Geometric Analysis on Minimal Representations

Representation Theory of Real Reductive Groups
University of Utah, Salt Lake City, USA, 27–31 July 2009

Toshiyuki Kobayashi
(the University of Tokyo)

http://www.ms.u-tokyo.ac.jp/~toshi/
Minimal representations

Oscillator rep. (= Segal–Shale–Weil rep.)
Minimal rep. of $Mp(n, \mathbb{R})$ (= double cover of $Sp(n, \mathbb{R})$)
\[\cdots\] split simple group of type C
Minimal representations

Oscillator rep. (= Segal–Shale–Weil rep.)
Minimal rep. of $Mp(n, \mathbb{R})$ (= double cover of $Sp(n, \mathbb{R})$)
  ... split simple group of type C

Today:
Minimal rep. of $O(p, q)$, $p + q$: even
  ... simple group of type D
Minimal representations

Oscillator rep. (= Segal–Shale–Weil rep.)
Minimal rep. of $Mp(n, \mathbb{R})$ (= double cover of $Sp(n, \mathbb{R})$)

... split simple group of type C

Today: Geometric and analytic aspects of Minimal rep. of $O(p, q)$, $p + q$: even

... simple group of type D
Minimal representations

Oscillator rep. (= Segal–Shale–Weil rep.)

Minimal rep. of $Mp(n, \mathbb{R})$ (= double cover of $Sp(n, \mathbb{R})$)

... split simple group of type C

Today: Geometric and analytic aspects of

Minimal rep. of $O(p, q), p + q$: even

... simple group of type D

(Ambitious) Project:

Use minimal reps as a guiding principle to find new interactions with other fields of mathematics.

If possible, try to formulate a theory in a wide setting without group, and prove it without representation theory.
Minimal rep of reductive groups

Minimal representations of a reductive group $G$
(Their annihilators are the Joseph ideal in $U(g)$)

Loosely, minimal representations are

- One of ‘building blocks’ of unitary reps.
- ‘Smallest’ infinite dimensional unitary rep.
- ‘Isolated’ among the unitary dual
  (finitely many) (continuously many)
- ‘Attached to’ the minimal nilpotent orbit
- Matrix coefficients are of bad decay
Minimal ⇔ Maximal

(Ambitious) Project:
Use minimal reps as a guiding principle to find new interactions with other fields of mathematics.
(Ambitious) Project:  
Use minimal reps as a guiding principle to find new interactions with other fields of mathematics.

Viewpoint:  
Minimal representation (group)  
≈ Maximal symmetries (rep. space)
Geometric analysis on minimal reps of $O(p, q)$

[1] Laguerre semigroup and Dunkl operators · · · preprint, 74 pp. [arXiv:0907.3749]


Collaborated with

S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers and B. Ørsted
Indefinite orthogonal group $O(p + 1, q + 1)$

Throughout this talk, $p, q \geq 1, p + q$: even $> 2$

$$G = O(p + 1, q + 1)$$

$$= \{ g \in GL(p + q + 2, \mathbb{R}) : {}^t g \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} g = \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} \}$$

... real simple Lie group of type D
Minimal representation of $G = O(p + 1, q + 1)$

$q = 1$

highest weight module $\oplus$ lowest weight module

the bound states of the Hydrogen atom
Minimal representation of $G = O(p + 1, q + 1)$

- $q = 1$
  - highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom

- $p = q$
  - spherical case
  - $\leftrightarrow$ realized in scalar valued functions on the Riemannian symmetric space $G/K$

- $p = q = 3$ case: Kostant (1990)
Minimal representation of $G = O(p + 1, q + 1)$

- $q = 1$
  - highest weight module $\oplus$ lowest weight module
  - the bound states of the Hydrogen atom

- $p = q$
  - spherical case
  - realized in scalar valued functions on the Riemannian symmetric space $G/K$

- $p = q = 3$ case: Kostant (1990)

- $p$, $q$: general
  - non-highest, non-spherical

  - subrepresentation of most degenerate principal series
    (Howe–Tan, Binegar–Zierau)

  - dual pair correspondence
    $(Sp(1, \mathbb{R}) \times O(p + 1, q + 1)$ in $Sp(p + q + 2, \mathbb{R}))$ (Huang–Zhu)
Two constructions of minimal reps.

1. Conformal model
   Theorem B

2. $L^2$ model
   (Schrödinger model)
   Theorem D
Two constructions of minimal reps.

Group action  Hilbert structure

1. Conformal model
   Theorem B  Clear
   v.s.
2. $L^2$ model
   (Schrödinger model)  ?
   Theorem D  Clear

Clear ... advantage of the model
Two constructions of minimal reps.

Group action  Hilbert structure

1. Conformal model
   Theorem B  Clear  Theorem C
   v.s.

2. $L^2$ model
   (Schrödinger model)  Theorem E  Clear
   Theorem D

Clear ··· advantage of the model
Two constructions of minimal reps.

1. Conformal model
   Theorem B
   v.s.
   Theorem C

2. $L^2$ model
   (Schrödinger model)
   Theorem D
   Theorem E

Clear ... advantage of the model

3. Deformation of Fourier transforms
   (Theorems F, G, H)
   (interpolation, Dunkl operators, special functions)
Geometric analysis on minimal reps of $O(p, q)$

[1] Laguerre semigroup and Dunkl operators ···

[2] Special functions associated to a fourth order differential equation ···


[4] Schrödinger model of minimal rep. ···

[5] Inversion and holomorphic extension ···

[6] Analysis on minimal representations ···

Collaborated with
    S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers and B. Ørsted
§1 Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

\[ \text{holomorphic} \circ \text{holomorphic} = \text{holomorphic} \]
§1 Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

\[
\text{holomorphic } \circ \text{ holomorphic } = \text{ holomorphic}
\]

taking real parts

\[
\text{harmonic } \circ \text{ conformal } = \text{ harmonic}
\]
on \( \mathbb{C} \sim \mathbb{R}^2 \)
§1 Conformal construction of minimal reps.

Idea: Composition of holomorphic functions
\[ \text{holomorphic} \circ \text{holomorphic} = \text{holomorphic} \]

taking real parts

\[ \text{harmonic} \circ \text{conformal} = \text{harmonic} \]
on \( \mathbb{C} \sim \mathbb{R}^2 \)

make sense for general Riemannian manifolds.
§1 Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

\[ \text{holomorphic} \circ \text{holomorphic} = \text{holomorphic} \]

\[ \downarrow \text{taking real parts} \]

\[ \text{harmonic} \circ \text{conformal} = \text{harmonic} \]

on \( \mathbb{C} \cong \mathbb{R}^2 \)

make sense for general Riemannian manifolds.

But \( \text{harmonic} \circ \text{conformal} \neq \text{harmonic} \)
in general
§1 Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

\[ \text{holomorphic} \circ \text{holomorphic} = \text{holomorphic} \]

\[ \downarrow \text{taking real parts} \]

\[ \text{harmonic} \circ \text{conformal} = \text{harmonic} \quad \text{on } \mathbb{C} \sim \mathbb{R}^2 \]

make sense for general Riemannian manifolds.

But \[ \text{harmonic} \circ \text{conformal} \neq \text{harmonic} \quad \text{in general} \]

\[ \Rightarrow \text{Try to modify the definition!} \]
$\text{Conf}(X, g) \supset \text{Isom}(X, g)$

$(X, g)$ Riemannian manifold

$\varphi \in \text{Diffeo}(X)$
\( \text{Conf}(X, g) \supset \text{Isom}(X, g) \)

\((X, g)\) Riemannian manifold
\(\varphi \in \text{Diffeo}(X)\)

**Def.**

\(\varphi\) is isometry \(\iff\) \(\varphi^* g = g\)

\(\varphi\) is conformal \(\iff\) \(\exists\) positive function \(C_\varphi \in C^\infty(X)\) s.t.

\[ \varphi^* g = C_\varphi^2 g \]

\(C_\varphi\) : conformal factor
\[ \text{Def.} \]

\( \varphi \) is isometry \iff \( \varphi^* g = g \)

\( \varphi \) is conformal \iff \exists \text{ positive function } C_\varphi \in C^\infty(X) \text{ s.t. } \varphi^* g = C_\varphi^2 g \)

\( C_\varphi \) : conformal factor
Conf\((X, g) \supset \text{Isom}(X, g)\)

\((X, g)\) pseudo-Riemannian manifold
\(\varphi \in \text{Diffeo}(X)\)

**Def.**

\(\varphi\) is isometry \(\iff \varphi^* g = g\)

\(\varphi\) is conformal \(\iff \exists\text{ positive function } C_\varphi \in C^\infty(X) \text{ s.t. }\)

\[\varphi^* g = C_\varphi^2 g\]

\(C_\varphi: \text{conformal factor}\)

\(\text{Diffeo}(X) \supset \text{Conf}(X, g) \supset \text{Isom}(X, g)\)

Conformal group \quad isometry group
Harmonic $\circ$ conformal $\neq$ harmonic

Modification
\[ \varphi \in \text{Conf}(X^n, g), \quad \varphi^* g = C_{\varphi}^2 g \]
Harmonic o conformal $\neq$ harmonic

Modification

$\varphi \in \text{Conf}(X^n, g), \quad \varphi^* g = C^2 g$

- pull-back $\sim\sim$ twisted pull-back

$$f \circ \varphi \sim\sim C^\varphi f \circ \varphi$$

conformal factor
Harmonic $\circ$ conformal $\neq$ harmonic

Modification
$\varphi \in \text{Conf}(X^n, g), \quad \varphi^* g = C_{\varphi}^2 g$

pull-back $\rightsquigarrow$ twisted pull-back

\[ f \circ \varphi \quad \rightsquigarrow \quad C_{\varphi}^{-\frac{n-2}{2}} f \circ \varphi \]

conformal factor

\[ \text{Sol}(\Delta_X) = \{ f \in C^\infty(X) : \Delta_X f = 0 \} \quad \text{(harmonic functions)} \]

\[ \rightsquigarrow \quad \text{Sol}(\overline{\Delta_X}) = \{ f \in C^\infty(X) : \overline{\Delta_X} f = 0 \} \]

\[ \overline{\Delta_X} := \Delta_X + \frac{n-2}{4(n-1)} \kappa \]

Yamabe operator \quad Laplacian \quad scalar curvature
Distinguished rep. of conformal groups

\[ \text{harmonic} \circ \text{conformal} \div \text{harmonic} \]

\[ \Downarrow \text{Modification} \]
Distinguished rep. of conformal groups

\[ \text{harmonic} \circ \text{conformal} \Downarrow \text{harmonic} \]

\[ \downarrow \text{Modification} \]

**Theorem A ([6, Part I])**

\( (X^n, g) \) Riemannian mfd

\[ \Rightarrow \text{Conf}(X, g) \text{ acts on } Sol(\Delta_X) \text{ by } f \mapsto C_{\varphi}^{-\frac{n-2}{2}} f \circ \varphi \]
Modification

Theorem A ([6, Part I]) \((X^n, g)\) Riemannian mfd

\[ \implies \text{Conf}(X, g) \text{ acts on } \text{Sol}(\Delta_X) \text{ by } f \mapsto C \frac{n-2}{2} f \circ \varphi \]

Point \(\Delta_X = \Delta_X + \frac{n-2}{4(n-2)} \kappa\)

\(\Delta_X\) is not invariant by \(\text{Conf}(X, g)\).

But \(\text{Sol}(\Delta_X)\) is invariant by \(\text{Conf}(X, g)\).
Distinguished rep. of conformal groups

\[ \text{harmonic} \circ \text{conformal} \not\subset \text{harmonic} \]

\[ \downarrow \text{Modification} \]

**Theorem A** ([6, Part I]) \((X^n, g)\) Riemannian mfd

\[ \implies \text{Conf}(X, g) \text{ acts on } \text{Sol}(\Delta_X) \text{ by } f \mapsto C_{\varphi}^{-\frac{n-2}{2}} f \circ \varphi \]

---

**Point** \(\Delta_X = \Delta_X + \frac{n-2}{4(n-2)} \kappa\)

\(\Delta_X\) is not invariant by \(\text{Conf}(X, g)\).

But \(\text{Sol}(\Delta_X)\) is invariant by \(\text{Conf}(X, g)\).

\[ \text{Diffeo}(X) \supset \text{Conf}(X, g) \supset \text{Isom}(X, g) \]

Conformal group \quad isometry group
Distinguished rep. of conformal groups

\[ \text{harmonic} \circ \text{conformal} \overset{\text{Modification}}{=} \text{harmonic} \]

**Theorem A** ([6, Part I]) \((X^n, g)\) pseudo-Riemannian mfd

\[ \therefore \text{Conf}(X, g) \text{ acts on } Sol(\tilde{\Delta_X}) \text{ by } f \mapsto C_{\varphi}^{-\frac{n-2}{2}} f \circ \varphi \]

Point \(\tilde{\Delta_X} = \Delta_X + \frac{n-2}{4(n-2)} \kappa\)
\(\Delta_X\) is not invariant by \(\text{Conf}(X, g)\).
But \(Sol(\tilde{\Delta_X})\) is invariant by \(\text{Conf}(X, g)\).

\[ \text{Diffeo}(X) \supset \text{Conf}(X, g) \supset \text{Isom}(X, g) \]
Conformal group isometry group
Application of Theorem A

\[(X, g) := (S^p \times S^q, \underbrace{+ \cdots +}_{p} \underbrace{- \cdots -}_{q})\]
Application of Theorem A

\[(X, g) := (S^p \times S^q, \underbrace{+_\cdots+}_{p} \underbrace{-\cdots-}_{q})\]

**Theorem B ([6, Part III])**

0) \(\text{Conf}(X, g) \simeq O(p + 1, q + 1)\)

1) \(\text{Sol}(\Delta_X) \neq \{0\} \iff p + q \text{ even}\)

2) If \(p + q \text{ is even and } > 2\), then
   \(\text{Conf}(X, g) \sim \text{Sol}(\Delta_X)\) is irreducible,
   and for \(p + q > 6\) it is a minimal rep of \(O(p + 1, q + 1)\).
Application of Theorem A

\[(X, g) := (S^p \times S^q, \underbrace{+ \cdots +}_{p} \underbrace{- \cdots -}_{q})\]

Theorem B ([6, Part II])

0) \(\text{Conf}(X, g) \simeq O(p + 1, q + 1)\)

1) \(\text{Sol}(\Delta_X) \neq \{0\} \iff p + q \text{ even}\)

2) If \(p + q\) is even and \(> 2\), then
   \(\text{Conf}(X, g) \sim \text{Sol}(\Delta_X)\) is irreducible,
   and for \(p + q > 6\) it is a minimal rep of \(O(p + 1, q + 1)\).

1) (conformal geometry) \(\iff\) (representation theory)
   characterizing subrep in \(\text{Ind}_{P_{\max}}^G (\mathbb{C}_\lambda)\) (\(K\)-picture)
   by means of differential equations
Application of Theorem A

\[(X, g) := (S^p \times S^q, + \cdots + \frac{1}{p}, - \cdots - \frac{1}{q})\]

Theorem B ([6, Part III])

0) \(\text{Conf}(X, g) \cong O(p + 1, q + 1)\)

1) \(\text{Sol}(\Delta_X) \neq \{0\} \iff p + q \text{ even}\)

2) If \(p + q\) is even and \(\geq 2\), then
   \(\text{Conf}(X, g) \sim \text{Sol}(\Delta_X)\) is irreducible,
   and for \(p + q > 6\) it is a minimal rep of \(O(p + 1, q + 1)\).

\[\exists a \text{ Conf}(X, g)\)-invariant inner product, and
take the Hilbert completion
Flat model

Stereographic projection

\[ S^n \rightarrow \mathbb{R}^n \cup \{\infty\} \quad \text{conformal map} \]
Flat model

Stereographic projection

\[ S^n \rightarrow \mathbb{R}^n \cup \{\infty\} \quad \text{conformal map} \]

More generally

\[ S^p \times S^q \leftrightarrow \mathbb{R}^{p+q} \quad \text{conformal embedding} \]

\[ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]
Flat model

Stereographic projection

\[ S^n \rightarrow \mathbb{R}^n \cup \{ \infty \} \text{ conformal map} \]

More generally

\[ S^p \times S^q \rightarrow \mathbb{R}^{p+q} \text{ conformal embedding} \]

\[ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

Functoriality of Theorem A

\[ \text{Sol}(\tilde{\Delta}_{S^p \times S^q}) \subset \text{Sol}(\tilde{\Delta}_{\mathbb{R}^{p+q}}) \]

\[ \text{Conf}(S^p \times S^q) \leftrightarrow \text{Conf}(\mathbb{R}^{p+q}) \]
Two constructions of minimal reps.

1. Conformal construction
   Theorem B
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

Clear ... advantage of the model
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \Delta_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

---

Unitarization of subrep (representation theory)

\[ \iff \]

Conservative quantity (differential eqn)
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

Unitarization of subrep (representation theory)

\[ \iff \]

Conservative quantity (differential eqn)

Unitarizability v.s. Unitarization
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \square_{p,q} \]

Unitarization of subrep (representation theory)

\[ \iff \]

Conservative quantity (differential eqn)

Unitarizability v.s. Unitarization

- Easy formulation
- Challenging formulation
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \Delta_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

**Problem**  Find an ‘intrinsic’ inner product on (a ‘large’ subspace of) \( Sol(\Box_{p,q}) \) if exists.
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{H}^{p,q} = \mathbb{H}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{H}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

Problem
Find an ‘intrinsic’ inner product on (a ‘large’ subspace of) \( \text{Sol}(\Box_{p,q}) \)
if exists.

Easy: if allowed to use the integral representation of solutions

Cf. (representation theory)
by using the Knapp–Stein intertwining formula

Challenging: to find the intrinsic formula
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

\( q = 1 \quad \text{wave operator} \)
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

\[ q = 1 \quad \text{wave operator} \]

energy \cdots \text{conservative quantity for wave equations w.r.t. time translation} \quad \mathbb{R} \]
Conservative quantity for ultra-hyperbolic eqn.

\[ \mathbb{R}^{p,q} = \mathbb{R}^{p+q}, \quad ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2 \]

\[ \tilde{\Delta}_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \Box_{p,q} \]

\( q = 1 \) wave operator

energy \( \cdots \) conservative quantity for wave equations

w.r.t. time translation \( \mathbb{R} \)

\[ \downarrow \]

? \( \cdots \) conservative quantity for ultra-hyperbolic eqs

w.r.t. conformal group \( O(p+1, q+1) \)
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \text{Sol}(\Box_{p,q})$

$$ (f, f) := \int_{\alpha} Q_{\alpha} f $$

\ldots \ldots \textcircled{1}
Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in Sol(\square_{p,q})$

\[
(f, f) := \int_\alpha Q_\alpha f
\]

\\ \[\text{Theorem C (6, Part III$^+$)}\]

1) $\textcircled{1}$ is independent of hyperplane $\alpha$. 
Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in Sol(\square_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f$$  \hspace{1cm} \cdots \cdots \circled{1}$$

**Theorem C ([6, Part III] + ε)**

1) $\circled{1}$ is independent of hyperplane $\alpha$.
2) $\circled{1}$ gives the unique inner product (up to scalar)
   which is invariant under $O(p + 1, q + 1)$.  

Geometric Analysis on Minimal Representations – p.18/49
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \text{Sol}(\Box_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f \quad \quad \quad \cdots \cdots \textcircled{1}$$

Theorem C ([6, Part III]+c)

1) \textcircled{1} is independent of hyperplane $\alpha$.

2) \textcircled{1} gives the unique inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$.

$$O(p, q) \quad \sim \quad \mathbb{R}^{p,q} \quad \text{(linear)}$$
Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \text{Sol}(\square_{p,q})$

\[ (f, f) := \int_{\alpha} Q_{\alpha} f \]

\[ \ldots \ldots \text{①} \]

**Theorem C ([6, Part III] + ε)**

1) ① is independent of hyperplane $\alpha$.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$.

\[ O(p, q) \quad \sim \quad \mathbb{R}^{p,q} \quad \text{(linear)} \]

\[ O(p + 1, q + 1) \quad \text{(Möbius transform)} \]
Parametrization of non-characteristic hyperplane

Fix \( v \in \mathbb{R}^{p,q} \) s.t. \( (v, v)_{\mathbb{R}^{p,q}} = \pm 1 \)

\( c \in \mathbb{R} \)

\[ \mathbb{R}^{p,q} \supset \alpha \equiv \alpha_{v,c} := \{ x \in \mathbb{R}^{p+q} : (x, v)_{\mathbb{R}^{p,q}} = c \} \]

non-characteristic hyperplane
‘Intrinsic’ inner product

Point: \[ f = f_+ + f_- \] (idea: Sato’s hyperfunction)
‘Intrinsic’ inner product

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)
‘Intrinsic’ inner product

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)

$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. $v$
‘Intrinsic’ inner product

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at $\infty$

Point: $f = f_+ + f_-$ (idea: Sato’s hyperfunction)

$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. $v$

$Q_\alpha f := \frac{1}{i} \left( f_+ \overline{f'_+} - f_- \overline{f'_-} \right)$
Conservative quantity for $\square_{p,q}f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \text{Sol}(\square_{p,q})$

$$(f, f) := \int_{\alpha} Q_\alpha f \quad \cdots \quad ①$$

Theorem C ([6, Part III] + ε)

1) ① is independent of hyperplane $\alpha$.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$. 
Conservative quantity for $\Box_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane.

For $f \in \text{Sol}(\Box_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f \quad \cdots \cdots \text{(1)}$$

Theorem C ([6, Part III]$^+$)

1) (1) is independent of hyperplane $\alpha$.
2) (1) gives the unique inner product (up to scalar) which is invariant under $O(p + 1, q + 1)$.

non-trivial even for $q = 1$ (wave equation)

In space-time,

average in space (i.e. time $t = \text{constant}$)

$= \text{average in} \ (\text{any hyperplane in space}) \times \mathbb{R}_t \ (\text{time})$
Two constructions of minimal reps.

1. Conformal construction
   Theorem B
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

Clear ... advantage of the model
Two constructions of minimal reps.

1. Conformal construction
   Theorems A, B

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

Conservative quantity

Group action

Hilbert structure

Clear ··· advantage of the model
Conformal model $\implies L^2$-model

$$
\square_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}
$$

$$
\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$
Conformal model $\iff L^2$-model

$$\square_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$= \bigtimes \quad \text{(figure for } (p, q) = (2, 1))$$

Geometric Analysis on Minimal Representations – p.24/49
Conformal model $\implies L^2$-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}$$

$$\Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi$$

Fourier trans.
Conformal model \implies L^2\text{-model}

\begin{align*}
\Box_{p,q} &= \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \\
\Xi &:= \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}
\end{align*}

\[ \Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi \]

\[ \mathcal{F} : S'(\mathbb{R}^{p,q}) \xrightarrow{\sim} S'(\mathbb{R}^{p,q}) \]
Conformal model $\implies L^2$-model

\[ \Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \]

\[ \Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \} \]

\[ \Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi \]

\[ \mathcal{F} : S'(\mathbb{R}^{p,q}) \overset{\sim}{\longrightarrow} S'(\mathbb{R}^{p,q}) \]

\[ \cup \]

\[ \text{Sol}(\Box_{p,q}) \]
Conformal model $\implies \mathbb{L}^2$-model

\[ \Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \]

\[ \Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \} \]

\[ \Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi \]

\[ \mathcal{F} : S'(\mathbb{R}^{p,q}) \xrightarrow{\sim} S'(\mathbb{R}^{p,q}) \]

\[ \overline{\text{Sol}(\Box_{p,q})} \xrightarrow{\sim} \square \]

\[ \square \text{ denotes the closure with respect to the inner product.} \]
Conformal model $\implies L^2$-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}$$

$$\Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi$$

Fourier trans.

$$\mathcal{F} : \quad S'(\mathbb{R}^{p,q}) \sim \mathcal{U} \quad \text{U} \quad \mathcal{U} \quad S'(\mathbb{R}^{p,q})$$

Theorem D ([6, Part III])

$$\text{Sol}(\Box_{p,q}) \sim L^2(\Xi)$$
Conformal model $\implies L^2$-model

$$\Box_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

$$\Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}$$

$$\Box_{p,q} f = 0 \implies \text{Supp } \mathcal{F} f \subset \Xi$$

$$\mathcal{F} : S'(\mathbb{R}^{p+q}) \sim \bigcup S'(\mathbb{R}^{p+q})$$

**Theorem D ([6, Part III])**

$$\overline{\text{Sol}(\Box_{p,q})} \sim L^2(\Xi)$$

conformal model $\implies L^2$-model
Two constructions of minimal reps.

1. Conformal construction
   Theorems A, B
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

Clear $\cdots$ advantage of the model
§2 $L^2$-model of minimal reps.

Theorem D ([6, Part III])

\[
\text{Sol}(\Box_{p,q}) \xrightarrow{\sim} L^2(\Sigma)
\]

conformal model $L^2$-model
§2 $L^2$-model of minimal reps.

Theorem D ([6, Part III])

$$\text{Sol}(\square_{p,q}) \sim L^2(\Xi)$$

conformal model  $L^2$-model

$p + q$: even $> 2$

$$G = O(p + 1, q + 1) \sim L^2(\Xi)$$ unitary rep.
§2 $L^2$-model of minimal reps.

Theorem D ([6, Part III])

$\text{Sol}(\Box_{p,q}) \sim \rightarrow L^2(\Xi)$

conformal model $L^2$-model

$p + q$: even $> 2$

$G = O(p + 1, q + 1) \sim \rightarrow L^2(\Xi)$ unitary rep.

Point: $\Xi$ is too small to be acted by $G$. 
§2 $L^2$-model of minimal reps.

**Theorem D** ([6, Part III]) \( \text{Sol}(\Box_{p,q}) \sim L^2(\Xi) \)

**conformal model** \hspace{1cm} **$L^2$-model**

\( p + q: \text{even} > 2 \)

\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \] \hspace{1cm} **unitary rep.**

**Point:** \( \Xi \) is too small to be acted by \( G \).

\[ \Xi \subset \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1} \]
§2 $L^2$-model of minimal reps.

Theorem D ([6, Part III])

\[
\begin{align*}
\text{conformal model} & \quad \sim \\
\text{$L^2$-model} & \quad \sim
\end{align*}
\]

$p + q$: even $> 2$

\[
G = O(p + 1, q + 1) \quad \sim \quad L^2(\Xi) \quad \text{unitary rep.}
\]

Point: $\Xi$ is too small to be acted by $G$.

\[
O(p, q) \quad \sim \quad \Xi \quad \subset \quad \mathbb{R}^{p,q} \quad \subset \quad \mathbb{R}^{p+1,q+1}
\]
§2 $L^2$-model of minimal reps.

Theorem D ([6, Part III])

\[
\text{Sol}(\Box_{p,q}) \sim L^2(\Xi)
\]

\[\text{conformal model} \quad \text{$L^2$-model}\]

\[p + q: \text{ even > 2}\]

minimal rep.

\[G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{unitary rep.}\]

\[O(p + 1, q + 1) \not\subset \Xi \subset \mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q+1}\]

\[\Xi \not\subset L^2(\Xi)\]

\[\text{Point: } \Xi \text{ is too small to be acted by } G.\]
Inversion element

\[ G = PGL(2, \mathbb{C}) \sim \mathbb{P}^1 \mathbb{C} \simeq \mathbb{C} \cup \{\infty\} \]

Möbius transform
Inversion element

\[ G = PGL(2, \mathbb{C}) \] \quad \sim \quad \mathbb{P}^1 \mathbb{C} \simeq \mathbb{C} \cup \{\infty\}

Möbius transform

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \; b \in \mathbb{C} \right\} \quad z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \quad \text{(inversion)} \]
**Inversion element**

\[ G = PGL(2, \mathbb{C}) \]

\[ \begin{array}{cccc}
0 & -1 \\
1 & 0 \\
\end{array} \]

\[ \Rightarrow \quad \mathbb{P}^1 \mathbb{C} \cong \mathbb{C} \cup \{\infty\} \]

Möbius transform

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \quad z \mapsto a z + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \quad \text{ (inversion)} \]

\[ G \text{ is generated by } P \text{ and } w. \]
Inversion element

\[ G = PGL(2, \mathbb{C}) \quad \bowtie \quad \mathbb{P}^1 \mathbb{C} \cong \mathbb{C} \cup \{ \infty \} \]

Möbius transform

\[ \divides O(3, 1) \quad \divides \mathbb{R}^{2,0} \]

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \quad z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \quad \text{(inversion)} \]

\( G \) is generated by \( P \) and \( w \).
Inversion element

\[ G = PGL(2, \mathbb{C}) \cong \mathbb{P}^{1} \mathbb{C} \cong \mathbb{C} \cup \{\infty\} \]

\[ \cong O(3, 1) \cong \mathbb{R}^{2,0} \]

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C} \right\} \quad z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \quad \text{(inversion)} \]

\[ G \text{ is generated by } P \text{ and } w. \]

\[ G = O(p + 1, q + 1) \cong \mathbb{R}^{p,q} \]

\[ P = \{(A, b) : A \in O(p, q) \cdot \mathbb{R}^\times, b \in \mathbb{R}^{p+q}\} \quad x \mapsto Ax + b \]

\[ w = \begin{pmatrix} I_p & -I_q \end{pmatrix} \quad \text{(inversion)} \]
Inversion element

\[ G = \text{PGL}(2, \mathbb{C}) \sim \mathbb{P}^1 \mathbb{C} \sim \mathbb{C} \cup \{ \infty \} \]

\[ \cong O(3, 1) \quad \cong \mathbb{R}^{2,0} \]

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C}^\times, \ b \in \mathbb{C} \right\} \quad z \mapsto az + b \]

\[ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad z \mapsto -\frac{1}{z} \text{ (inversion)} \]

\( G \) is generated by \( P \) and \( w \).

\[ G = O(p + 1, q + 1) \sim \mathbb{R}^{p,q} \]

\[ \text{Möbius transform} \]

\[ P = \{(A, b) : A \in O(p, q) \cdot \mathbb{R}^\times, \ b \in \mathbb{R}^{p+q}\} \quad x \mapsto Ax + b \]

\[ w = \begin{pmatrix} I_p \\ -I_q \end{pmatrix} : (x', x'') \mapsto \frac{4}{|x'|^2 - |x''|^2}(-x', x'') \text{ (inversion)} \]
Unitary inversion operator \( \mathcal{F}_\Xi \)

\[ p + q : \text{even} > 2 \]
\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{minimal rep.} \]
Unitary inversion operator $\mathcal{F}_\Xi$

\[ p + q : \text{even} > 2 \]
\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{minimal rep.} \]

$P$-action \quad \cdots \quad \text{translation and multiplication}

$w$-action \quad \cdots \quad \mathcal{F}_\Xi \quad \text{(unitary inversion operator)}
Unitary inversion operator $\mathcal{F}_\Xi$

$p + q$: even $> 2$

$G = O(p + 1, q + 1) \hat{\sim} L^2(\Xi)$  
minimal rep.

$P$-action $\cdots$ translation and multiplication

$w$-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem Find the unitary operator $\mathcal{F}_\Xi$ explicitly.
Unitary inversion operator $\mathcal{F}_\Xi$

$p + q$: even $> 2$

$$G = O(p + 1, q + 1) \overset{\sim}{\longrightarrow} L^2(\Xi)$$  minimal rep.

$P$-action $\cdots$ translation and multiplication

$\omega$-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem  Find the unitary operator $\mathcal{F}_\Xi$ explicitly.

Easy: express it as a composition of integral transforms and a known formula for other models (e.g. conformal model)

Challenging: to find a single and explicit formula in $L^2$ model
Unitary inversion operator $\mathcal{F}_\Xi$

$p + q$: even $> 2$

$$G = O(p + 1, q + 1) \overset{\sim}{\to} L^2(\Xi)$$

minimal rep.

$P$-action $\cdots$ translation and multiplication

$w$-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem Find the unitary operator $\mathcal{F}_\Xi$ explicitly.

Cf. Analogous operator for the oscillator rep.

$$M_p(n, \mathbb{R}) \overset{\sim}{\to} L^2(\mathbb{R}^n)$$

unitary inversion operator coincides with Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ (up to scalar)!

Geometric Analysis on Minimal Representations – p.28/49
Fourier transform $\mathcal{F}_\Xi$ on $\Xi$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$= \quad \text{(figure for } (p, q) = (2, 1))$$
Fourier transform $\mathcal{F}_\Xi$ on $\Xi$

$$\Xi := \{ \xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0 \}$$

= \phantom{=} (\text{figure for } (p, q) = (2, 1))

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$ \\
$\mathcal{F}_\Xi$ on $\Xi = \phantom{=}$
Fourier transform $\mathcal{F}_\Xi$ on $\Xi$

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$$= \bigcap \quad \text{(figure for } (p, q) = (2, 1))$$

**Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$**

**$\mathcal{F}_\Xi$ on $\Xi = \bigcap$**

**Problem** Define $\mathcal{F}_\Xi$ and find its formula.
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi = \mathcal{C}_1$
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}^4 = \text{id}$

$\mathcal{F}_\Xi$ on $\Xi = \bigcirc$
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}^4 = \text{id}$

$\mathcal{F}_\Xi$ on $\Xi = \mathcal{F}_\Xi^2 = \text{id}$
Fourier trans. $\mathcal{F}_{R^n}$ on $\mathbb{R}^n$

$Q_j \leftrightarrow -P_j$

$P_j \leftrightarrow Q_j$

$Q_j = x_j$  \hspace{1cm} (multiplication by coordinate function)

$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$$
\begin{align*}
Q_j & \mapsto -P_j \\
P_j & \mapsto Q_j
\end{align*}
$$

$Q_j = x_j$ (multiplication by coordinate function)

$$
P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}
$$

$R_j = \Xi$ second order differential op. on $\Xi$

$R_j \mapsto Q_j$

Geometric Analysis on Minimal Representations – p.30/49
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

- $Q_j \leftrightarrow -P_j$
- $P_j \leftrightarrow Q_j$

$Q_j = x_j$ (multiplication by coordinate function)

$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$

$R_j = \Xi$ second order differential op. on $\Xi$

Bargmann–Todorov’s operators
‘Fourier transform’ $\mathcal{F}_\Xi$ on $\Xi$

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

\[
\begin{align*}
Q_j &\quad \mapsto \quad -P_j \\
P_j &\quad \mapsto \quad Q_j
\end{align*}
\]

$Q_j = x_j$ \hspace{1cm} (multiplication by coordinate function)

$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$

$R_j = \Xi$ \hspace{1cm} second order differential op. on $\Xi$

\[
\begin{align*}
Q_1^2 + \cdots + Q_p^2 - Q_{p+1}^2 - \cdots - Q_{p+q}^2 &= 0 \\
R_1^2 + \cdots + R_p^2 - R_{p+1}^2 - \cdots - R_{p+q}^2 &= 0
\end{align*}
\] on $\Xi$
**Unitary inversion operator** $\mathcal{F}_\Xi$

$p + q$: even $> 2$

\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \quad \text{minimal rep.} \]

$P$-action $\cdots$ translation and multiplication on $L^2(\Xi)$

$w$-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

**Problem** Find the unitary operator $\mathcal{F}_\Xi$ explicitly.
Unitary inversion operator $\mathcal{F}_\Xi$

$p + q$: even $> 2$

$$G = O(p + 1, q + 1) \sim L^2(\Xi)$$  minimal rep.

$P$-action $\cdots$ translation and multiplication on $L^2(\Xi)$

$w$-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

**Problem** Find the unitary operaotr $\mathcal{F}_\Xi$ explicitly.

Cf. Euclidean case $\varphi(t) = e^{-it}$ (one variable)

$$\mathcal{F}_{\mathbb{R}^N} f(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy$$
Unitary inversion operator $\mathcal{F}_\Xi$

\[ p + q: \text{even} > 2 \]
\[ G = O(p + 1, q + 1) \sim L^2(\Xi) \]

$P$-action \hspace{1cm} translation and multiplication on $L^2(\Xi)$

$w$-action \hspace{1cm} $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem \hspace{1cm} Find the unitary operator $\mathcal{F}_\Xi$ explicitly.

Cf. Euclidean case \hspace{1cm} $\varphi(t) = e^{-it}$ (one variable)

\[ \mathcal{F}_{\mathbb{R}^N} f(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy \]

Theorem E \hspace{1cm} Suppose $p + q: \text{even} > 2$

\[ (\mathcal{F}_\Xi f)(x) = c \int_{\Xi} \Phi_{\frac{1}{2}(p+q-4)}(\langle x, y \rangle) f(y) dy \]
Idea: Apply Mellin–Barnes type integral to distributions.

Fix $m \in \mathbb{N}$. Take a contour $L_m$ s.t.
Mellin–Barnes type integral

Idea: Apply Mellin–Barnes type integral to distributions.

Fix \( m \in \mathbb{N} \). Take a contour \( L_m \) s.t.

1) \( L_m \) starts at \( \gamma - i\infty \)
2) passes the real axis at \( s \)
3) ends at \( \gamma + i\infty \)

where

\[-m - 1 < s < -m\]

\[-1 < \gamma < 0\]
Explicit formula of $\mathcal{F}_\Xi$ on $\Xi$

**Theorem E** ([4]) Suppose $p + q$: even $> 2$

$$(\mathcal{F}_\Xi f)(x) = c \int_\Xi \Phi_{\frac{1}{2}(p+q-4)}(\langle x, y \rangle) f(y) dy$$
Explicit formula of $\mathcal{F}_\Xi$ on $\Xi$

**Theorem E ([4])**

Suppose $p + q$: even $> 2$

$$(\mathcal{F}_\Xi f)(x) = c \int_\Xi \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)}(\langle x, y \rangle) f(y)\,dy$$

Here, $\varepsilon(p, q) = \begin{cases} 
0 & \text{if } \min(p, q) = 1, \\
1 & \text{if } p, q > 1 \text{ are both odd,} \\
2 & \text{if } p, q > 1 \text{ are both even.}
\end{cases}$
Explicit formula of $\mathcal{F}_\Xi$ on $\Xi$

**Theorem E** ([41]) Suppose $p + q$: even > 2

$$(\mathcal{F}_\Xi f)(x) = c \int_{\Xi} \Phi_{\frac{1}{2}(p+q-4)}^{\varepsilon(p,q)}(\langle x, y \rangle) f(y) dy$$

Here, $\varepsilon(p, q) = \begin{cases} 0 & \text{if } \min(p, q) = 1, \\ 1 & \text{if } p, q > 1 \text{ are both odd,} \\ 2 & \text{if } p, q > 1 \text{ are both even.} \end{cases}$

$$\Phi_{m}^{\varepsilon}(t) = \begin{cases} \int_{L_0} \frac{\Gamma(-\lambda)}{\Gamma(\lambda + 1 + m)} (2t)^{\lambda} d\lambda \\ \int_{L_m} \frac{\Gamma(-\lambda)}{\Gamma(\lambda + 1 + m)} (2t)^{\lambda} d\lambda \\ \int_{L_m} \frac{\Gamma(-\lambda)}{\Gamma(\lambda + 1 + m)} \left( \frac{(2t)^{\lambda}}{\tan(\pi\lambda)} + \frac{(2t)^{\lambda}}{\sin(\pi\lambda)} \right) d\lambda \end{cases} \begin{array}{ll} \text{for } \varepsilon = 0 \\ \text{for } \varepsilon = 1 \\ \text{for } \varepsilon = 2 \end{array}$$
Regularity of $\Phi^\varepsilon_m(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R}) \cap \cdots$
Regularity of $\Phi^\varepsilon_m(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R}) \cap \cdots$

Recall two distributions on $\mathbb{R}$

$\delta(t)$: Dirac’s delta function

$t^{-1}$: Cauchy’s principal value

$$= \lim_{s \to 0} \left( \int_{-\infty}^{-s} + \int_{s}^{\infty} \right) \langle \frac{1}{t}, \cdot \rangle dt$$

these are not in $L^1_{\text{loc}}(\mathbb{R})$
Regularity of $\Phi^\varepsilon_m(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R}) \cap \cdots$

Prop. ([4]) We have the identities mod $L^1_{\text{loc}}(\mathbb{R})$

$$\Phi^\varepsilon_m(t) \equiv \begin{cases} 
0 & (\varepsilon = 0) \\
-\pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m - l - 1)!} \delta^{(l)}(t) & (\varepsilon = 1) \\
-\frac{l!}{2^l(m - l - 1)!} t^{-l-1} & (\varepsilon = 2)
\end{cases}$$
Regularity of $\Phi^\varepsilon_m(t)$

Cf. Euclidean Fourier transform $e^{-it} \in \mathcal{A}(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R}) \cap \cdots$

\[ \Phi^\varepsilon_m(t) \equiv \begin{cases} 
0 & (\varepsilon = 0) \\
-\pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!} \delta^{(l)}(t) & (\varepsilon = 1) \\
-i \sum_{l=0}^{m-1} \frac{l!}{2^l(m-l-1)!} t^{-l-1} & (\varepsilon = 2) 
\end{cases} \]

Cor. $\mathcal{F}_{\Xi}$ has a locally integrable kernel if and only if $G$ is $O(p + 1, 2)$, $O(2, q + 1)$, or $O(3, 3)$ ($\cong SL(4, \mathbb{R})$).
Prop. ([4]) \( \Phi_m^\varepsilon(t) \) solves the differential equation
\[
(\theta^2 + m\theta + 2t)u = 0
\]
where \( \theta = t \frac{d}{dt} \).
Prop. (4) \( \Phi_{m}^{\varepsilon}(t) \) solves the differential equation
\[
(\theta^2 + m\theta + 2t)u = 0
\]
where \( \theta = t\frac{d}{dt} \).

Explicit forms
\[
\Phi_{m}^{0}(t) = 2\pi i (2t)^{m/2} J_{m}(2\sqrt{2t+})
\]
\[
\Phi_{m}^{1}(t) = \Phi_{m}^{0}(t) - \pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!} \delta^{(l)}(t)
\]
Prop. ([4]) \( \Phi_{m}(t) \) solves the differential equation

\[
(\theta^2 + m\theta + 2t)u = 0
\]

where \( \theta = t \frac{d}{dt} \).

Explicit forms

\[
\Phi_{m}^{0}(t) = 2\pi i (2t)^{-\frac{m}{2}} J_{m}(2\sqrt{2t+})
\]

\[
\Phi_{m}^{1}(t) = \Phi_{m}^{0}(t) - \pi i \sum_{l=0}^{m-1} \frac{(-1)^{l}}{2^{l}(m-l-1)!} \delta^{(l)}(t)
\]

\[
\Phi_{m}^{2}(t) = 2\pi i (2t)^{-\frac{m}{2}} Y_{m}(2\sqrt{2t+})
\]

\[
+ 4(-1)^{m+1} i (2t)^{-\frac{m}{2}} K_{m}(2\sqrt{2t-})
\]
Two constructions of minimal reps.

1. Conformal construction
   Theorems A, B
   v.s.

2. $L^2$ construction
   (Schrödinger model)
   Theorem D

3. Deformation of Fourier transforms
   (Theorems F, G, H)

Group action
Hilbert structure

Conservative quantity

‘Fourier transform’ $\mathcal{F}_\Xi$
Two constructions of minimal reps.

1. Conformal construction
   Theorems A, B
   
   v.s.

2. \( L^2 \) construction
   (Schrödinger model)
   Theorem D
   
   Clear \( \cdots \) advantage of the model

3. Deformation of Fourier transforms
   (Theorems F, G, H)
Deformation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

\begin{align*}
\mathcal{F}_\Xi & \quad \ldots \quad \text{`Fourier transform' on } \Xi \subset \mathbb{R}^{p,q} \\
\mathcal{F}_{\mathbb{R}^N} & \quad \ldots \quad \text{Fourier transform on } \mathbb{R}^N
\end{align*}
Assume $q = 1$. Set $p = N$. 

\[ \mathbb{R}^{N,1} \supset \Xi = \begin{array}{c} \text{projection} \\ \downarrow \end{array} \mathbb{R}^N \]
Assume $q = 1$. Set $p = N$. 

$$\mathbb{R}^{N,1} \supset \Xi = \text{projection} \rightarrow \mathbb{R}^N$$
Deformation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_\Xi \quad \ldots \quad \text{‘Fourier transform’ on } \Xi \subset \mathbb{R}^{p,q}$

$\mathcal{F}_{\mathbb{R}^N} \quad \ldots \quad \text{Fourier transform on } \mathbb{R}^N$

Assume $q = 1$. Set $p = N$.

$\mathbb{R}^{N,1} \supset \Xi = \begin{array}{c}
\text{projection} \\
\text{deform}
\end{array} \rightarrow \mathbb{R}^N$

$a = 1 \quad \quad \quad \quad \quad \quad a = 2$
(k, a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\mathcal{F}_{\mathbb{R}^N} = c \exp \left( \frac{\pi i}{4} \left( \Delta - |x|^2 \right) \right)$$
(k, a)-deformation of \( \exp \frac{t}{2}(\Delta - |x|^2) \)

**Fourier transform**

self-adjoint op. on \( L^2(\mathbb{R}^N) \)

\[
F_{\mathbb{R}^N} = c \exp \left( \frac{\pi i}{4} \left( \Delta - |x|^2 \right) \right)
\]

phase factor \( \Delta \) Laplacian

\[
= e^{\frac{\pi i N}{4}}
\]
(k, a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

self-adjoint op. on $L^2(\mathbb{R}^N)$

$$\mathcal{F}_{\mathbb{R}^N} = c \exp \left( \frac{\pi i}{4} (\Delta - |x|^2) \right)$$

phase factor \quad Laplacian

$$= e^{\frac{\pi i N}{4}}$$

Hermite semigroup

$$I(t) := \exp \frac{t}{2}(\Delta - |x|^2)$$

R. Howe (oscillator semigroup, 1988)
\[(k, a)\text{-deformation of } \exp \frac{t}{2}(\Delta - |x|^2)\]

Hankel-type transform on \(\Xi\)

self-adjoint op. on \(L^2(\mathbb{R}^N, \frac{dx}{|x|})\)

\[\mathcal{F}_\Xi = c \exp \left( \frac{\pi i}{2} (|x|\Delta - |x|) \right)\]

phase factor \(\varphi\) and Laplacian

\[= e^{\frac{\pi i (N-1)}{2}}\]

“Laguerre semigroup” ([5], 2007 Howe 60th birthday volume)

\[\mathcal{I}(t) := \exp t(|x|\Delta - |x|)\]
\((k, a)\)-deformation of \(\exp \frac{t}{2} (\Delta - |x|^2)\)

\((k, a)\)-generalized Fourier transform

self-adjoint op. on \(L^2(\mathbb{R}^N, \vartheta_{k,a}(x) dx)\)

\[
\mathcal{F}_{k,a} = \begin{cases} 
    c \exp \left( \frac{\pi i}{2a} \left( |x|^{2-a} \Delta_k - |x|^a \right) \right) 
\end{cases}
\]

phase factor \(= e^{i \frac{\pi (N+2(k)+a-2)}{2a}}\)

Dunkl Laplacian

\((k, a)\)-deformation of Hermite semigroup ([11], 2009)

\[
\mathcal{I}_{k,a}(t) := \exp \frac{t}{a} \left( |x|^{2-a} \Delta_k - |x|^a \right)
\]

\(k\): multiplicity on root system \(\mathcal{R}\), \(a > 0\)
Special values of holomorphic semigroup $I_{k,a}(t)$

$(k, a)$-generalized Fourier transform $F_{k,a}$

\[ t \rightarrow \frac{\pi i}{2} \]

Holomorphic semigroup $I_{k,a}(t)$

- $a \rightarrow 2$
- $t \rightarrow \frac{\pi i}{2}$
- $k \rightarrow 0$

Dunkl transform

Hermite semigroup

Laguerre semigroup

- $a \rightarrow 1$
- $t \rightarrow \frac{\pi i}{2}$
- $k \rightarrow 0$

- $k \rightarrow 0$
- \( t \rightarrow \frac{\pi i}{2} \)

Fourier transform

Hankel transform

Geometric Analysis on Minimal Representations – p.41/49
Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$(k, a)$-generalized Fourier transform $\mathcal{F}_{k,a}$

$t \rightarrow \frac{\pi i}{2}$

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$a \rightarrow 2$

$a \rightarrow 1$

$\mathcal{I}_{k,2}(t)$

$\mathcal{I}_{k,1}(t)$

Dunkl transform

Hermite semigroup

Laguerre semigroup

$\mathcal{F}_{k,1}$

$t \rightarrow \frac{\pi i}{2}$

$k \rightarrow 0$

$k \rightarrow 0$

Fourier transform

Hankel transform

$a \rightarrow 2$

$a \rightarrow 1$
Special values of holomorphic semigroup \( I_{k,a}(t) \)

\[(k, a)\)-generalized Fourier transform \( F_{k,a} \)

\[ t \rightarrow \frac{\pi i}{2} \]

Holomorphic semigroup \( I_{k,a}(t) \)

\[ a \rightarrow 2 \]

\[ a \rightarrow 1 \]

\( I_{k,2}(t) \)

\[ t \rightarrow \frac{\pi i}{2} \]

\[ k \rightarrow 0 \]

Dunkl transform

\( k \rightarrow 0 \)

Hermite semigroup

\[ t \rightarrow \frac{\pi i}{2} \]

Laguerre semigroup

\[ k \rightarrow 0 \]

Fourier transform

\[ \leftrightarrow \text{‘unitary inversion operator’} \Rightarrow \]

the Weil representation of the metaplectic group \( Mp(N, \mathbb{R}) \)

Hankel transform

\[ \leftrightarrow \text{the minimal representation of the conformal group } O(N + 1, 2) \]
(k, a)-deformation of Hermite semigroup

\[ k = (k_\alpha): \text{ multiplicity of root system } \mathcal{R} \text{ in } \mathbb{R}^N \]

\[ \mathcal{H}_{k,a} := L^2(\mathbb{R}^N, |x|^{a-2} \prod_{\alpha \in \mathcal{R}} |\langle x, \alpha \rangle|^{k_\alpha} \, dx) \]
\[(k, a)\)-deformation of Hermite semigroup

\[ k = (k_\alpha) : \text{multiplicity of root system } \mathcal{R} \text{ in } \mathbb{R}^N \]

\[ \mathcal{H}_{k,a} := L^2(\mathbb{R}^N, |x|^{a-2} \prod_{\alpha \in \mathcal{R}} |\langle x, \alpha \rangle|^{k_\alpha} \, dx) \]

**Thm F ([11])** Assume \(a > 0\) and \(a + \sum k_\alpha + N - 2 > 0\).

\[ \mathcal{I}_{k,a}(t) := \exp \frac{t}{a} (|x|^{2-a} \Delta_k - |x|^a) \text{ is a holomorphic semigroup} \]

on \( \mathcal{H}_{k,a} \) for \( \Re t > 0 \).
\((k, a)\)-deformation of Hermite semigroup

\( k = (k_\alpha) \): multiplicity of root system \( \mathcal{R} \) in \( \mathbb{R}^N \)

\( \mathcal{H}_{k,a} := L^2(\mathbb{R}^N, |x|^{a-2} \prod_{\alpha \in \mathcal{R}} |\langle x, \alpha \rangle|^{k_\alpha} \, dx) \)

**Thm F ([11])** Assume \( a > 0 \) and \( a + \sum k_\alpha + N - 2 > 0 \).

\( \mathcal{I}_{k,a}(t) := \exp \frac{t}{a} (|x|^{2-a} \Delta_k - |x|^a) \) is a holomorphic semigroup on \( \mathcal{H}_{k,a} \) for \( \text{Re} \, t > 0 \).

\[ \mathcal{I}_{k,a}(t_1) \circ \mathcal{I}_{k,a}(t_2) = \mathcal{I}_{k,a}(t_1 + t_2) \quad \text{for Re} \, t_1, t_2 \geq 0 \]

\((\mathcal{I}_{k,a}(t)f, g)\) is holomorphic for \( \text{Re} \, t > 0 \), for \( \forall f, \forall g \)
\((k, a)\)-deformation of Hermite semigp

\(k = (k_\alpha)\): multiplicity of root system \(R\) in \(\mathbb{R}^N\)

\[\mathcal{H}_{k,a} := L^2(\mathbb{R}^N, |x|^{a-2} \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k_\alpha} \, dx)\]

**Thm F ([11])** Assume \(a > 0\) and \(a + \sum k_\alpha + N - 2 > 0\).

\[\mathcal{I}_{k,a}(t) := \exp \frac{t}{a} (|x|^{2-a} \Delta_k - |x|^a)\]

is a holomorphic semigp on \(\mathcal{H}_{k,a}\) for \(\Re t > 0\).

**Point:** The unitary rep on \(\mathcal{H}_{k,a}\) is \(SL(2, \mathbb{R})\)-admissible (i.e. discretely decomposable and finite multiplicities)
$(k, a)$-deformation of Hermite semigroup

\[ k = (k_\alpha): \text{multiplicity of root system } \mathcal{R} \text{ in } \mathbb{R}^N \]

\[ \mathcal{H}_{k,a} := L^2(\mathbb{R}^N, |x|^{a-2} \prod_{\alpha \in \mathcal{R}} |\langle x, \alpha \rangle|^{k_\alpha} \, dx) \]

**Thm F ([11])** Assume $a > 0$ and $a + \sum k_\alpha + N - 2 > 0$.

\[ \mathcal{I}_{k,a}(t) := \exp \frac{t}{a} (|x|^{2-a} \Delta_k - |x|^a) \text{ is a holomorphic semigroup} \]

on $\mathcal{H}_{k,a}$ for $\Re t > 0$.

Point: The unitary rep on $\mathcal{H}_{k,a}$ is $SL(2, \mathbb{R})$-admissible (i.e. discretely decomposable and finite multiplicities)

\[ \Rightarrow \forall \text{ Spectrum of } |x|^{2-a} \Delta_k - |x|^a \text{ is discrete and negative} \]
\[(k, a)\text{-deformation of Hermite semigroup}

\begin{align*}
k &= (k_\alpha) : \text{multiplicity of root system } \mathcal{R} \text{ in } \mathbb{R}^N \\
\mathcal{H}_{k,a} &:= L^2(\mathbb{R}^N, |x|^{a-2} \prod_{\alpha \in \mathcal{R}} |\langle x, \alpha \rangle|^{k_\alpha} \, dx)
\end{align*}

**Thm F (11)** Assume \(a > 0\) and \(a + \sum k_\alpha + N - 2 > 0\).

\[\mathcal{I}_{k,a}(t) := \exp \frac{t}{a} \left( |x|^{2-a} \Delta_k - |x|^a \right)\]
is a holomorphic semigroup on \(\mathcal{H}_{k,a}\) for \(\text{Re } t > 0\).

**Point:** The unitary rep on \(\mathcal{H}_{k,a}\) is \(SL(2, \mathbb{R})\)-admissible

(i.e. discretely decomposable and finite multiplicities)

\[\implies\text{automorphisms of the ring of operators.}\]

\[a = 1 \implies SL(2, \mathbb{Z})\text{ action on degenerate DAHA (Cherednik)}\]
$$(k, a)$$-deformation of Hermite semigroup

$$k = (k_\alpha):$$ multiplicity of root system $${\mathcal R}$$ in $${\mathbb R}^N$$

$${\mathcal H}_{k,a} := L^2({\mathbb R}^N, |x|^{a-2} \prod_{\alpha \in {\mathcal R}} |\langle x, \alpha \rangle|^{k_{\alpha}} \, dx)$$

**Thm F (11)** Assume $a > 0$ and $a + \sum k_\alpha + N - 2 > 0$.

$${\mathcal I}_{k,a}(t) := \exp \frac{t}{a} (|x|^{2-a} \Delta_k - |x|^a)$$ is a holomorphic semigroup on $${\mathcal H}_{k,a}$$ for $\Re t > 0$.

$${\mathcal F}_{k,a} := \int_{c} {\mathcal I}_{k,a}(\frac{\pi i}{2})$$

phase factor
$(k, a)$-deformation of Hermite semigp

\[ k = (k_\alpha): \text{ multiplicity of root system } \mathcal{R} \text{ in } \mathbb{R}^N \]
\[ \mathcal{H}_{k,a} := L^2(\mathbb{R}^N, |x|^{a-2} \prod_{\alpha \in \mathcal{R}} |\langle x, \alpha \rangle|^{k_\alpha} \, dx) \]

Thm E ([11]) Assume $a > 0$ and $a + \sum k_\alpha + N - 2 > 0$. $\mathcal{I}_{k,a}(t) := \exp \left( \frac{t}{a} (|x|^{2-a} \Delta_k - |x|^a) \right)$ is a holomorphic semigp on $\mathcal{H}_{k,a}$ for $\text{Re } t > 0$.

\[ \mathcal{F}_{k,a} := c \mathcal{I}_{k,a}(\frac{\pi i}{2}) \]

phase factor
\[ e^{i \frac{\pi (N+2 \langle k \rangle + a - 2)}{2a}} \]
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \mathcal{I}_{k,a}(\frac{\pi i}{2})$$
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \mathcal{I}_{k,a}(\frac{\pi i}{2}) = c \exp \left( \frac{\pi i}{2a} \left( |x|^{2-a} \Delta_k - |x|^a \right) \right)$$

**Thm G 1)** $\mathcal{F}_{k,a}$ is a unitary operator
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \mathcal{I}_{k,a} \left( \frac{\pi i}{2} \right) = c \exp \left( \frac{\pi i}{2a} (|x|^{2-a} \Delta_k - |x|^a) \right)$$

**Thm G**

1) $\mathcal{F}_{k,a}$ is a unitary operator

2) $\mathcal{F}_{0,2} =$ Fourier transform on $\mathbb{R}^N$

3) $\mathcal{F}_{k,a}$ is of finite order $\iff a \in \mathbb{Q}$

4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a} \Delta_k$
Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c \mathcal{I}_{k,a}(\frac{\pi i}{2}) = c \exp\left(\frac{\pi i}{2a}(|x|^{2-a} \Delta_k - |x|^a)\right)$$

**Thm G**  
1) $\mathcal{F}_{k,a}$ is a unitary operator
2) $\mathcal{F}_{0,2}$ = Fourier transform on $\mathbb{R}^N$  
   $F_{k,a}$ = Dunkl transform on $\mathbb{R}^N$  
   $\mathcal{F}_{0,1}$ = Hankel transform on $L^2(\mathbb{R}_+)$
3) $\mathcal{F}_{k,a}$ is of finite order $\iff a \in \mathbb{Q}$
4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a} \Delta_k$

$\implies$ generalization of classical identities such as Hecke identity, Bochner identity, Parseval–Plancherel formulas, Weber’s second exponential integral, etc.
Application to special functions

Minimal reps (↔ group)
Application to special functions

Minimal reps (≤ group)
≈ Maximal symmetries (≤ space)

⇒
Application to special functions

Minimal reps (≅ group) 
≈ Maximal symmetries (≅ space)

⇒ ‘Special functions’, ‘orthogonal polynomials’ associated to 4th order differential eqn $[2a, 2b]$
Application to special functions

Minimal reps (↔ group)
≈ Maximal symmetries (↔ space)

⇒ ‘Special functions’, ‘orthogonal polynomials’
associated to 4th order differential eqn \([2a, 2b]\)
with 4 parameters

\[
\begin{pmatrix}
p, q \\
l, m
\end{pmatrix}
\]

dimension branching laws (multiplicity-free)

Special case \(q = 1\): Laguerre polynomials \(4 = 2 \times 2\)
Heisenberg-type inequality

\[ \| | x |^{\alpha} f(x) \|_k \| \| \xi |^{\alpha} (\mathcal{F}_k, a f)(\xi) \|_k \geq \frac{2 \langle k \rangle + N + a - 2}{2} \| f(x) \|_k^2 \]

\( k \equiv 0, \ a = 2 \) \hspace{1cm} \cdots \ \text{Weyl–Pauli–Heisenberg inequality for Fourier transform } \mathcal{F}_{\mathbb{R}^N}

\( k: \text{ general}, \ a = 2 \) \hspace{1cm} \cdots \ \text{Heisenberg inequality for Dunkl transform } \mathcal{D}_k \ (\text{Rösler, Shimeno})

\( k \equiv 0, \ a = 1, \ N = 1 \) \hspace{1cm} \cdots \ \text{Heisenberg inequality for Hankel transform}
Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$(k, a)$-generalized Fourier transform $\mathcal{F}_{k,a}$

$t \rightarrow \frac{\pi i}{2}$

(a → 2)

Holomorphic semigroup $\mathcal{I}_{k,a}(t)$

$a \rightarrow 1$

$\mathcal{I}_{k,2}(t)$

$t \rightarrow \frac{\pi i}{2}$

$k \rightarrow 0$

(a → 1)

$\mathcal{I}_{k,1}(t)$

$t \rightarrow \frac{\pi i}{2}$

$k \rightarrow 0$

Dunkl transform

Hermite semigroup

Fouier transform

Laguerre semigroup

Hankel transform

$\mathcal{F}_{k,1}$

$\leftrightarrow$ ‘unitary inversion operator’ $\Rightarrow$

the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of the conformal group $O(N + 1, 2)$
Hidden symmetries in $L^2(\mathbb{R}^N, \mathcal{V}_{k,a}(x) \, dx)$

Coxeter group

$k \to 0$

$(k, a : \text{general})$

\[ \mathfrak{c} \times \widetilde{SL(2, \mathbb{R})} \]

$O(N) \times \widetilde{SL(2, \mathbb{R})}$

\[ O(N + 1, 2) \sim \]

$a \to 1$

$a \to 2$

\[ Mp(N, \mathbb{R}) \]
Bessel functions

\[ J_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j \left( \frac{z}{2} \right)^{2j}}{j! \Gamma(j + \nu + 1)} \]

\[ I_\nu(z) := e^{-\frac{\sqrt{-1} \nu \pi}{2}} J_\nu \left( e^{\frac{\sqrt{-1} \pi}{2}} z \right) \]

\[ Y_\nu(z) := \frac{J_\nu(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi} \] (second kind)

\[ K_\nu(z) := \frac{\pi}{2 \sin \nu \pi} \left( I_{-\nu}(z) - I_\nu(z) \right) \] (third kind)
Geometric analysis on minimal reps of $O(p, q)$

[1] Laguerre semigroup and Dunkl operators · · ·

[2] Special functions associated to a fourth order differential equation · · ·


[4] Schrödinger model of minimal rep. · · ·

[5] Inversion and holomorphic extension · · ·

[6] Analysis on minimal representations · · ·

Collaborated with

S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers and B. Ørsted