thm G (with Ben Saïd, Ørsted, to appear)**

1) $\mathcal{F}_{k,a}$ is a unitary operator

2) $\mathcal{F}_{0,2} =$ Fourier transform on $\mathbb{R}^N$
   
   $F_{k,a} =$ Dunkl transform on $\mathbb{R}^N$
   
   $\mathcal{F}_{0,1} =$ Hankel-type transform on $L^2(\mathbb{R}^N)$

3) $\mathcal{F}_{k,a}$ is of finite order $\iff a \in \mathbb{Q}$

4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a} \Delta_k$
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \mathcal{I}_{k,a}(\frac{\pi i}{2}) = c \exp \left( \frac{\pi i}{2a} (|x|^{2-a} \Delta_k - |x|^a) \right)$$

**Thm G** *(with Ben Saïd, Ørsted, to appear)*

1) $\mathcal{F}_{k,a}$ is a unitary operator
2) $\mathcal{F}_{0,2}$ = Fourier transform on $\mathbb{R}^N$
   $F_{k,a}$ = Dunkl transform on $\mathbb{R}^N$
   $\mathcal{F}_{0,1}$ = Hankel-type transform on $L^2(\mathbb{R})$
3) $\mathcal{F}_{k,a}$ is of finite order $\iff a \in \mathbb{Q}$
4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a} \Delta_k$

$\implies$ generalization of classical identities such as Hecke identity, Bochner identity, Parseval–Plancherel formulas, Weber's second exponential integral, etc.
Heisenberg-type inequality

**Thm H ([2]) (Heisenberg inequality)**

\[
\| |x|^\frac{q}{2} f(x)\|_k \| |\xi|^\frac{q}{2} (\mathcal{F}_{k,a}f)(\xi)\|_k \geq \frac{2^{(k)+N+a-2}}{2} \| f(x) \|_k^2
\]

\( k \equiv 0, a = 2 \) \hspace{1cm} \cdots \text{Weyl–Pauli–Heisenberg inequality for Fourier transform } \mathcal{F}_{\mathbb{R}^N} \)

\( k: \text{general, } a = 2 \) \hspace{1cm} \cdots \text{Heisenberg inequality for Dunkl transform } \mathcal{D}_k \text{ (Rösler, Shimeno)}

\( k \equiv 0, a = 1, N = 1 \) \hspace{1cm} \cdots \text{Heisenberg inequality for Hankel transform}
Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$(k, a)$-generalized Fourier transform $\mathcal{F}_{k,a}$

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$\mathcal{I}_{k,2}(t)$

Dunkl transform

Hermite semigroup

Laguerre semigroup

$\mathcal{F}_{k,1}$

Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

Hankel-type transform $\mathcal{F}_{\Xi}$

$\leftarrow$ ‘unitary inversion operator’ $\rightarrow$

the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of the conformal group $O(N + 1, 2)$
Hidden symmetries in $L^2(\mathbb{R}^N, \mathcal{C}_k(x)dx)$

Coxeter group

$\mathbb{C} \times S\hat{L}(2, \mathbb{R})$

(k, a : general)

$k \to 0$

$O(N) \times S\hat{L}(2, \mathbb{R})$

$a \to 1$

$a \to 2$

$O(N + 1, 2)\sim$

$Mp(N, \mathbb{R})$
Minimal ⇔ Maximal

(Ambitious) Project:
Use minimal reps to get an inspiration in finding new interactions with other fields of mathematics.

Viewpoint:
Minimal representation ($\Leftrightarrow$ group)
$\approx$ Maximal symmetries ($\Leftrightarrow$ rep. space)
Geometric analysis on minimal reps

[2] Laguerre semigroup and Dunkl operators ⋅⋅⋅
[3] Schrödinger model of minimal representations of $O(p, q)$ ⋅⋅⋅
[4] Algebraic analysis on minimal representations ⋅⋅⋅
[5] Geometric analysis of small unitary reps of $GL(n, \mathbb{R})$ ⋅⋅⋅
[6] Special functions associated to a fourth order differential equation ⋅⋅⋅
    Ramanujan J. Math (2011)
[7] Analysis on minimal representations ⋅⋅⋅
[8] Generalized Fourier transforms $\mathcal{F}_{k,a}$ ⋅⋅⋅
    C.R.A.S. Paris 2009
[9] Inversion and holomorphic extension ⋅⋅⋅

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