Global Geometry and Analysis on Locally Symmetric Spaces

beyond the Riemannian case

Differential Equations and Symmetric Spaces

Conference in honor of Toshio Oshima's 60th birthday

Tokyo, 15 January 2009

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Compact-like actions

compact groups

(very nice behaviors)

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Non-compact Lie groups

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Non-compact Lie groups

occasionally behave nicely

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Compact-like linear/non-linear actions

$\boxed{\begin{array}{l}\mathcal{H}\text{: Hilbert space} \\ L \curvearrowright \mathcal{H}\end{array}}$

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Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space
$L \curvearrowright \mathcal{H}$

L: compact $\Longrightarrow$ unitarizable

Unitarizability might be interpreted as one of "compact-like properties".

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$L \curvearrowright M$ proper actions

i.e. $L \times M \to M \times M, \quad (g, x) \mapsto (x, g \cdot x)$ is proper

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$\mathcal{H} = L^2(G/H), L^2(G/\Gamma)$: Hilbert space

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Decomposition into irreducible reps

Two important cases

$$\underset{\text{subgroup}}{G' \subset G}$$

1) Induction

2) Restriction

Decomposition into irreducible reps

Two important cases

$$G' \underset{\text{subgroup}}{\subset} G$$

1) Induction: $G' \uparrow G$

Plancherel Formula

(e.g. Analysis on homo. space G/G')

2) Restriction: $G \downarrow G'$

Branching Law

(e.g. Tensor product, . . .)

Special restrictions $\Longrightarrow L^2(G/H)$

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Many other restrictions $\pi|_G$ cannot be reduced to $L^2(G/H)$

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Question (to find "nicest settings")

- When does the restriction $\pi|_{G'}$ decompose discretely?
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Say the restriction $\pi|_{G'}$ is G'-admissible if both are fulfilled.

Two closed cones

$$G \quad \supset \quad K \quad \supset \quad T$$

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$$\widehat{G} \ni \pi \rightsquigarrow \quad \mathrm{AS}_K(\pi) \quad \text{asymptotic } K\text{-support}$$

$$\cap \quad \text{(Kashiwara–Vergne)}$$

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 & \sqrt{-1}\mathfrak{t}^* & \\
 & \cup & \\
G \supset G' \rightsquigarrow & \mu(T^*(K/K')) & \text{momentum image} \\
\cup \quad\ \ \cup & & \mu : T^*(K/K') \to \sqrt{-1}\mathfrak{t}^* \\
K \supset K' & &
\end{array}$$

Criterion of admissible restriction

Theorem A (Criterion) (K– Ann Math '98, Progr Math '05)

Let $G' \underset{\text{reductive}/\mathbb{R}}{\subset} G$ and $\pi \in \widehat{G}$. If

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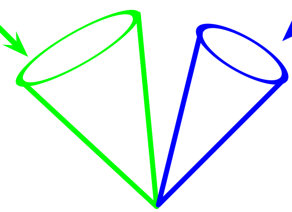

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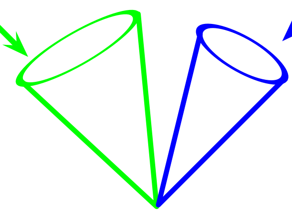

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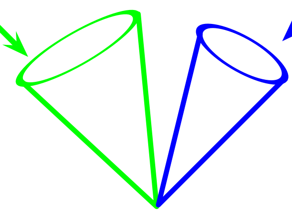

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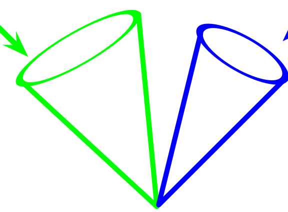

$\mathbb{R}^n$

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$\cdots$ compact-like linear actions

Special cases of Thm A

Ex.1 $\quad \mu(T^*(K/K')) = \{0\} \iff K = K' \iff G' \supset K$

$\Longrightarrow$ Harish-Chandra's admissibility thm

Ex.2 $\mathrm{AS}_K(\pi) = \{0\} \iff \dim \pi < \infty$

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$\implies \mu(T^*(K/K')) =$ positive Weyl chamber

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Ex.5 $\pi = A_{\mathfrak{q}}(\lambda)$ (e.g. discrete series)
$\Longrightarrow \mathrm{AS}_K(\pi) \subset \mathbb{R}_+$-span of $\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t})$
$(\mathfrak{q} = \mathfrak{l} + \mathfrak{u},\ \mathfrak{g} = \mathfrak{k} + \mathfrak{p})$

Criterion for compact-like actions

Some further developments in this framework
(compact-like branching laws)

by D. Gross–N. Wallach, S.-T. Lee–H. Loke,
M. Duflo–J. Vargas, B. Ørsted–B. Speh,
J. S. Huang–D. Vogan, K–T. Oda, . . .

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G : reductive Lie group $\supset K$

$\cup \qquad\qquad \cup$

$G' \qquad\qquad \supset K'$

$\mu : \quad T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*$ momentum map

<u>Thm A</u> $\pi \in \widehat{G}$

$$\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$$

$\Longrightarrow \pi|_{G'}$ is discrete decomposable.

$L \subset G \supset H$

$\nu : G \to \mathfrak{a}$ (Cartan projection)

Thm B (proper action)

$L \pitchfork H$ in $G \Longleftrightarrow \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$

Proper action

$$\begin{array}{ccc} & \text{action} & \\ L & \curvearrowright & X \\ \text{top. gp} & & \text{top. sp (locally compact)} \end{array}$$

$$\begin{array}{rcl} X & & L \\ \text{subset} \cup & \rightsquigarrow & \cup \\ S & & L_S := \{\gamma \in L : \gamma S \cap S \neq \phi\} \end{array}$$

$S = \{p\} \implies L_S =$ stabilizer of p

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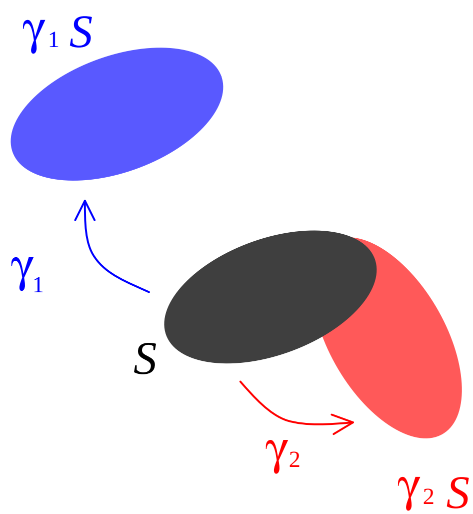

$$\Longrightarrow \quad \begin{array}{l} \gamma_1 \not\in L_S \\ \gamma_2 \in L_S \end{array}$$

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Def. $L \curvearrowright X$ is proper $\iff$ L_S is compact ($^\forall S$: compact)

$L \curvearrowright X$ is free $\iff$ $\#L_{\{p\}} = 1 \ (\forall p \in X)$

Delicate examples

$$L \curvearrowright X$$

$\boxed{\begin{array}{ll} \text{(A)} & \text{free action} \overset{?}{\Longrightarrow} \text{proper action} \\ \text{(B)} & \text{all orbits are closed} \overset{?}{\Longrightarrow} L\backslash X \text{ Hausdorff} \end{array}}$

Delicate examples

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(A) free action $\not\Longrightarrow$ proper action

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Counterexamples to (A) & (B) even for

$$L \simeq \mathbb{R}^k, \ X = G/H \quad \text{where} \quad L \subset \underset{\text{Lie groups}}{G} \supset H$$

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<u>Ex.</u> ($G = SL(2,\mathbb{R})$)

$L = \mathbb{R} \curvearrowright X = \mathbb{R}^2 \setminus \{0\}$ (Lorentz isometry)

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<u>Ex.</u> ($G =$ 1-conn. nilpotent Lie gp)

$$L = \mathbb{R}^2 \curvearrowright X = \mathbb{R}^5$$ (nilmanifolds)

(Yoshino 2004, counterexample to Lipsman's conjecture)

proper + discrete = properly discont.

$$\boxed{\begin{array}{c}\text{properly discont. action}\\ \parallel \\ \text{proper action}\\ + \\ \text{group is discrete}\end{array}}$$

proper + discrete = properly discont.

action | properly discont. action

$\|$

action | proper action

$+$

group is discrete

Criterion for discontinuous groups

Setting

$$L \quad \subset \; G \; \supset \quad H$$

discrete subgp — closed subgp

General Problem

Find effective methods to determine whether $L \curvearrowright G/H$ is properly discont.

Criterion for discontinuous groups

Setting

$L \subset G \supset H$

~~discrete~~ subgp closed subgp

General Problem

Find effective methods to determine whether

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Idea: forget even that L and H are group

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Def. (K–)

1) $L \pitchfork H \iff \overline{L \cap SHS}$ is compact
for $^\forall$compact $S \subset G$

2) $L \sim H \iff {}^\exists$ compact $S \subset G$
s.t. $L \subset SHS$ and $H \subset SLS$.

H

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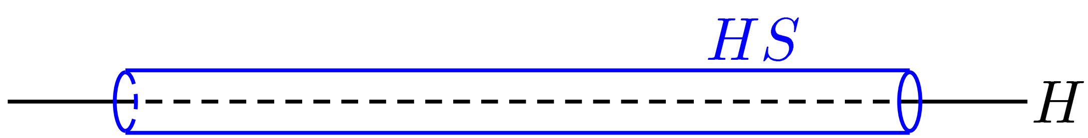

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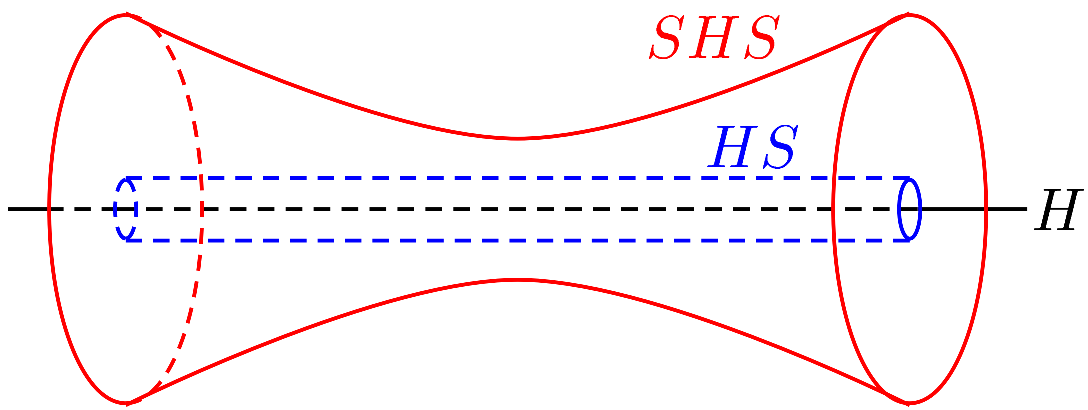

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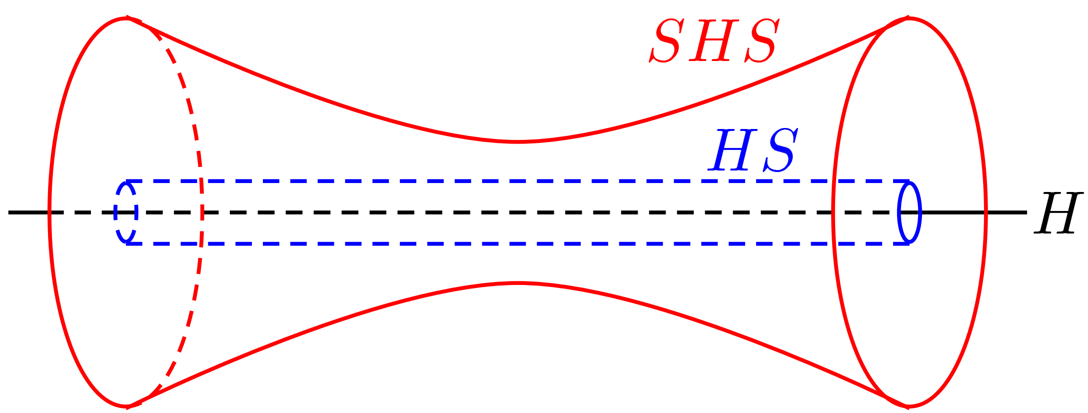

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E.g. $G = \mathbb{R}^n$; L, H subspaces

$$L \pitchfork H \iff L \cap H = \{0\}.$$

$$L \sim H \iff L = H.$$

$\pitchfork$ and $\sim$

$$L \quad \subset \quad G \quad \supset \quad H$$

Forget even that L and H are group

1) $L \pitchfork H \iff$ generalization of proper actions
2) $L \sim H \iff$ economy in considering

Meaning of $\pitchfork$:

$$\boxed{L \pitchfork H \iff L \curvearrowright G/H \text{ proper action}}$$

for closed subgroups L and H

$\sim$ provides economies in considering $\pitchfork$

$$H \sim H' \implies \boxed{H \pitchfork L \iff H' \pitchfork L}$$

$\pitchfork$ and $\sim$

$$L \quad \subset \quad G \quad \supset \quad H$$

Forget even that L and H are group

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$$H \sim H' \Longrightarrow \boxed{H \pitchfork L \iff H' \pitchfork L}$$

Criterion for $\pitchfork$ and $\sim$

G: real reductive Lie group
$G = K \exp(\mathfrak{a}) K$: Cartan decomposition
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E.g. $\nu: \quad GL(n,\mathbb{R}) \to \mathbb{R}^n$

$$g \quad \mapsto \frac{1}{2}(\log\lambda_1, \cdots, \log\lambda_n)$$

Here, $\lambda_1 \geq \cdots \geq \lambda_n \, (> 0)$ are the eigenvalues of ${}^t gg$.

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Thm B (K– , Benoist)

1) $L \sim H$ in G $\iff$ $\nu(L) \sim \nu(H)$ in $\mathfrak{a}$.

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Special cases include

(1)'s $\Rightarrow$: Uniform bounds on errors in eigenvalues when a matrix is perturbed.

(2)'s $\Leftrightarrow$: Criterion for properly discont. actions.

Criterion for compact-like actions

G : reductive Lie group $\supset K$

$\cup \qquad\qquad \cup$

$G' \qquad\qquad \supset K'$

$\mu: \quad T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*$ momentum map

<u>Thm A</u> $\pi \in \widehat{G}$, $G' \subset G$

$$\mu(T^*(K/K')) \cap \mathrm{AS}_K(\pi) = \{0\}$$

$\Longrightarrow \pi|_{G'}$ is discrete decomposable.

Criterion for compact-like actions

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$$L \subset G \supset H$$

$\nu : G \to \mathfrak{a}$ (Cartan projection)

<ins>Thm B</ins> (proper action)

$L \pitchfork H$ in $G \Longleftrightarrow \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$

Compact-like linear/non-linear actions

$\mathcal{H}$: Hilbert space

$L \curvearrowright \mathcal{H}$ discrete decomposability

$\cdots$ L behaves nicely in $U(\mathcal{H})$ (unitary operators)

as if it were a compact group

M: topological space

$L \curvearrowright M$ proper actions

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$\mathcal{H} = L^2(G/H), L^2(G/\Gamma)$: Hilbert space

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'nice behavior' (topological action)

Compact-like non-linear/linear actions

$$L \subset G \supset H$$

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'nice behavior' (topological action)

$$\Downarrow ?$$

$\begin{cases} H \curvearrowright L^2(G/L) & \text{(Margulis, Oh)} \\ \overline{L} \curvearrowright L^2(G/H) & \text{(K–)} \end{cases}$

'nice behavior' (representation theory)

Interacting example

Ex. (K– 1988) $(G, L) = (SO(4,2),\ SO(4,1))$

π: discrete series of G with GK-dim 5

(quarternionic discrete series)

$\Longrightarrow \pi|_L$ is L-admissible

Idea: Tessellation of pseudo-Riemannian mfd X

$$X = SO(4,2)/U(2,1) \quad (\underset{\text{open}}{\subset} \mathbb{P}^3\mathbb{C})$$

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$$\begin{tikzcd}
\Gamma \arrow[dr, hook] \arrow[dd, phantom, "\cap"', "\text{lattice}\ "' near start] & & X \\
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L \arrow[ur, hook] & &
\end{tikzcd}$$

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discont. gp

$\Gamma \curvearrowright X$

lattice $\cap \quad G \curvearrowright \boxed{\pi \subset L^2(X)}$ (discrete series)

L

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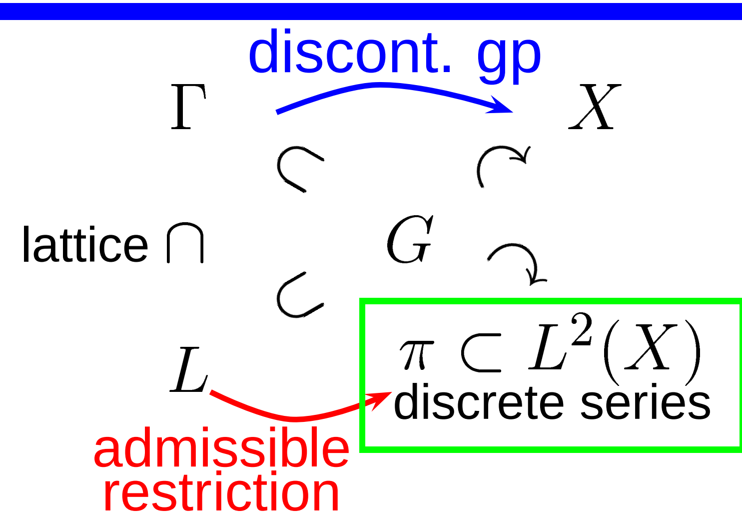

Interacting examples

Pseudo-Riemannian manifold X

$$X = G/H = SO(4,2)/U(2,1) \quad (\underset{\text{open}}{\subset} \mathbb{P}^3\mathbb{C})$$

Interacting examples

$$\boxed{\begin{array}{c}\text{Pseudo-Riemannian manifold } X \\ X = G/H = SO(4,2)/U(2,1) \quad (\underset{\text{open}}{\subset} \mathbb{P}^3\mathbb{C})\end{array}}$$

- Cocompact discontinuous group for $X = G/H$

$$\boxed{\underline{\text{Thm}} \quad G/H \text{ admits a cocompact, discontinuous gp } \Gamma.}$$

Proof. Take $\Gamma \underset{\text{cocompact}}{\subset} L = SO(4,1)$. ∎

Interacting examples

Pseudo-Riemannian manifold X

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- Cocompact discontinuous group for $X = G/H$

<u>Thm</u> G/H admits a cocompact, discontinuous gp Γ.

Proof. Take $\Gamma \underset{\text{cocompact}}{\subset} L = SO(4,1)$. ∎

- Function space on $X = G/H$

<u>Thm</u> If $\pi \in \widehat{G}$ is realized in $L^2(G/H)$, then $\pi|_L$ decomposes discretely.

Compact-like linear/non-linear actions

$\mathcal{H} = L^2(G/H), L^2(G/\Gamma)$: Hilbert space

$L \curvearrowright \mathcal{H}$ discrete decomposability

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proper $+$ discrete $=$ properly discont.

action	properly discont. action
	$\|\|$
action	proper action
	$+$
	group is discrete

Local to global

$$\Gamma \subset G \supset H$$

Knowledge of discrete subgp Γ

$\Bigg\Downarrow \Leftarrow$ criterion of $\pitchfork$ (Thm B)

Knowledge of Γ-actions on G/H

Local to global

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$$\boxed{\begin{array}{l}\text{Knowledge of discrete subgp } \Gamma \\ \qquad \Downarrow \Leftarrow \text{criterion of } \pitchfork \text{ (Thm B)} \\ \text{Knowledge of } \Gamma\text{-actions on } G/H\end{array}}$$

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E.g. existence problem of cocompact discont. gp

rigidity / deformation

$\ldots$

Local to global

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$$\underbrace{G/H}_{\text{local geometric structure}} \qquad \to \qquad \underbrace{\Gamma\backslash}_{\text{global}} G/H$$

Rigidity/deformation

- Positivity of ‘metric’ is crucial?

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$$\Gamma \underset{\text{lattice}}{\subset} L$$

Rigidity/deformation

- Positivity of 'metric' is crucial?

$$
\begin{array}{ccc}
\Gamma & \underset{\text{lattice}}{\subset} & L \\
 & & \cap \\
 & & G
\end{array}
$$

Rigidity/deformation

- Positivity of ‘metric’ is crucial?

$$\begin{array}{ccccc} \Gamma & \underset{\text{lattice}}{\subset} & L & & \\ & & \cap & & \\ & & G & \overset{\text{isometry}}{\curvearrowright} & \overset{\text{pseudo-Riemannian}}{X} \end{array}$$

Rigidity/deformation

- Positivity of ‘metric’ is crucial?

$$\begin{array}{ccccc} & \text{rigid} & & & \\ \Gamma & \subset & L & & \\ & \text{lattice} & & & \\ \text{deform} & \searrow & \cap & & \text{pseudo-Riemannian} \\ & & G & \overset{\text{isometry}}{\curvearrowright} & X \end{array}$$

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\end{array}$$

proper discontinuity
may break down

Rigidity, stability, and deformation

$$G \curvearrowright X$$

Rigidity, stability, and deformation

$$\begin{tikzcd}[column sep=small]
\Gamma \arrow[dr, phantom, sloped, "\subset"] \arrow[rrd, bend left, "\text{properly discont.}"] & & \\
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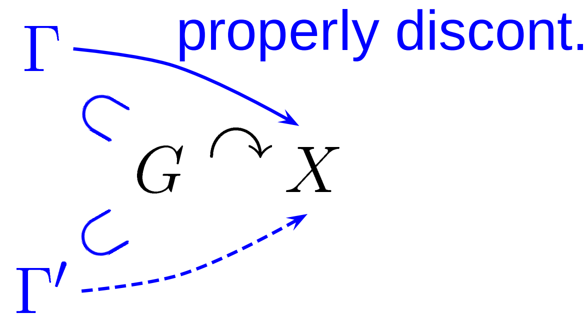
\end{tikzcd}$$

Rigidity, stability, and deformation

$$\begin{tikzcd}
\Gamma \arrow[dr, phantom, sloped, "\subset"] \arrow[drr, bend left, "\text{properly discont.}"] & & \\
& G \arrow[r, phantom, "\curvearrowright"] & X \\
\Gamma' \arrow[ur, phantom, sloped, "\subset"] & &
\end{tikzcd}$$

Suppose Γ' is ‘close to’ Γ

Rigidity, stability, and deformation



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Rigidity, stability, and deformation

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\Gamma' \arrow[ru, phantom, "\subset" sloped] \arrow[rrru, dashed, bend right] & & &
\end{tikzcd}$$

Suppose Γ' is ‘close to’ Γ

$$\boxed{\text{(R) (local rigidity)} \quad \Gamma' = g\Gamma g^{-1} \ ({}^{\exists}g \in G)}$$

Rigidity, stability, and deformation

$$\begin{tikzcd}
\Gamma \arrow[rd, phantom, sloped, "\subset"] \arrow[rrd, bend left, "\text{properly discont.}"] & & \\
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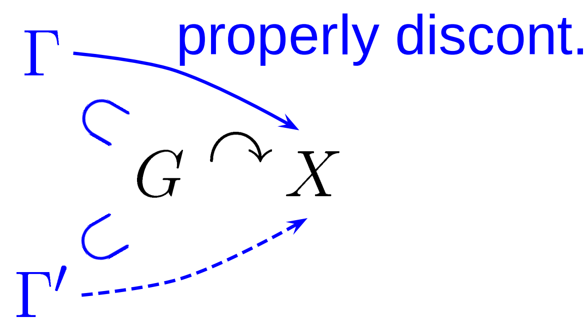
\Gamma' \arrow[ru, phantom, sloped, "\supset"] \arrow[rru, bend right, dashed] & &
\end{tikzcd}$$

Suppose Γ' is ‘close to’ Γ

(R) (local rigidity)	$\Gamma' = g\Gamma g^{-1}$ $({}^{\exists}g \in G)$
(S) (stability)	$\Gamma' \curvearrowright X$ properly discont.

Rigidity, stability, and deformation



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In general,

Rigidity, stability, and deformation

$$\begin{tikzcd}[column sep=small] \Gamma \arrow[dr, phantom, sloped, "\subset"] \arrow[drr, bend left, "\text{properly discont.}"] & & \\ & G \curvearrowright & X \\ \Gamma' \arrow[ur, phantom, sloped, "\subset"] \arrow[urr, bend right, dashed] & & \end{tikzcd}$$

Suppose Γ' is ‘close to’ Γ

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Rigidity, stability, and deformation

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In general,

- (R) $\Rightarrow$ (S).
- (S) may fail (so does (R)).

Local rigidity and deformation

$\Gamma \subset G \curvearrowright X = G/H$ cocompact, discontinuous gp

<u>General Problem</u>

1. When does local rigidity (R) fail?
2. Does stability (S) still hold?

Local rigidity and deformation

$\Gamma \subset G \curvearrowright X = G/H$ cocompact, discontinuous gp

General Problem

1. When does local rigidity (R) fail?
2. Does stability (S) still hold?

Point: for non-compact H

1. (good aspect) There may be large room for deformation of Γ in G.
2. (bad aspect) Properly discontinuity may fail under deformation.

Rigidity Theorem

$$\boxed{① \qquad G/\{e\} \quad \simeq \quad (G \times G)/\Delta G \qquad ②}$$

$\Gamma \subset G$ simple Lie gp

Rigidity Theorem

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Fact (Selberg–Weil's local rigidigy, 1964)
$^{\exists}$uniform lattice Γ admitting continuous deformations for ①
$\iff G \approx SL(2, \mathbb{R})$ (loc. isom).

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$\iff$ trivial representation is not isolated in the unitary dual (not having Kazhdan's property (T))

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Local rigidity (R) may fail for pseudo-Riemannian symm. sp. even for high and irreducible case!

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Method: use the criterion of $\pitchfork$
($\Rightarrow$ criterion for properly discontinuous actions)

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$\cdots$ Solution to Goldman's stability conjecture (1985), 3-dim case

Existence problem of compact quotients

$$G \supset H$$

Existence problem of compact quotients

$$(\Gamma \subset)\ G \supset H$$

General Problem For which pair (G, H) does there exist a discrete subgroup Γ s.t.

- $\Gamma \curvearrowright G/H$ properly discont, freely,
- $\Gamma\backslash G/H$ is compact (or of finite volume) ?

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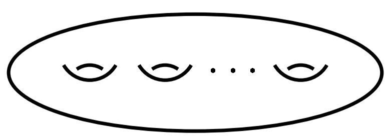
$G/H = SL(2,\mathbb{R})/SO(2)$ (Riemannian symm. sp.)

$\Gamma\backslash G/H \simeq$  $(g \geq 2)$

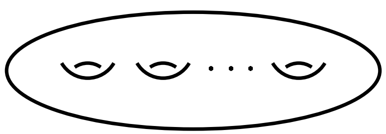
Existence problem of compact quotients

$$(\Gamma \subset)\ G \supset H$$

General Problem For which pair (G, H) does there exist a discrete subgroup Γ s.t.

- $\Gamma \curvearrowright G/H$ properly discont, freely,
- $\Gamma \backslash G/H$ is compact (or of finite volume) ?

$G/H = SL(2,\mathbb{R})/SO(2)$ (Riemannian symm. sp.)

$\Gamma \backslash G/H \simeq$  $(g \geq 2)$

Consider the case when H is non-compact.

Space forms (definition)

(M, g) : pseudo-Riemannian mfd,
geodesically complete

Def. (M, g) is a space form
$\iff$ sectional curvature κ is constant

Space forms (examples)

Space form $\cdots$ $\begin{cases} \text{Signature } (p,q) \text{ of pseudo-Riemannian metric } g \\ \text{Curvature } \kappa \in \{+, 0, -\} \end{cases}$

<u>E.g.</u>	$q = 0$ (Riemannian mfd)		
	sphere S^n	$\mathbb{R}^n$	hyperbolic sp
	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

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<u>E.g.</u> $q = 1$ (Lorentz mfd)

de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

Space form problem

Space form problem for pseudo-Riemannian mfds

Local Assumption

signature (p, q), curvature $\kappa \in \{+, 0, -\}$

$$\Downarrow$$

Global Results

- Do compact quotients exist?
- What groups can arise as their fundamental groups?

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  Is the universe closed?

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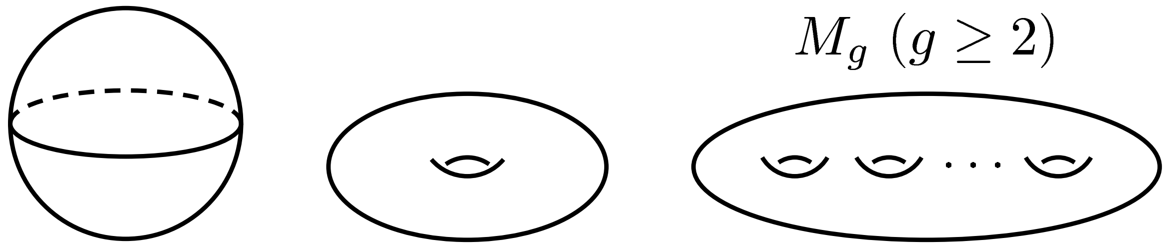
2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2, 0)$)



curvature	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

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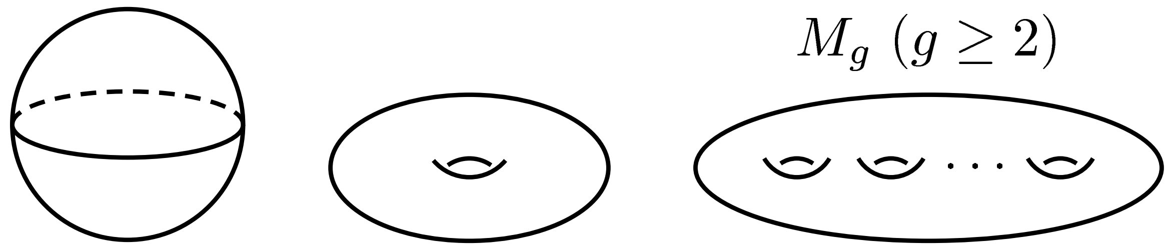
M_g $(g \geq 2)$



curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Lorentz case ($\Longleftrightarrow$ signature $(1, 1)$)

compact forms do NOT exist

for $\kappa > 0$ and $\kappa < 0$

Compact space forms ($\kappa < 0$)

Geometry $\iff$ Group theoretic formulation

Compact space forms exist
for $\kappa < 0$ and signature (p, q)

$\iff$ Cocompact discont. gps exist
for symmetric sp $O(p, q+1)/O(p, q)$

Compact space forms ($\kappa < 0$)

- Riemannian case $\cdots$ hyperbolic space

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Exist by $\underbrace{\text{Siegel, Borel}}_{\text{arithmetic}}$, $\underbrace{\text{Vinberg, Gromov–Piateski-Shapiro}}_{\text{non-arithmetic}}$ $\cdots$

Space form conjecture $\kappa < 0$

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$\Longleftarrow$ True (Proved (1950–2005))

①② (Riemmanian)

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$\Longrightarrow$ Partial answers:

$q = 1$, $p \le q$, or pq is odd

Hirzebruch's proportionality principle (K–Ono)

Methods

Understanding proper actions ($\pitchfork, \sim$),
cohomology of discrete groups

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Construction of lattice

- Find a connected subgp L that acts on G/H properly and cocompactly.
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Obstruction of lattice

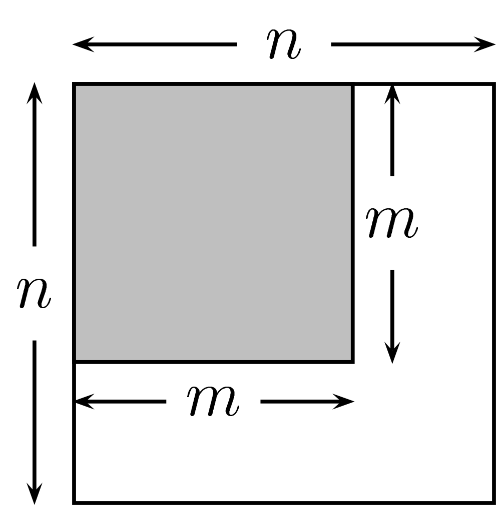
- Characteristic classes
- Comparison theorem: $\Gamma \curvearrowright G/H \iff \Gamma \curvearrowright G/H'$

Compact quotients for $SL(n)/SL(m)$

Problem: Does there exist compact Hausdorff quotients of

$$SL(n,\mathbb{F})/SL(m,\mathbb{F}) \quad (n > m,\ \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$$

by discrete subgps of $SL(n,\mathbb{F})$?



$SL(n)/SL(m)$ **case**

Conjecture $SL(n)/SL(m)$ $(n > m > 1)$
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K–	criterion of proper actions	$\frac{n}{3} > [\frac{m+1}{2}]$
Zimmer	orbit closure thm (Ratner)	$n > 2m$
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Margulis	unitary representation	$(n \geq 5, m = 2)$
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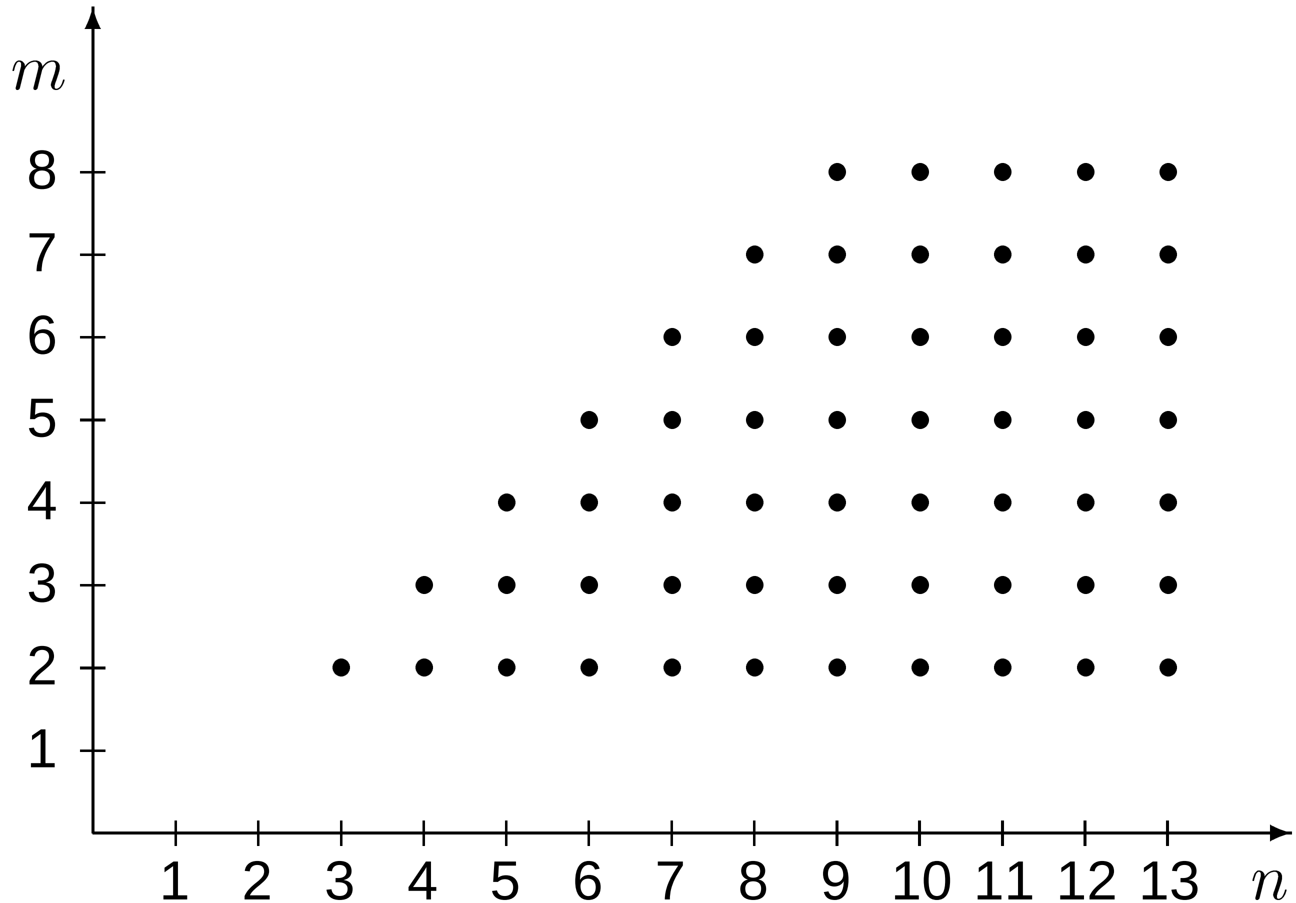

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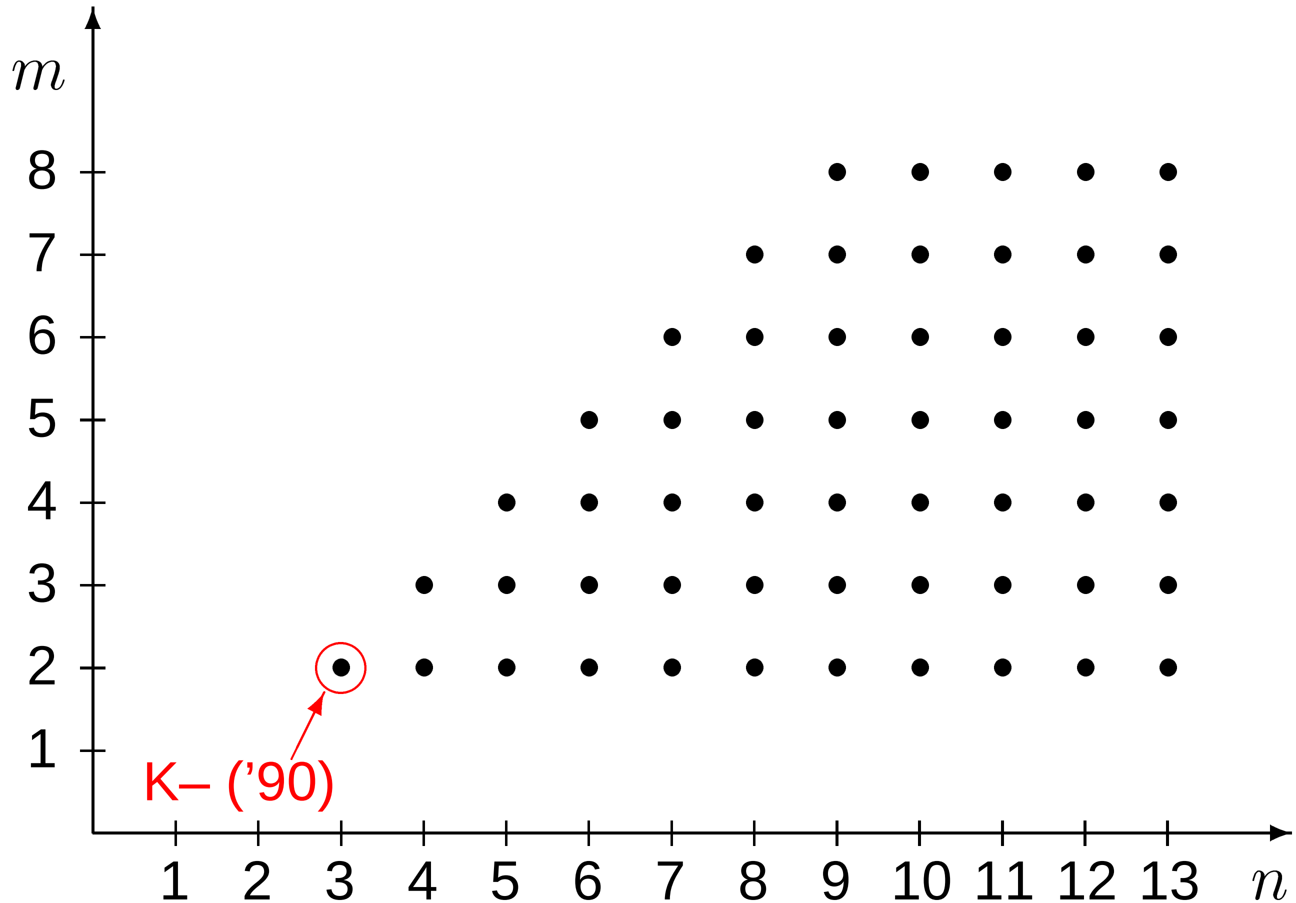

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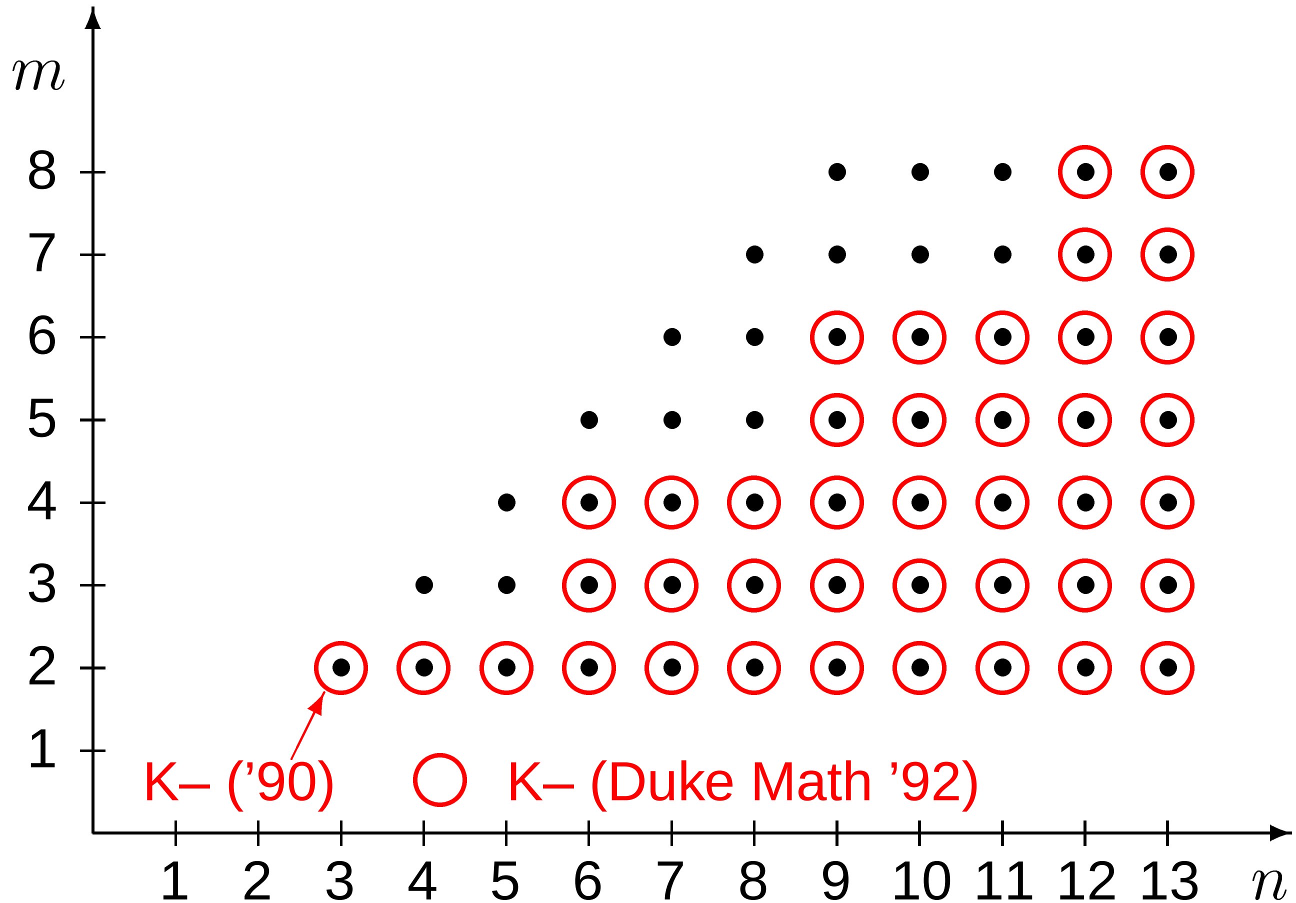

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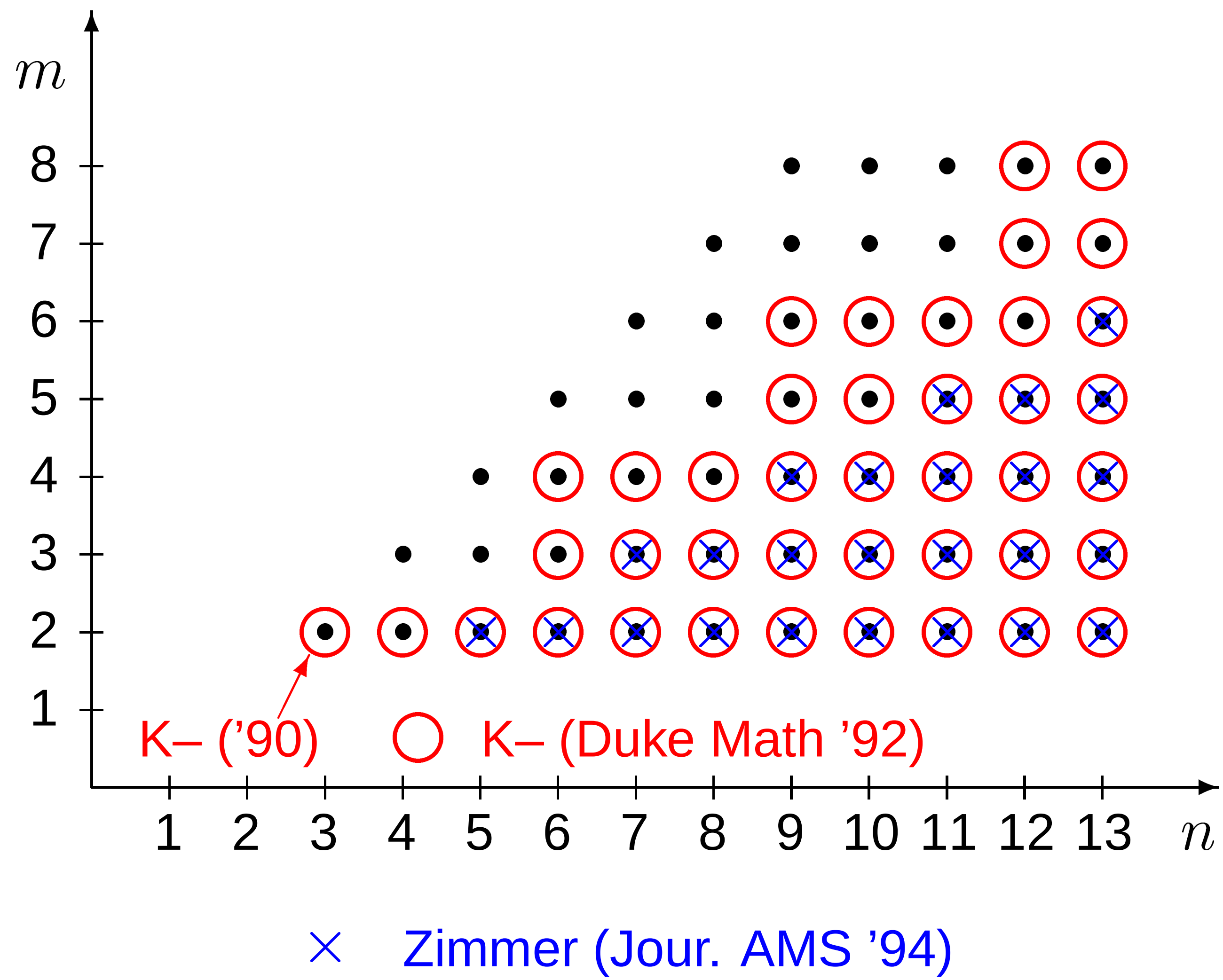

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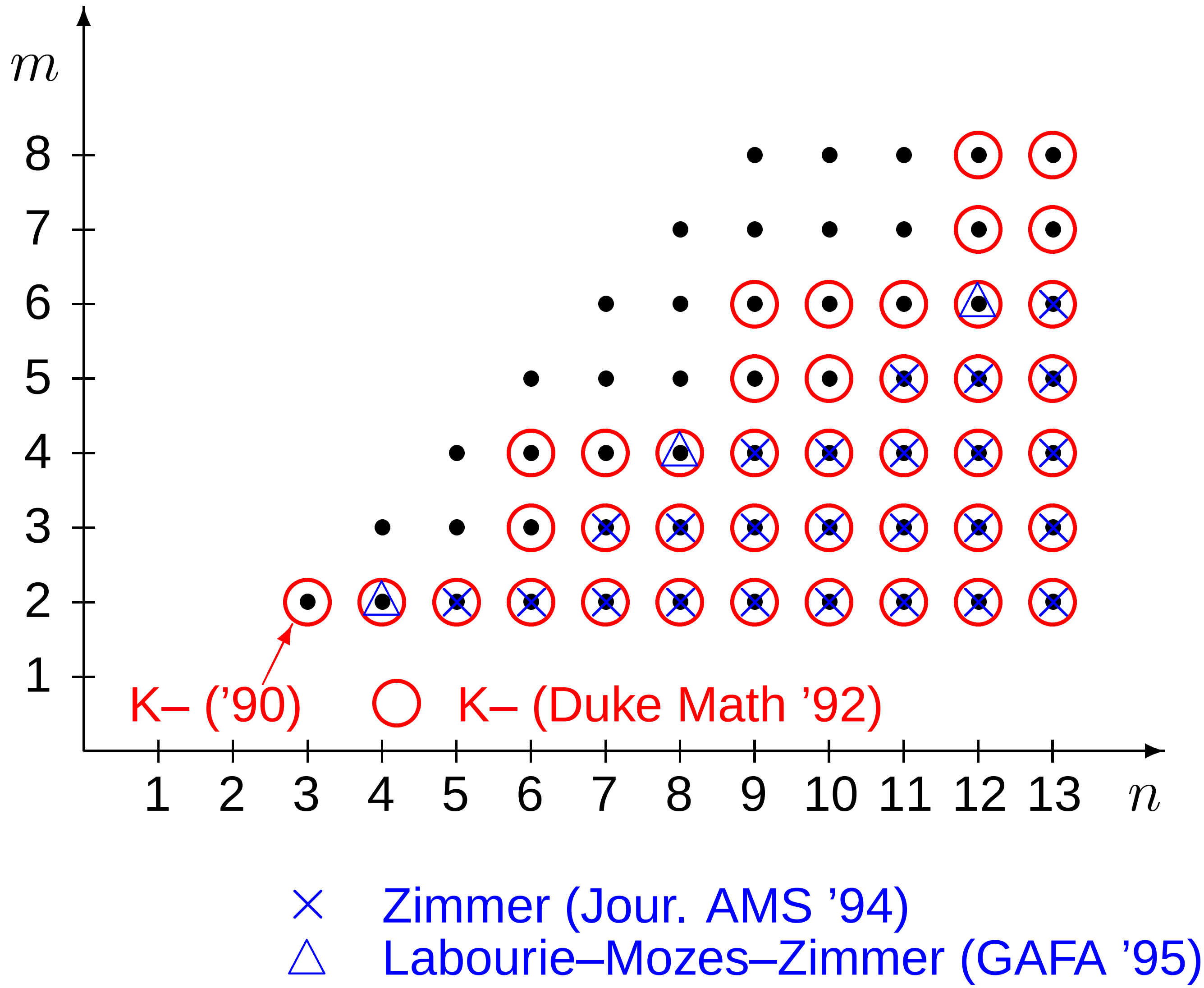

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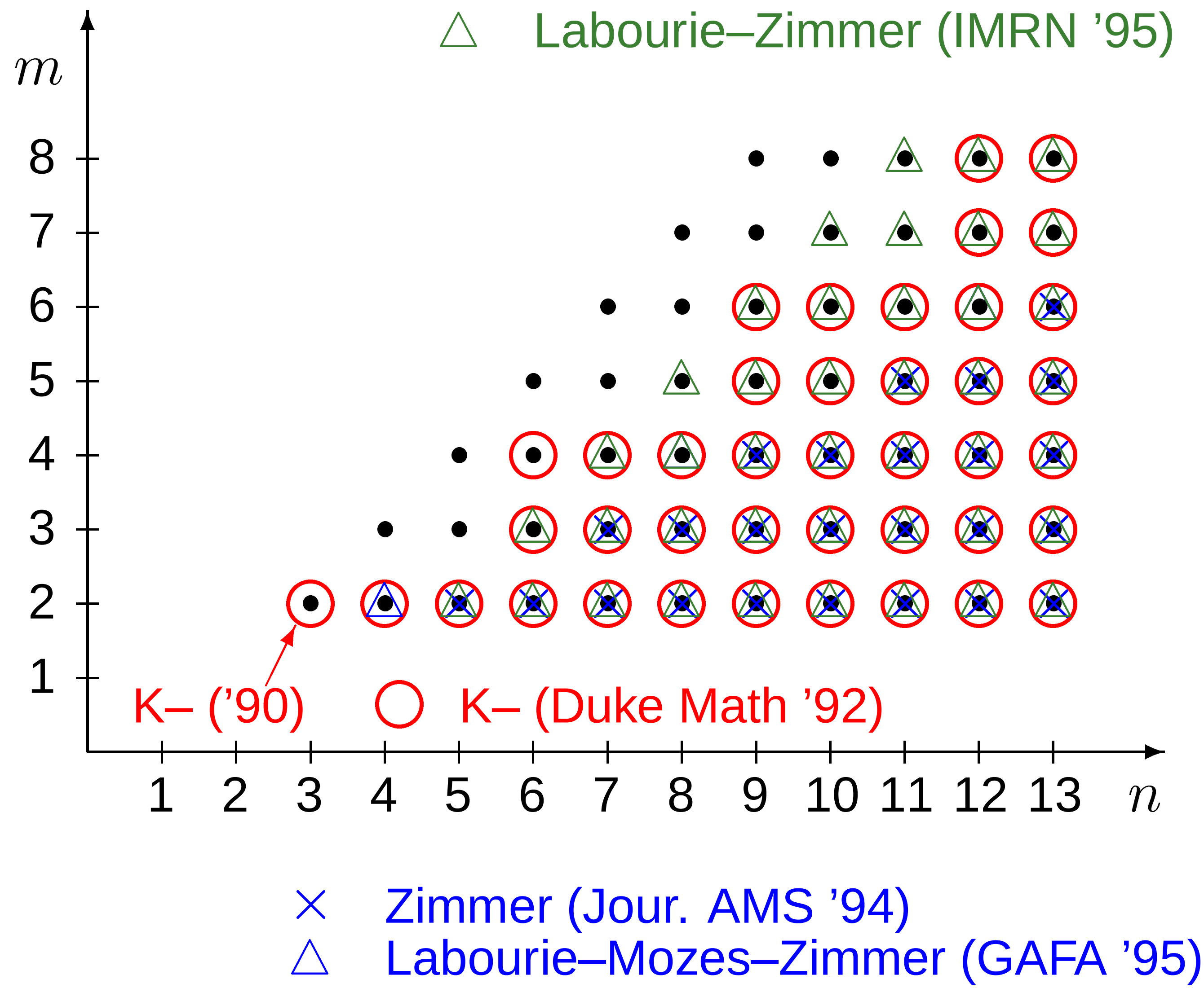

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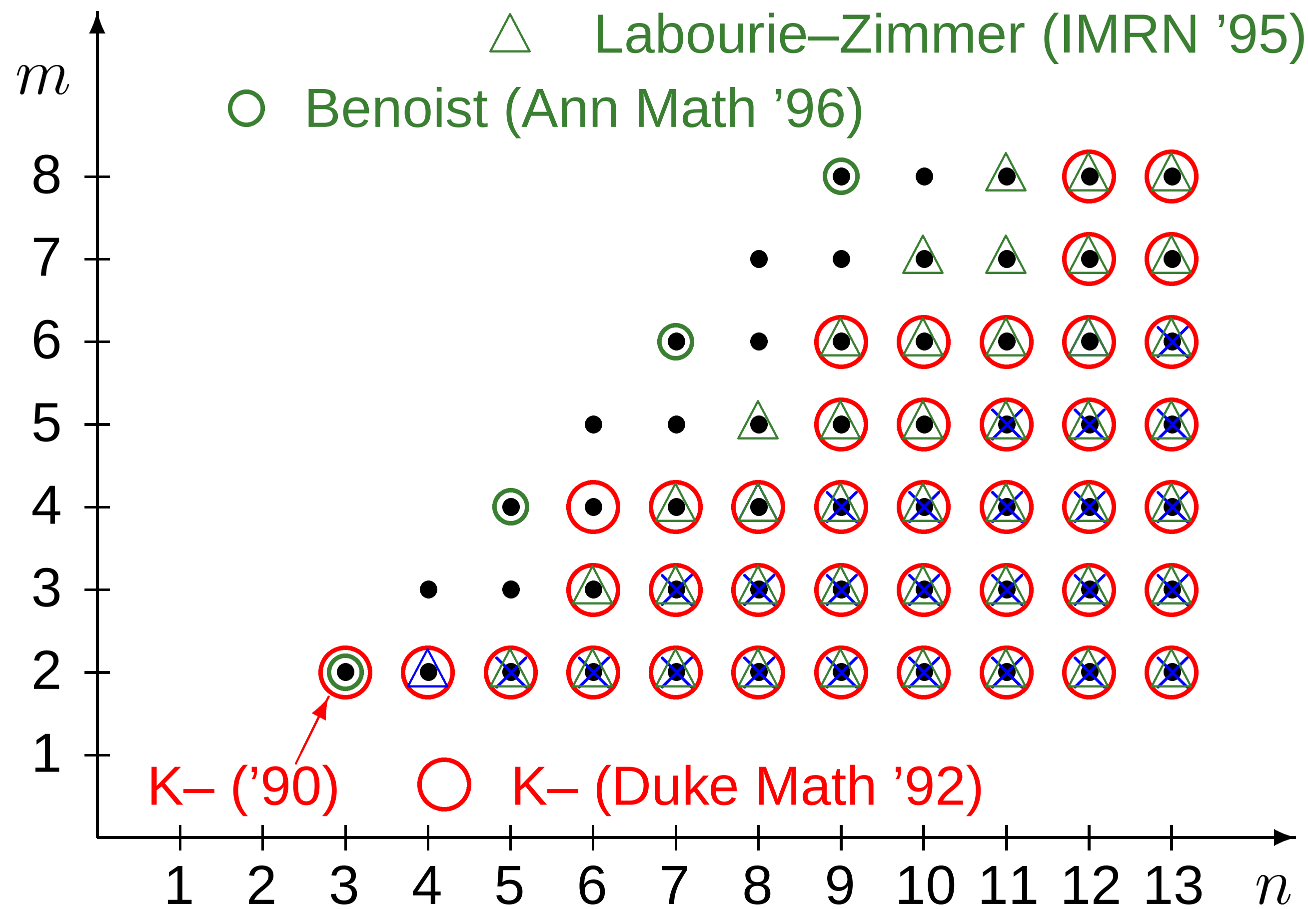

× Zimmer (Jour. AMS '94)
△ Labourie–Mozes–Zimmer (GAFA '95)

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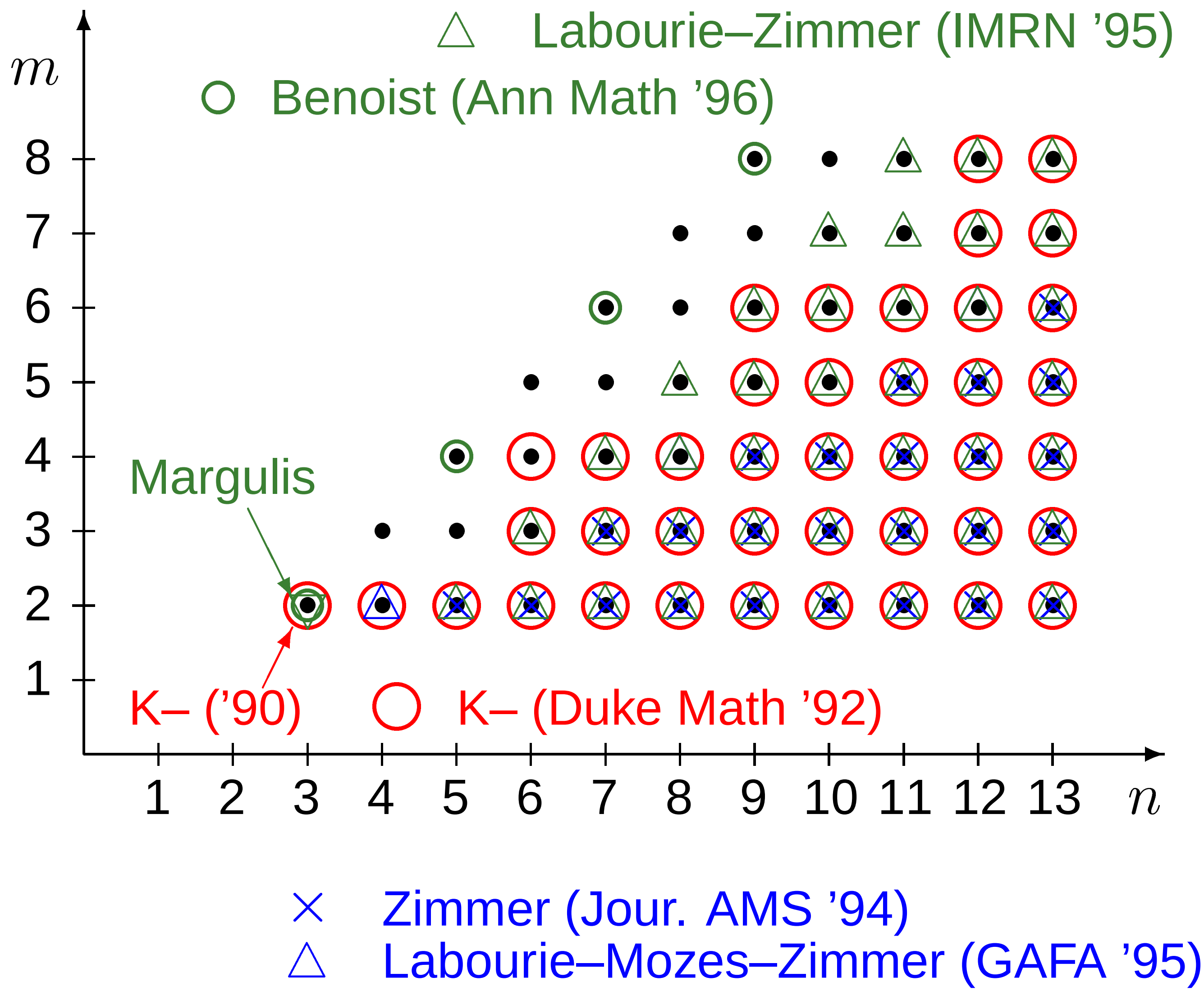

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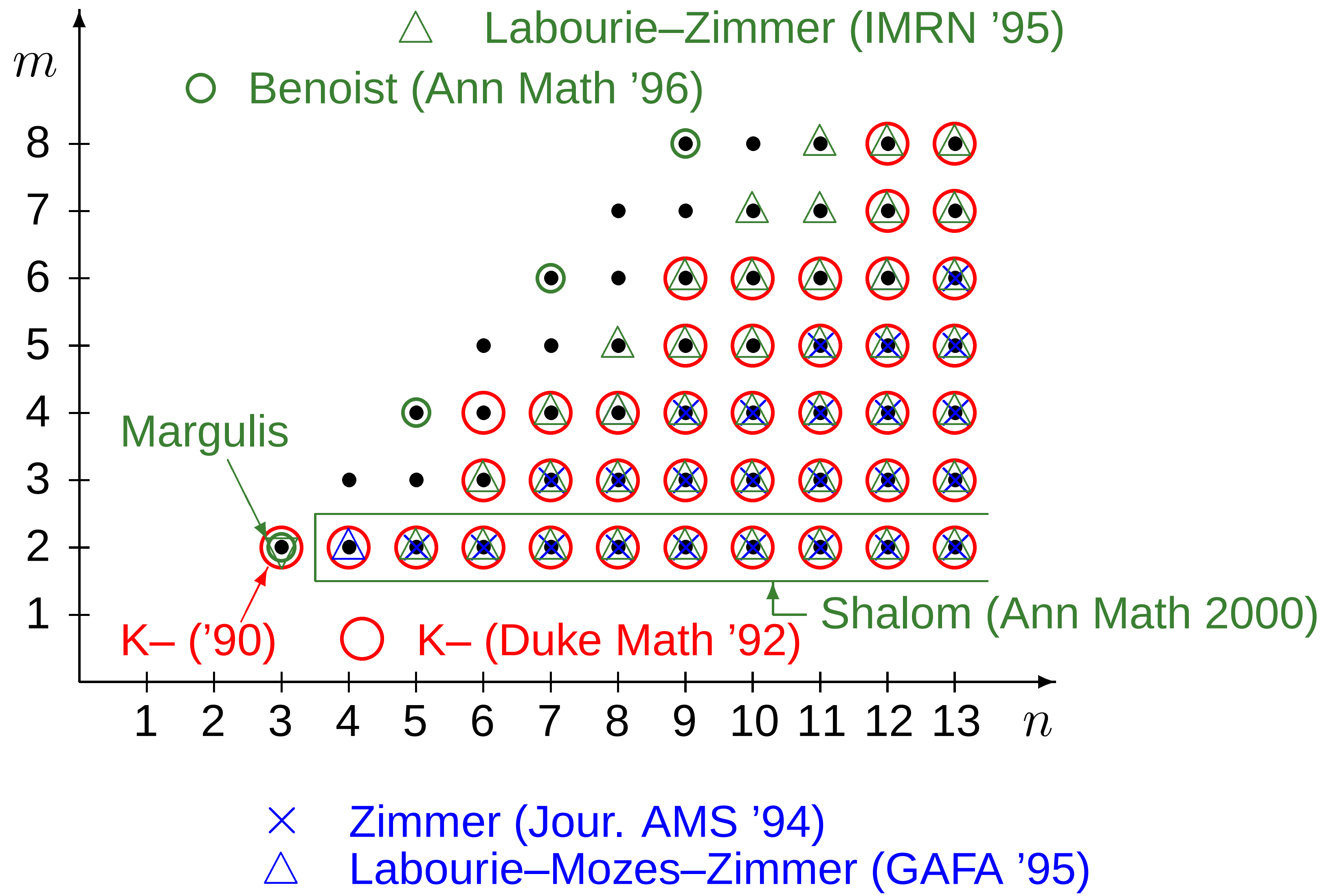

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G/K: Riemannian symmetric space

$$\Downarrow \text{complexification}$$

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$\Leftarrow$ proved by K–Yoshino 05,
$\Rightarrow$ remaining case $S^{4k-1}_{\mathbb{C}}$, $k \geq 3$ (Benoist, K–)

Existence of compact locally symm. sp

Theorem Exists a uniform lattice for the following G/H:
Exists a non-uniform lattice for G/H, too.

space form indefinite-Kähler complex symmetric

	G/H	
1	$SU(2,2n)/Sp(1,n)$	$n = 1, 2, 3, \ldots$
2	$SU(2,n)/U(1,n)$	$n = 2, 4, 6, \ldots$
3	$SO(2,2n)/U(1,n)$	$n = 1, 2, 3, \ldots$
4	$SO(2,n)/SO(1,n)$	$n = 2, 4, 6, \ldots$
5	$SO(4,n)/SO(3,n)$	$n = 4, 8, 12, \ldots$
6	$SO(4,4)/SO(4,1) \times SO(3)$	
7	$SO(4,3)/SO(4,1) \times SO(2)$	
8	$SO(8,8)/SO(7,8)$	
9	$SO(8,\mathbb{C})/SO(7,\mathbb{C})$	
10	$SO(8,\mathbb{C})/SO(7,1)$	
11	$SO^*(8)/U(3,1)$	
12	$SO^*(8)/SO^*(6) \times SO^*(2)$	

Global analysis on $\Gamma\backslash G/H$

What can we expect?

$$\begin{array}{rc} & G/H \\ \text{covering} & \downarrow \\ & \Gamma\backslash G/H \end{array}$$

G-invariant diff. op. $\widetilde{D}$

e.g. Laplacian

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Difficulties for the non-compact H case

- Laplacian is not elliptic
- $\mathrm{volume}(\Gamma\backslash G) = \infty$

Observation for $\mathbb{R}^{p,q}$

$$\mathbb{R}^{p,q} = (\mathbb{R}^{p+q},\ dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2)$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

Γ : lattice for $\mathbb{R}^{p+q}$ $\quad (\simeq \mathbb{Z}^{p+q})$

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$X_\Gamma := \Gamma \backslash \mathbb{R}^{p+q}$ $\quad (\simeq \mathbb{T}^{p+q})$

Observation $\quad \mathrm{Spec}(X_\Gamma, \Delta) \subset \mathbb{R}$

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because L^1 eigenfunction of Laplacian must be zero!

Construction of eigenfunction on $\Gamma\backslash G/H$

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Idea works for semisimple symmetric sp. G/H !

under the Flensted-Jensen – Matsuki–Oshima condition

$\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$

Universal $\mathrm{Spec}_\Delta(\Gamma\backslash G/H)$

$$G/H = U(2,2)/U(1)\times U(1,2)$$

$$\simeq \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^3\mathbb{C} : |z_1|^2 + |z_2|^2 > |z_3|^2 + |z_4|^2\}$$

complex 3-dim'l (real 6-dim'l preudo-Riemannian mfd)

Γ: torsion free, cocompact lattice of $Spin(4,1)$

Note $\mathrm{Vol}(\Gamma\backslash G) = \infty$, Δ: ultrahyperbolic operator

Thm C 1) $M_\Gamma := \Gamma\backslash G/H$ is a 6-dim'l compact mfd with indefinite metric of signature $(4,2)$.
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Idea of proof

$$\Gamma \underset{\text{lattice}}{\subset} Spin(4,1) \subset U(2,2) \supset U(1) \times U(1,2)$$

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$$\underset{\text{lattice}}{\Gamma \quad \subset} \quad L \quad \subset \quad G \quad \supset \quad H$$

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$$\Gamma \underset{\text{lattice}}{\subset} L \subset G \supset H$$

Induction

Restriction

$$\begin{tikzcd}
\boxed{\text{Analysis on } G/H} \arrow[dr, Rightarrow] & & \boxed{\text{Branching problem } G \downarrow L} \arrow[dl, Rightarrow] \\
& \boxed{\text{Thm C } \mathrm{Spec}_{\Delta}(\Gamma\backslash G/H)} & \\
& \boxed{\text{Geometry of } \Gamma\backslash G/H} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Flensted-Jensen, Oshima
Matsuki, Sekiguchi, Delorme
van den Ban, Schlichtkrull

Thm A

Thm B

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$\boxed{\text{Analysis on } G/H}$

Flensted-Jensen, Oshima
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Restriction

$\boxed{\text{Branching problem } G \downarrow L}$

Thm A

$\Searrow$ $\Swarrow$

$\boxed{\text{Thm C } \operatorname{Spec}_{\Delta}(\Gamma\backslash G/H)}$

$\Uparrow$

$\boxed{\text{Geometry of } \Gamma\backslash G/H}$ Thm B

Happy Birthday to Professor Oshima!