Analysis of minimal representations

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§1 What are minimal representations?

§2 Conformal model of minimal representations

§3 Geometric quantization of minimal nilpotent orbits and L^2 model

§4 Deformation of Fourier transform

What are minimal reps?

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  (finitely many) (continuously many)
- 'attached to' minimal nilpotent orbits (orbit method)
- matrix coefficients are of bad decay

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$\uparrow$ “induction”, etc.

finitely many “very small” irred. unitary reps.
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My (ambiguous and ambitious) project

collaborated with S.Ben Saïd, J.Hilgert, G.Mano, J.Möllers, B.Ørsted, M.Pevzner:

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Observation. ϖ: minimal rep of G

$\mathrm{DIM}(\varpi)$ (Gelfand–Kirillov dimension)

$= \frac{1}{2}$ dimension of minimal nilpotent orbits

$<$ dimension of any non-trivial G-space

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My viewpoint:

Minimal representation ($\Leftarrow$ group)

$\approx$ Maximal symmetries ($\Leftarrow$ rep. space)

Geometric analysis on minimal reps

[1] Minimal representations via Bessel operators $\cdots$ 66 pp. arXiv:1106.3621
[2] Laguerre semigroup and Dunkl operators $\cdots$
Compositio Math (to appear), 74 pp.
[3] Schrödinger model of minimal representations of $O(p,q)$ $\cdots$
Memoirs of Amer. Math. Soc. (2011), no.1000, 132 pp.
[4] Algebraic analysis on minimal representations $\cdots$
Publ. RIMS (2011), 28 pp.
[5] Geometric analysis of small unitary reps of $GL(n,\mathbb{R})$ $\cdots$
J. Funct. Anal. (2011)
[6] Special functions associated to a fourth order differential equation $\cdots$
Ramanujan J. Math (2011)
[7] Analysis on minimal representations $\cdots$
Adv. Math. (2003) I, II, III, 110 pp.
[8] Generalized Fourier transforms $\mathcal{F}_{k,a}$ $\cdots$ C.R.A.S. Paris 2009
[9] Inversion and holomorphic extension $\cdots$
R. Howe 60th birthday volume (2007), 65 pp.

Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers, Ørsted and M. Pevzner

Indefinite orthogonal group $O(p+1, q+1)$

Throughout this talk, $p, q \geq 1$, $p+q$: even > 2

$$\begin{aligned} G &= O(p+1, q+1) \\ &= \{g \in GL(p+q+2, \mathbb{R}) : {}^t g \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} g = \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} \} \end{aligned}$$

$\cdots$ real simple Lie group of type D

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  non-highest, non-spherical
  - algebraic construction (e.g. dual pair)
    (Binegar–Zierau, Howe–Tan, Huang–Zhu)
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    (L^2 model of minimal reps of some other groups Sahi, Hilgert–K–Möllers)

Two constructions of minimal reps.

1. Conformal model
   Theorem B

2. L^2 model
(Schrödinger model)
   Theorem D

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	Group action	Hilbert structure
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Clear Picture $\cdots$ advantage of the model

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3. Deformation of Fourier transforms (Theorems F, G, H)
(interpolation, special functions, Dunkl operators)

Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

holomorphic $\circ$ holomorphic $=$ holomorphic

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$\Downarrow$ taking real parts

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$\Longrightarrow$ Try to modify the definition!

$\mathrm{Conf}(X, g) \supset \mathrm{Isom}(X, g)$

(X, g) Riemannian manifold
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<u>Def.</u>

φ is isometry $\iff \varphi^* g = g$

φ is conformal $\iff$ $^\exists$positive function $C_\varphi \in C^\infty(X)$ s.t.

$$\varphi^* g = C_\varphi^2\, g$$

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$\mathrm{Diffeo}(X) \supset \underset{\text{Conformal group}}{\mathrm{Conf}(X, g)} \supset \underset{\text{isometry group}}{\mathrm{Isom}(X, g)}$

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(X, g) pseudo-Riemannian manifold
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Harmonic $\circ$ conformal $\neq$ harmonic

Modification

$\varphi \in \mathrm{Conf}(X^n, g), \;\; \varphi^* g = C_\varphi^2 g$

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- pull-back $\rightsquigarrow$ twisted pull-back

  $f \circ \varphi \quad \rightsquigarrow \quad \underbrace{C_\varphi^{-\frac{n-2}{2}}}_{\text{conformal factor}} f \circ \varphi$

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conformal factor

- $\mathcal{S}ol(\Delta_X) = \{f \in C^\infty(X) : \Delta_X f = 0\}$ (harmonic functions)

$$\rightsquigarrow \mathcal{S}ol(\widetilde{\Delta_X}) = \{f \in C^\infty(X) : \widetilde{\Delta_X} f = 0\}$$

$$\widetilde{\Delta_X} \quad := \quad \Delta_X \quad + \frac{n-2}{4(n-1)} \kappa$$

Yamabe operator — Laplacian — scalar curvature

Distinguished rep. of conformal groups

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$$\Downarrow \text{Modification}$$

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$\Big\Downarrow$ Modification

Theorem A ([K–Ørsted 03]) (X^n, g): Riemannian mfd
$\Longrightarrow \mathrm{Conf}(X, g)$ acts on $\mathcal{S}ol(\widetilde{\Delta_X})$ by $f \mapsto C_\varphi^{-\frac{n-2}{2}} f \circ \varphi$

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Point $\widetilde{\Delta_X} = \Delta_X + \frac{n-2}{4(n-1)}\kappa$
$\widetilde{\Delta_X}$ is not invariant by $\mathrm{Conf}(X, g)$.
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Application of Theorem A

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Theorem B ([7, Part I]) $\widetilde{\Delta_X} = \Delta_{S^p} - \Delta_{S^q} + \text{const.}$

0) $\mathrm{Conf}(X, g) \simeq O(p+1, q+1)$

1) $\mathcal{S}ol(\widetilde{\Delta_X}) \neq \{0\} \iff p + q$ even

2) If $p + q$ is even and > 2, then
$\mathrm{Conf}(X, g) \curvearrowright \mathcal{S}ol(\widetilde{\Delta_X})$ is irreducible,
and for $p + q > 6$ it is a minimal rep of $O(p+1, q+1)$.

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Remark 1. $\mathcal{S}ol \cdots$ in the sense of C^∞, hyperfunction, etc (the result (1) remains true)

Remark 2. For $q = 1$, take a double cover X' of X. Then $\mathcal{S}ol(\widetilde{\Delta_{X'}}) \neq \{0\}$ for all p.

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$\uparrow$

$^\exists$a $\mathrm{Conf}(X, g)$-invariant inner product, and
take the Hilbert completion

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction		
Theorem B	Clear	$\boxed{?}$
v.s.		
2. L^2 construction		
(Schrödinger model) Theorem D	$\boxed{?}$	Clear

Clear $\cdots$ advantage of the model

Hilbert structure on $Sol(\widetilde{\Delta})$

Three methods:

1. Parseval-type formula [7, Part II]
2. Using Green function [7, Part III]
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Flat model

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$$\underset{+\cdots+}{S^p} \times \underset{-\cdots-}{S^q} \hookrightarrow \underset{ds^2=dx_1^2+\cdots+dx_p^2-dx_{p+1}^2-\cdots-dx_{p+q}^2}{\mathbb{R}^{p+q}} \quad \text{conformal embedding}$$

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Functoriality of Theorem A

$$\boxed{\begin{array}{ccc} \mathcal{S}ol(\widetilde{\Delta}_{S^p\times S^q}) & \subset & \mathcal{S}ol(\widetilde{\Delta}_{\mathbb{R}^{p,q}}) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Conf}(S^p \times S^q) & \hookrightarrow & \mathrm{Conf}(\mathbb{R}^{p,q}) \end{array}}$$

Conservative quantity for ultra-hyperbolic eqn.

$$\mathbb{R}^{p,q} = \mathbb{R}^{p+q},\ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$$

$$\widetilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q}$$

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Problem Find an 'intrinsic' inner product on (a 'large' subspace of) $\mathcal{S}ol(\Box_{p,q})$ if exists.

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Unitarization of subrep (representation theory)

$\Longleftrightarrow$

Conservative quantity (differential eqn)

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<u>energy</u> $\cdots$ conservative quantity for wave equations
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$$\Downarrow$$

$\boxed{\ ?\ }$ $\cdots$ conservative quantity for ultra-hyperbolic eqs
w.r.t. conformal group $O(p+1, q+1)$

Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

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Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \quad \text{(to be defined soon)} \qquad \cdots\cdots ①$$

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Theorem C ([7, Part III]+K– 2011)

1) ① is independent of hyperplane α.

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Theorem C ([7, Part III]+K– 2011)

1) ① is independent of hyperplane α.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

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$$O(p, q) \curvearrowright \mathbb{R}^{p,q} \quad \text{(linear)}$$

Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\square_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \quad \text{(to be defined soon)} \qquad \cdots\cdots ①$$

<u>Theorem C</u> ([7, Part III]+K– 2011)

1) ① is independent of hyperplane α.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

$\cancel{O(p,q)}$ $O(p+1, q+1)$ $\curvearrowright$ $\mathbb{R}^{p,q}$ ~~(linear)~~ (Möbius transform)

Construction of $Q_\alpha f$

$$\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2)$$

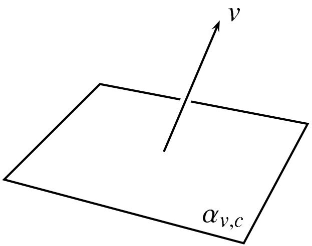
Fix $v \in \mathbb{R}^{p,q}$ s.t. $(v,v)_{\mathbb{R}^{p,q}} = \pm 1$

$c \in \mathbb{R}$

$\Downarrow$

$$\mathbb{R}^{p,q} \supset \alpha \equiv \alpha_{v,c} := \{x \in \mathbb{R}^{p+q} : (x,v)_{\mathbb{R}^{p,q}} = c\}$$

non-characteristic hyperplane



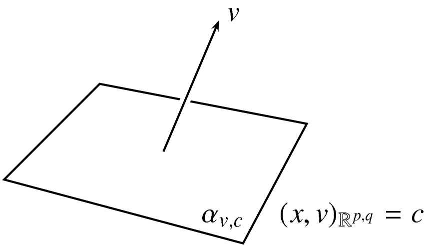
Construction of $Q_\alpha f$

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)

Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)

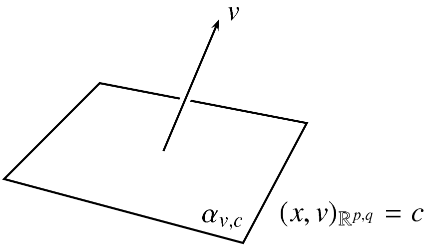


Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)

$f_\pm$ extends holomorphically to the direction $\pm\sqrt{-1}v$

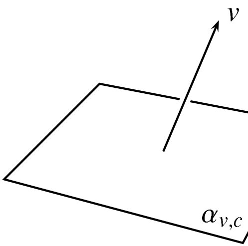


Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)

$$f_\pm(x;v) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mp f(x-tv)}{t \pm i0} dt$$



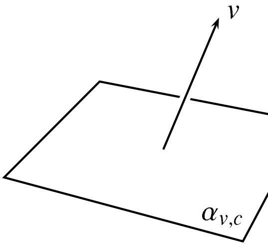
$(x,v)_{\mathbb{R}^{p,q}} = c$

Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)

$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. v



$(x, v)_{\mathbb{R}^{p,q}} = c$

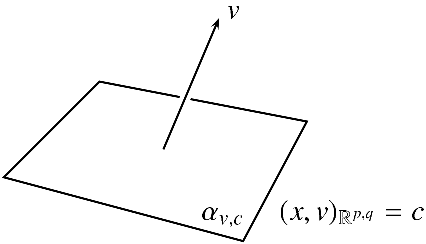
Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)

$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. v

$$Q_\alpha f := \frac{1}{i}\left(f_+\overline{f'_+} - f_-\overline{f'_-}\right)$$



Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\Box_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \qquad \cdots\cdots ①$$

Theorem C

1) ① is independent of hyperplane α.
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Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\square_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \qquad \cdots\cdots ①$$

Theorem C

1) ① is independent of hyperplane α.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

Theorem C is non-trivial even for $q = 1$ (wave equation)

In space-time $\mathbb{R}^{p+1} = \mathbb{R}^p_x \times \mathbb{R}_t$,

average in space (i.e. time t = constant)

= average in (any hyperplane in space) $\times \mathbb{R}_t$ (time)

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction		
Theorems A, B	Clear	$\boxed{?}$
v.s.		
2.		
$\boxed{?}$	$\boxed{?}$	$\boxed{?}$

Clear $\cdots$ advantage of the model

Two constructions of minimal reps.

		Group action	Hilbert structure
1.	Conformal construction Theorems A, B	Clear	$\boxed{\text{conservative quantity Theorem C}}$
	v.s.		
2.	$\boxed{?}$	$\boxed{?}$	$\boxed{?}$

Clear $\cdots$ advantage of the model

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	$\boxed{\text{conservative quantity Theorem C}}$
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	$\boxed{?}$	Clear

Clear $\cdots$ advantage of the model

Conformal model $\Longrightarrow$ L^2-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

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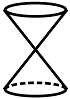
$=$  (figure for $(p,q) = (2,1)$)

Conformal model $\Longrightarrow$ L^2-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$\boxed{\Box_{p,q} f = 0 \quad \underset{\text{Fourier trans.}}{\Longrightarrow} \quad \operatorname{Supp} \mathcal{F} f \subset \Xi}$$

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$$\mathcal{F} : \quad \mathcal{S}'(\mathbb{R}^{p,q}) \xrightarrow{\sim} \mathcal{S}'(\mathbb{R}^{p,q})$$

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$$\begin{array}{cccc} \mathcal{F} : & \mathcal{S}'(\mathbb{R}^{p,q}) & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}^{p,q}) \\ & \cup & & \cup \\ & \mathcal{S}ol(\Box_{p,q}) & & \end{array}$$

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$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

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$$\begin{array}{cccc} \mathcal{F}: & \mathcal{S}'(\mathbb{R}^{p,q}) & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}^{p,q}) \\ & \cup & & \cup \\ & \overline{Sol(\Box_{p,q})} & \xrightarrow{\sim} & \boxed{?} \end{array}$$

$\overline{\cdot}$ denotes the Hilbert completion w.r.t. the invariant inner product.

Conformal model $\Longrightarrow$ L^2-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

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$$\begin{array}{lccc} \mathcal{F}: & \mathcal{S}'(\mathbb{R}^{p,q}) & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}^{p,q}) \\ & \cup & & \cup \\ \text{Theorem D ([7, Part III])} & \overline{\mathcal{S}ol(\Box_{p,q})} & \xrightarrow{\sim} & L^2(\Xi) \end{array}$$

Conformal model $\Longrightarrow$ L^2-model

$$\square_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

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Theorem D ([7, Part III])

conformal model $\qquad$ L^2-model

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	$\boxed{\text{conservative quantity}}$
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	$\boxed{?}$	Clear

Clear $\cdots$ advantage of the model

L^2-model of minimal reps.

$\boxed{\text{Theorem D} \quad p+q>2, \text{ even.} \quad \overline{Sol(\Box_{p,q})} \xrightarrow{\sim} L^2(\Xi)}$

conformal model $\qquad$ L^2-model

L^2-model of minimal reps.

Theorem D $p+q>2$, even. $\overline{Sol(\square_{p,q})} \xrightarrow{\sim} L^2(\Xi)$

conformal model $\quad$ L^2-model

minimal rep.

$$G = O(p+1, q+1) \curvearrowright L^2(\Xi) \quad \text{unitary rep.}$$

L^2-model of minimal reps.

$\boxed{\text{Theorem D} \quad p+q>2, \text{ even.} \quad \overline{Sol(\Box_{p,q})} \xrightarrow{\sim} L^2(\Xi)}$

conformal model $\qquad$ L^2-model

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$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ unitary rep.

$\boxed{\dim \Xi = p+q-1 \implies \Xi \text{ is too small to be acted by } G.}$

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$$\Xi \subset \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1}$$

L^2-model of minimal reps.

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conformal model L^2-model

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$\boxed{\dim \Xi = p+q-1 \implies \Xi \text{ is too small to be acted by } G.}$

$$\Xi \subset \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1}$$

$\boxed{\text{What is } \Xi\text{?}}$

Geometric quantization of minimal nilpotent orbit

$$G \not\curvearrowright \Xi \quad \text{but} \quad G \curvearrowright L^2(\Xi)$$

$$\boxed{\text{What is } \Xi \text{ ?}}$$

Geometric quantization of minimal nilpotent orbit

$\mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda$ coadjoint orbit

$\rightsquigarrow$? "geometric quantization"

$\widehat{G} \ni \pi$ irred. unitary rep of G

Geometric quantization of minimal nilpotent orbit

$$\boxed{\begin{array}{lll} \mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda & \text{coadjoint orbit} & \\ \qquad\quad \rightsquigarrow \ ? & & \text{“geometric quantization”} \\ \widehat{G} \ni \pi & \text{irred. unitary rep of } G & \end{array}}$$

hyperbolic orbit ⇝ real polarization ⇝ principal series reps (induced reps from parabolic subgroups)

elliptic orbit ⇝ complex polarization ⇝ Zuckerman derived functor modules/Dolbeault cohomologies on non-compact homogeneous complex manifolds

(Kostant, Schmid, Zuckerman, Vogan, Schmid, Bernstein, . . . late 1970s–1990s)

Geometric quantization of minimal nilpotent orbit

Difficult case

$\mathfrak{g}^* \supset \mathcal{O}_{\min} = \mathrm{Ad}^*(G)\lambda$ minimal nilp. orbit

$\quad\rightsquigarrow$? "geometric quantization"

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hyperbolic orbit $\rightsquigarrow$ real polarization $\rightsquigarrow$ principal series reps (induced reps from parabolic subgroups)

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Geometric quantization of minimal nilpotent orbit

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hyperbolic orbit $\rightsquigarrow$ real polarization $\rightsquigarrow$ principal series reps (induced reps from parabolic subgroups)

elliptic orbit $\rightsquigarrow$ complex polarization $\rightsquigarrow$ Zuckerman derived functor modules/Dolbeault cohomologies on non-compact homogeneous complex manifolds

(Kostant, Schmid, Zuckerman, Vogan, Schmid, Bernstein, . . . late 1970s–1990s)

In the previous theorem, Ξ is Lagrangian submanifold of $\mathcal{O}_{\min}$ for $G = O(p+1, q+1)$!

L^2-model of minimal rep.

V: simple Jordan algebra
$G =$ (a finite covering of) the conformal group of V

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

Ex 1 $V = \mathrm{Symm}(n, \mathbb{R})$
$G = Mp(n, \mathbb{R})$, a double cover of $Sp(n, \mathbb{R})$

Ex 2 $V = \mathbb{R}^{p,q+1}$
$G = O(p+1, q+1)$

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

$\mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G

$\Xi := \mathcal{O}_{\min} \cap V$

<u>Observation</u>

$$\begin{array}{ccccc}
\mathfrak{g} & & \simeq & & \mathfrak{g}^* \\
\cup & & & & \cup \\
V & & & & \mathcal{O}_{\min} \\
& \supset & & \subset & \\
& & \Xi & &
\end{array}$$

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

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$\Xi := O_{\min} \cap V$

Ex 1 $\quad \mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$

$V = Symm(n, \mathbb{R}) \qquad O_{\min}$

$\cup \qquad \cup$ Lagrangian

$\Xi = \{X : X = {}^tX, \ \operatorname{rank} X \leq 1\}$

L^2-model of minimal rep.

V: simple Jordan algebra

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<u>Ex 2</u> $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$

$V = \mathbb{R}^{p+q+1}$ $\qquad\qquad \mathcal{O}_{\min}$

$\circlearrowright \qquad \circlearrowleft$ Lagrangian

$\Xi = \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

$\mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G

$\Xi := \mathcal{O}_{\min} \cap V$

Assume that a maximal euclidean Jordan subalgebra of V is simple, and $V \not\simeq \mathbb{R}^{p,q+1}$ with $p+q$: odd.

Theorem (with Hilgert, Moellers, arXiv:1106.3621)

1) Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$.

L^2-model of minimal rep.

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1) Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$.

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1) Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$.
2) We get an irreducible unitary representation of G on $L^2(\Xi)$.
3) The Gelfand–Kirillov dimension attains its minimum among all (∞-dim'l) irreducible unitary representations of G.
4) The annihilator of the differential rep $d\pi$ is the Joseph ideal in $U(\mathfrak{g})$ if V is split and $\mathfrak{g} \not\simeq A_n$.

L^2-model of minimal rep.

V: simple Jordan algebra
G = (a finite covering of) the conformal group of V

$O_{\min}$: minimal nilpotent coadjoint orbit of G
$\Xi := O_{\min} \cap V$

<u>Ex 1</u> $V = Symm(n, \mathbb{R})$
$G = Mp(n, \mathbb{R})$
$\Longrightarrow$ Schrödinger model of the Weil representation
$G \curvearrowright L^2(\mathbb{R}^n)_{\text{even}} \simeq L^2(\Xi)$

<u>Ex 2</u> $V = \mathbb{R}^{p,q+1}$, $p + q$: even
$G = O(p + 1, q + 1)$
$\Longrightarrow G \curvearrowright L^2(\Xi)$
(Theorem D)

Inversion element

$$G = PGL(2,\mathbb{C}) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

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$$P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\} \qquad z \mapsto az + b$$

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\frac{1}{z} \qquad \text{(inversion)}$$

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G is generated by P and w.

Inversion element

$$G = PGL(2, \mathbb{C}) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

$$\doteqdot O(3,1) \qquad \doteqdot \mathbb{R}^{2,0}$$

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$$G = PGL(2,\mathbb{C}) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

$$\doteqdot O(3,1) \qquad\qquad \doteqdot \mathbb{R}^{2,0}$$

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\} \qquad z \mapsto az + b$$

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\frac{1}{z} \qquad \text{(inversion)}$$

G is generated by P and w.

$$G = O(p+1, q+1) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{R}^{p,q}$$

$$P = \{(A,b) : A \in O(p,q) \cdot \mathbb{R}^\times,\ b \in \mathbb{R}^{p+q}\} \quad x \mapsto Ax + b$$

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Inversion element

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Towards a global formula

$p+q$: even > 2

$$G = O(p+1, q+1) \curvearrowright L^2(\Xi) \qquad \text{minimal rep.}$$

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Problem What is $\mathcal{F}_\Xi$?

Towards a global formula

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Problem What is $\mathcal{F}_\Xi$?

Cf. Analogous operator for the oscillator rep.

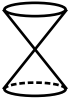
$$Mp(n, \mathbb{R}) \curvearrowright L^2(\mathbb{R}^n)$$

unitary inversion operator coincides with

Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ (up to scalar)!

New Fourier transform $\mathcal{F}_\Xi$ on Ξ

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$=$  (figure for $(p, q) = (2, 1)$)

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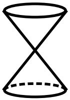
$=$  (figure for $(p, q) = (2, 1)$)

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on Ξ = 

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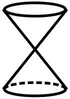
$=$  (figure for $(p, q) = (2, 1)$)

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$	$\mathcal{F}_\Xi$ on $\Xi =$ 

Problem
1. Algebraic properties of $\mathcal{F}_\Xi$
2. Analytic formula of $\mathcal{F}_\Xi$.

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi =$ 

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$\mathcal{F}^4 = \mathrm{id}$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

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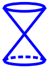
$\mathcal{F}_\Xi$ on $\Xi =$ 

$$\mathcal{F}_\Xi^2 = \mathrm{id}$$

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$\begin{aligned} Q_j &\mapsto -P_j \\ P_j &\mapsto Q_j \end{aligned}$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$$

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

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$R_j = {}^{\exists}$second order differential op. on Ξ

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

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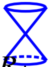
$\mathcal{F}_\Xi$ on $\Xi =$ 

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Rediscover Bargmann–Todorov’s operators (1977)

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

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$$Q_j \mapsto R_j$$
$$R_j \mapsto Q_j$$

$Q_j = x_j$ (multiplication by coordinates function)

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$R_j = {}^{\exists}$second order differential op. on Ξ

Notice
$$\left.\begin{array}{l} {Q_1}^2 + \cdots + {Q_p}^2 - {Q_{p+1}}^2 - \cdots - {Q_{p+q}}^2 = 0 \\ {R_1}^2 + \cdots + {R_p}^2 - {R_{p+1}}^2 - \cdots - {R_{p+q}}^2 = 0 \end{array}\right\} \text{ on } \Xi$$

Unitary inversion operator $\mathcal{F}_\Xi$

$p+q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem Find the unitary operaotr $\mathcal{F}_\Xi$ explicitly.

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Cf. Euclidean case $\varphi(t) = e^{-it}$ (one variable)
$\mathcal{F}_{\mathbb{R}^N} f(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y\rangle) f(y) dy$

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Thm E (K–Mano, Memoirs AMS, 2011, vol.1000)

$$(\mathcal{F}_\Xi f)(x) = c \int_\Xi \Phi(\langle x, y \rangle) f(y) dy$$

$\mathcal{F}_{\mathbb{R}^N}$ v.s. $\mathcal{F}_{\Xi}$

On $\mathbb{R}^N$

$$(\mathcal{F}_{\mathbb{R}^N} f)(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy$$

$\varphi(t) = e^{-it}$ satisfies

$$\left(\frac{d}{dt} + i\right) \varphi(t) = 0$$

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On Ξ $(\subset \mathbb{R}^{p,q})$

$$(\mathcal{F}_{\Xi} f)(x) = c \int_{\Xi} \Phi(\langle x, y \rangle) f(y) dy$$

$\Phi(t)$ satisfies

$$\left(\left(t\frac{d}{dt}\right)^2 + \frac{1}{2}(p+q-4)t\frac{d}{dt} + 2t\right)\Phi(t) = 0$$

Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{z}{2}\right)^{2j}}{j!\,\Gamma(j+\nu+1)}$$

$$I_\nu(z) := e^{-\frac{\sqrt{-1}\nu\pi}{2}} J_\nu\left(e^{\frac{\sqrt{-1}\pi}{2}} z\right)$$

$$Y_\nu(z) := \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \qquad \text{(second kind)}$$

$$K_\nu(z) := \frac{\pi}{2\sin\nu\pi}\left(I_{-\nu}(z) - I_\nu(z)\right) \qquad \text{(third kind)}$$

Bessel distribution

Prop. $\Phi_m^{\varepsilon}(t)$ solves the differential equation

$$(\theta^2 + m\theta + 2t)u = 0$$

where $\theta = t\frac{d}{dt}$.

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Explicit forms

$$\Phi_m^0(t) = 2\pi i (2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+})$$

$$\Phi_m^1(t) = \Phi_m^0(t) - \pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!}\delta^{(l)}(t)$$

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$$\Phi_m^2(t) = 2\pi i (2t)_+^{-\frac{m}{2}} Y_m(2\sqrt{2t_+}) + 4(-1)^{m+1} i\,(2t)_-^{-\frac{m}{2}} K_m(2\sqrt{2t_-})$$

Bessel distribution

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$$(\mathcal{F}_{\Xi} f)(x) = c \int_{\Xi} \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)}(\langle x, y \rangle) f(y) dy$$

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$$(\mathcal{F}_\Xi f)(x) = c\int_\Xi \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)}(\langle x,y\rangle)f(y)dy$$

Here, $\varepsilon(p,q) = \begin{cases} 0 & \text{if } \min(p,q) = 1, \\ 1 & \text{if } p,q > 1 \text{ are both odd,} \\ 2 & \text{if } p,q > 1 \text{ are both even.} \end{cases}$

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	$\boxed{\text{conservative quantity}}$
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	$\boxed{\text{'Fourier transform' } \mathcal{F}_{\Xi}}$	Clear

Clear $\cdots$ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction		
Theorems A, B	Clear	Theorem C
v.s.		
2. L^2 construction		
(Schrödinger model)	Theorem E	Clear
Theorem D		

Clear $\cdots$ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)

§1 What are minimal representations?

§2 Conformal model of minimal representations

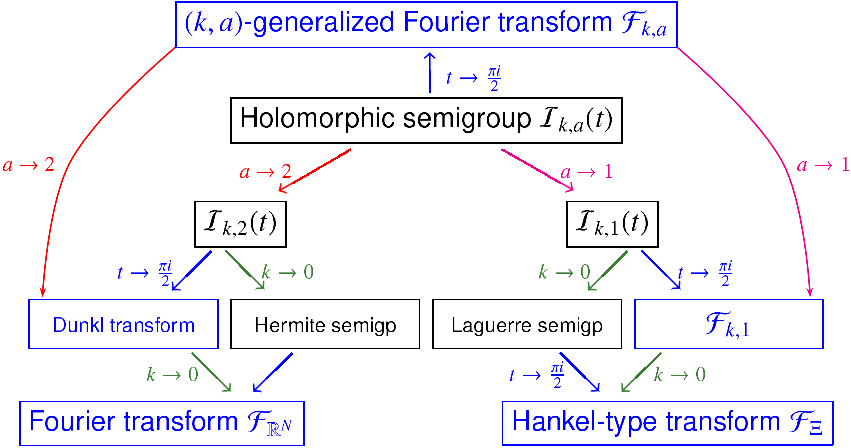
§3 Geometric quantization of minimal nilpotent orbits and L^2 model

§4 Deformation of Fourier transform

Deformation theory of Fourier transform

- Generalized Fourier transform $\mathcal{F}_{k,a}$ C.R.A.S. Paris (2009)
- Laguerre semigroup and Dunkl operators 74 pp.
  Compositio Math (to appear), with Ben Saïd and Bent Ørsted
- Inversion and holomorphic extension
  R. Howe 60th birthday volume (2007), 65 pp. with Mano

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



$\Leftarrow$ 'unitary inversion operator' $\Rightarrow$

the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of the conformal group $O(N + 1, 2)$

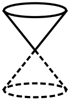
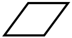
Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_{\Xi}$	$\cdots$	'Fourier transform' on $\Xi \quad \subset \mathbb{R}^{p,q}$
$\mathcal{F}_{\mathbb{R}^N}$	$\cdots$	Fourier transform on $\mathbb{R}^N$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$$\boxed{\begin{array}{lll} \mathcal{F}_{\Xi} & \cdots & \text{`Fourier transform' on } \Xi \quad \subset \mathbb{R}^{p,q} \\ \mathcal{F}_{\mathbb{R}^N} & \cdots & \text{Fourier transform on } \mathbb{R}^N \end{array}}$$

Assume $q = 1$. Set $p = N$.

$\mathbb{R}^{N,1} \supset \Xi =$  $\xrightarrow{\text{projection}}$  $= \mathbb{R}^N$

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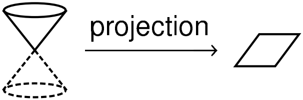
$\mathbb{R}^{N,1} \supset \Xi =$  $= \mathbb{R}^N$

$\mathcal{F}_{\Xi}$	$\mathcal{F}_{\mathbb{R}^N}$
$O(N+1,2)$	$Mp(N,\mathbb{R})$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

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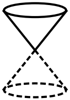
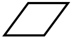
$\mathbb{R}^{N,1} \supset \Xi =$  $\xrightarrow{\text{projection}}$  $= \mathbb{R}^N$

$\mathcal{F}_\Xi$	interpolate ·········	$\mathcal{F}_{\mathbb{R}^N}$
$a = 1$		$a = 2$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\boxed{\mathcal{F}_{\mathbb{R}^N} = c \quad \exp\Big(\frac{\pi i}{4}(\quad \Delta \quad - |x|^2)\Big)}$$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\boxed{\mathcal{F}_{\mathbb{R}^N} = \underbrace{c}_{\substack{\text{phase factor} \\ = e^{\frac{\pi i N}{4}}}} \exp\Big(\frac{\pi i}{4}(\underbrace{\underbrace{\Delta}_{\text{Laplacian}} - |x|^2}_{\text{self-adjoint op. on } L^2(\mathbb{R}^N)})\Big)}$$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\boxed{\mathcal{F}_{\mathbb{R}^N} = \underbrace{c}_{\substack{\text{phase factor} \\ = e^{\frac{\pi i N}{4}}}} \exp\left(\frac{\pi i}{4}\overbrace{(\underbrace{\Delta}_{\text{Laplacian}} - |x|^2)}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N)}\right)}$$

Hermite semigroup

$$\boxed{I(t) := \exp \frac{t}{2}(\Delta - |x|^2)}$$

Mehler kernel using $\exp(-x^2)$

(k,a)-deformation of $\exp\frac{t}{2}(\Delta - |x|^2)$

(k,a)-generalized Fourier transform

$$\mathcal{F}_{k,a} = \underbrace{c}_{\text{phase factor}} \exp\left(\frac{\pi i}{2a}\left(\underbrace{|x|^{2-a}\Delta_k}_{\text{self-adjoint op. on } L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)} - |x|^a\right)\right)$$

phase factor $= e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}$

Dunkl Laplacian

(k,a)-deformation of Hermite semigroup

$$\boxed{\mathcal{I}_{k,a}(t) := \exp\frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a)}$$

Mehler kernel using $\exp(-x^2)$

k: multiplicity on root system $\mathcal{R}$, $a > 0$

(k,a)-deformation of $\exp\frac{t}{2}(\Delta - |x|^2)$

Hankel-type transform on Ξ

self-adjoint op. on $L^2(\mathbb{R}^N, \frac{dx}{|x|})$

$$\boxed{\mathcal{F}_\Xi = \underbrace{c}_{\text{phase factor}} \exp\Big(\frac{\pi i}{2}(\overbrace{\underbrace{|x|\Delta}_{\text{Laplacian}} - |x|}^{})\Big)}$$

phase factor $= e^{\frac{\pi i(N-1)}{2}}$

"Laguerre semigroup" ([K–Mano], 2007)

$$\boxed{\mathcal{I}(t) := \exp t(|x|\Delta - |x|)} \qquad \operatorname{Re} t > 0$$

closed formula using Bessel function

(k,a)-deformation of $\exp\frac{t}{2}(\Delta - |x|^2)$

(k,a)-generalized Fourier transform

$$\mathcal{F}_{k,a} = \underbrace{c}_{\substack{\text{phase factor} \\ = e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}}} \exp\Big(\frac{\pi i}{2a}\big(\overbrace{\underbrace{|x|^{2-a}\Delta_k}_{\text{Dunkl Laplacian}} - |x|^a}^{\text{self-adjoint op. on } L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)}\big)\Big)$$

(k,a)-deformation of Hermite semigroup ([with Ben Saïd, Ørsted])

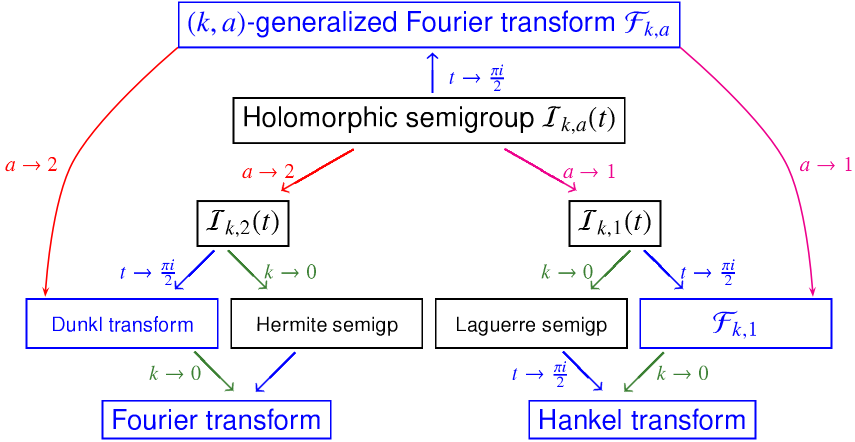
$$\boxed{\mathcal{I}_{k,a}(t) := \exp\frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a)} \qquad \operatorname{Re} t > 0$$

k: multiplicity on root system $\mathcal{R}$, $a > 0$

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$$\begin{tikzcd}
 & & & \boxed{(k,a)\text{-generalized Fourier transform } \mathcal{F}_{k,a}} \arrow[dddlll, bend right, "a \to 2"'] \arrow[dddrrr, bend left, "a \to 1"] & & & \\
 & & & \boxed{\text{Holomorphic semigroup } \mathcal{I}_{k,a}(t)} \arrow[u, "t \to \frac{\pi i}{2}"'] \arrow[dll, "a \to 2"'] \arrow[drr, "a \to 1"] & & & \\
 & \boxed{\mathcal{I}_{k,2}(t)} \arrow[dl, "t \to \frac{\pi i}{2}"'] \arrow[dr, "k \to 0"] & & & & \boxed{\mathcal{I}_{k,1}(t)} \arrow[dl, "k \to 0"'] \arrow[dr, "t \to \frac{\pi i}{2}"] & \\
\boxed{\text{Dunkl transform}} \arrow[dr, "k \to 0"'] & & \boxed{\text{Hermite semigp}} \arrow[dl, "t \to \frac{\pi i}{2}"] & & \boxed{\text{Laguerre semigp}} \arrow[dr, "t \to \frac{\pi i}{2}"'] & & \boxed{\mathcal{F}_{k,1}} \arrow[dl, "k \to 0"] \\
 & \boxed{\text{Fourier transform}} & & & & \boxed{\text{Hankel transform}} &
\end{tikzcd}$$

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



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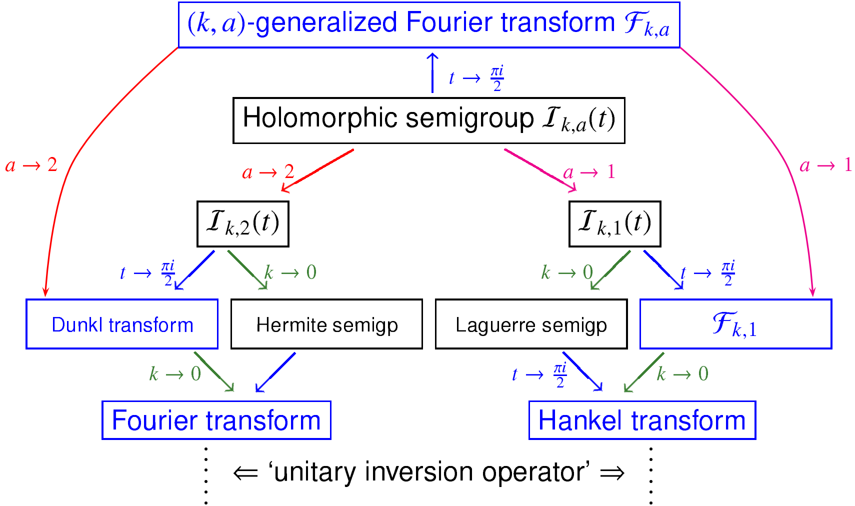

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the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of
the conformal group $O(N+1, 2)$

Generalized Fourier transform $\mathcal{F}_{k,a}$

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   $F_{k,a}$ = Dunkl transform on $\mathbb{R}^N$
   $\mathcal{F}_{0,1}$ = Hankel-type transform on $L^2($  $)$
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$\Longrightarrow$ generalization of classical identities such as Hecke identity, Bochner identity, Parseval–Plancherel formulas, Weber's second exponential integral, etc.

Heisenberg-type inequality

Thm H ([2]) (Heisenberg inequality)

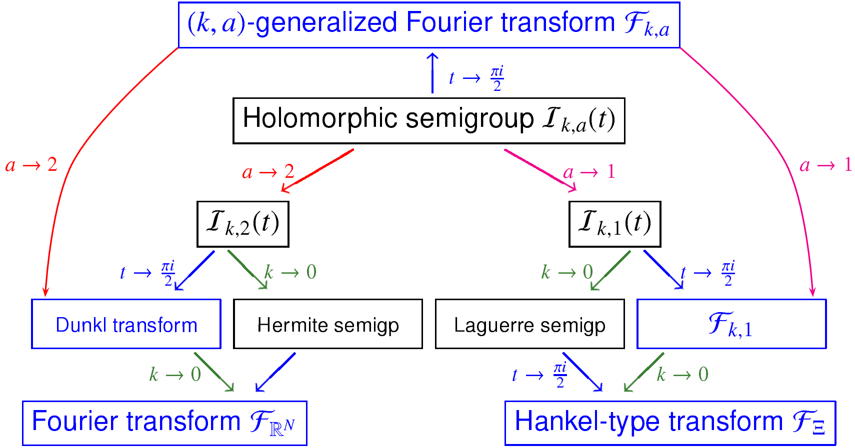
$$\| |x|^{\frac{a}{2}} f(x) \|_k \, \| |\xi|^{\frac{a}{2}} (\mathcal{F}_{k,a} f)(\xi) \|_k \geq \frac{2\langle k \rangle + N + a - 2}{2} \| f(x) \|_k^2$$

$k \equiv 0$, $a = 2$ $\cdots$ Weyl–Pauli–Heisenberg inequality for Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

k: general, $a = 2$ $\cdots$ Heisenberg inequality for Dunkl transform $\mathcal{D}_k$ (Rösler, Shimeno)

$k \equiv 0$, $a = 1$, $N = 1$ $\cdots$ Heisenberg inequality for Hankel transform

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



$\Leftarrow$ 'unitary inversion operator' $\Rightarrow$

the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of the conformal group $O(N+1, 2)$

Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

Coxeter group

$$\begin{tikzcd} & & \boxed{O(N+1,2)^{\sim}} \\ \boxed{\mathfrak{C}\times \widetilde{SL(2,\mathbb{R})}} \arrow[r, "k\to 0"] & \boxed{O(N)\times \widetilde{SL(2,\mathbb{R})}} \arrow[ur, "a\to 1"'] \arrow[dr, "a\to 2"] & \\ & & \boxed{Mp(N,\mathbb{R})} \end{tikzcd}$$

$(k, a$: general$)$

Minimal $\Leftrightarrow$ Maximal

(Ambitious) Project:
Use minimal reps to get an inspiration in finding new interactions with other fields of mathematics.

Viewpoint:
Minimal representation ($\Leftarrow$ group)
$\approx$ Maximal symmetries ($\Leftarrow$ rep. space)

Geometric analysis on minimal reps

[1] Minimal representations via Bessel operators $\cdots$ 66 pp. arXiv:1106.3621
[2] Laguerre semigroup and Dunkl operators $\cdots$
Compositio Math (to appear), 74 pp.
[3] Schrödinger model of minimal representations of $O(p,q)$ $\cdots$
Memoirs of Amer. Math. Soc. (2011), no.1000, 132 pp.
[4] Algebraic analysis on minimal representations $\cdots$
Publ. RIMS (2011), 28 pp.
[5] Geometric analysis of small unitary reps of $GL(n,\mathbb{R})$ $\cdots$
J. Funct. Anal. (2011)
[6] Special functions associated to a fourth order differential equation $\cdots$
Ramanujan J. Math (2011)
[7] Analysis on minimal representations $\cdots$
Adv. Math. (2003) I, II, III, 110 pp.
[8] Generalized Fourier transforms $\mathcal{F}_{k,a}$ $\cdots$ C.R.A.S. Paris 2009
[9] Inversion and holomorphic extension $\cdots$
R. Howe 60th birthday volume (2007), 65 pp.

Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers, Ørsted and M. Pevzner