NOTES ON RADON TRANSFORMS

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Abstract

These notes include the following selected topics: Discussion of Radon's paper (1917); totally geodesic Radon transforms on the sphere and associated analytic families of intertwining operators; Radon transforms on Grassmann manifolds and matrix spaces; the generalized Minkowski-Funk transform for non-central spherical sections, and small divisors for spherical harmonic expansions; the Busemann-Petty problem on sections of convex bodies. Basic classical ideas and some recent results are presented in a systematic form.

Contents

2000 *Mathematics Subject Classification.* Primary 44A12; Secondary 14M15, 53C65.

Key words and phrases. Radon transforms, fractional integrals, Grassmann manifolds, the Busemann-Petty problem .

1. Introduction

According I.M. Gelfand [Ge], one of the basic problems of integral geometry can be stated as follows. Given a manifold X, let Ξ be a certain family of submanifolds of X. We write $x \in X$, $\xi \in \Xi$, and consider the mapping

$$f(x) \to (Rf)(\xi) = \int\limits_{\xi} f \tag{1.1}$$

that assigns to each sufficiently good function f on X a collection of integrals of f over submanifolds $\xi \in \Xi$. The problem is to study mapping properties of (1.1) (range, kernel, norm estimates) and find explicit inversion formulas in appropriate function spaces. It is assumed that Ξ itself is endowed with the structure of a manifold.

The mapping (1.1) is usually called *the Radon transform of* f.

Example 1.1. $X = \mathbb{R}^n$; Ξ *is the family of all hyperplanes in* $\mathbb{R}^n$.

Example 1.2. $X = S^n$ *is the unit sphere in* $\mathbb{R}^{n+1}$; Ξ *is the family of all* $(n-1)$*-dimensional subspheres of* S^n *of radius 1.*

Example 1.3. $X = G_{n,k}$ *is the Grassmann manifold of* k*-dimensional subspaces of* $\mathbb{R}^n$, $1 \le k < n$; $\Xi = G_{n,k'}$ *is the similar manifold with* $k' > k$.

An idea to study manifolds of submanifolds goes back to the 19th century (J. Plücker, F. Klein, M.S. Lie).[1] The celebrated paper by J. Radon [Rad] contains fundamental ideas related to operators (1.1) in important special cases and paves the way to further developments. In this paper one can also find information about the history of the problem. Namely, the problem was suggested to Radon by Blaschke. Reconstruction of functions on S^2 from their integrals over big circles was studied by Minkowski [Min] (1904). In 1913 P. Funk [Fu1], who was a student of D. Hilbert, reproduced Minkowski's solution and showed that the problem reduces to Abel's integral equation.

This manuscript was prepared for our lectures at the University of Tokyo in Summer 2006. It represents an updated and extended version of our previous text [Ru13].

We start by reviewing basic ideas of the original paper by J. Radon [Rad], and then proceed to the Minkowski-Funk transform and its generalizations for lower-dimensional central sections and Grassmannians

[1]Lie's interest to the group theory was influenced by Klein, who was Plücker's student.

(Sections 2-5). All these transforms and analytic families of intertwining operators generated by them, are of primary importance in the integral geometry of star-shaped/convex bodies. In Section 6 we discuss the old problem of P. Ungar (1954) about injectivity of non-central modifications of the Minkowski-Funk transform. To the best of our knowledge, this problem is still unsolved. It has a number of reformulations and leads to the realm of number theory. We present some partial results which give a flavor of how challenging the problem is. Section 7 is devoted to the Busemann-Petty problem (1956) on sections of convex bodies. For the hyperplane sections, it was solved only recently due to the efforts of a number of people. The lower dimensional version of the Busemann-Petty problem when dimension of the section is 2 and 3 is still mysterious. There is a remarkable interplay between Radon transforms of different kinds behind this problem. We present the solution to the original Busemann-Petty problem in a clear and simple form, and connect it with known results from Section 3. Section 8 concludes our notes and deals with Radon transforms on the space of real rectangular matrices. Here we follow our recent works [OR1], [OR2], inspired by pioneering results due to Petrov [Pe1], [Pe2].

Acknowledgements. I am grateful to Professor Toshio Oshima for the invitation to talk at the University of Tokyo and the hospitality during my visit.

2. Radon's paper

Radon begins his paper with the 2-dimensional case of lines in the plane, and then proceeds to generalizations and other settings. We start by reviewing basic ideas of Radon in the context of Example 1.1 for n dimensions.

Each hyperplane ξ in $\mathbb{R}^n$ is defined by

$$\xi = \{x \in \mathbb{R}^n : x \cdot \theta = t\}, \qquad \theta \in S^{n-1}, \quad t \in \mathbb{R},$$

and the Radon transform (1.1) can be represented as

$$\begin{aligned} (2.1) \qquad (Rf)(\xi) &= \int\limits_{x\cdot\theta=t} f(x)dm(x) = \int_{\mathbb{R}^n} f(x)\delta(x\cdot\theta - t)dx \\ &= \int\limits_{\theta^\perp} f(t\theta + u)du \equiv (Rf)(\theta, t). \end{aligned}$$

Here $\delta(\cdot)$ is the usual delta function of one variable, $\theta^\perp$ is the $(n-1)$-dimensional subspace orthogonal to θ, $dm(x)$ and du stand for the relevant induced Lebesque measures. For simplicity, we suppose that f belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Clearly,

$$(2.2) \qquad (Rf)(\theta, t) = (Rf)(-\theta, -t) \qquad \text{(the symmetry property).}$$

The following statement follows immediately from (1.1).

Proposition 2.1. *The operator (2.1) commutes with rigid motions of* $\mathbb{R}^n$. *Namely, if* $\tau : x \to \gamma x + y$, $\gamma \in O(n)$, $y \in \mathbb{R}^n$, *then*

$$R : f(\tau x) \to (Rf)(\tau \xi).$$

This is the basic property of Rf. It means that in order to solve the equation $Rf = \varphi$, it suffices to recover f only at one point, say $x = 0$, and restrict the consideration to radial functions $f(x) \equiv f_0(|x|)$.

Let us pass to details. If $f(x) \equiv f_0(r)$, $r = |x|$, then by rotation (set $u = \gamma_\theta v$, where $\gamma_\theta \in SO(n)$, $\gamma_\theta e_n = \theta$, $e_n = (0, \ldots, 0, 1)$), (2.1) yields

$$(Rf)(\theta, t) = \int\limits_{\mathbb{R}^{n-1}} f_0(|te_n + v|)dv = \sigma_{n-2} \int\limits_0^\infty f_0(\sqrt{t^2 + s^2}) s^{n-2} ds,$$

$$\sigma_{n-2} = |S^{n-2}| = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}.$$

This gives the following important statement.

Proposition 2.2. *If* $f(x) \equiv f_0(|x|)$, *then* $(Rf)(\theta, t) \equiv \varphi_0(t)$, *where* $\varphi_0(t)$ *is an even function defined by*

$$(2.3) \qquad \varphi_0(t) = \sigma_{n-2} \int\limits_{|t|}^\infty f_0(r)\ (r^2 - t^2)^{(n-3)/2} r dr.$$

The integral (2.3) is of Abel type. To be more precise, we introduce Riemann-Liouville (or Weyl) fractional integrals of the form

$$(2.4) \qquad v(t) = (I_-^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^\infty u(r)(r-t)^{\alpha-1} dr, \qquad \alpha > 0.$$

The inverse of (2.4) is called a fractional derivative. For sufficiently good $v(t)$, it can be written as

$$(2.5) \quad u(t) = (\mathcal{D}_-^\alpha v) = \left(-\frac{d}{dt}\right)^m (I_-^{m-\alpha} v)(t), \qquad \forall m \in \mathbb{N}, \ m \geq \alpha.$$

If $\alpha \in \mathbb{N}$ one can set $m = \alpha$ and get $\mathcal{D}_-^\alpha v = \left(-\frac{d}{dt}\right)^m v$; see [SKM] for more details.

By changing variables, we write (2.3) as

$$\varphi_0(\sqrt{t}) = \pi^{(n-1)/2}(I_-^{(n-1)/2} f_0(\sqrt{\cdot}))(t), \qquad t > 0,$$

so that

$$f_0(r) = \pi^{(1-n)/2}(\mathcal{D}_-^{(n-1)/2} \varphi_0(\sqrt{\cdot}))(r^2). \tag{2.6}$$

This is the inversion formula for the Radon transform in the radial case. If n is odd, then

$$f_0(r) = \pi^{(1-n)/2} \left(-\frac{1}{2r}\frac{d}{dr} \right)^{(n-1)/2} \varphi_0(r). \tag{2.7}$$

Now let us recover $f(x)$ from $(Rf)(\theta, t) = \varphi(\theta, t)$ in the general case. Fix x, and denote $f_x(y) = f(x+y)$. By (2.1),

$$(Rf_x)(\theta, t) = \varphi(\theta, t + x \cdot \theta), \tag{2.8}$$

and therefore

$$\int\limits_{SO(n)} (Rf_x)(\gamma\theta, t) d\gamma = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} \varphi(\theta, t + x \cdot \theta) d\theta. \tag{2.9}$$

The right hand side of (2.9) is the mean value of φ over all hyperplanes at distance $|t|$ from x. We denote

$$(M_t^* \varphi)(x) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} \varphi(\theta, t + x \cdot \theta) d\theta. \tag{2.10}$$

Since R commutes with rotations, the left hand side of (2.9) is the Radon transform of the radial function $f_0(|y|) = \int_{SO(n)} f_x(\gamma y) d\gamma$. The latter can be written as the spherical mean

$$f_0(r) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} f(x + r\theta) d\theta \overset{\text{def}}{=} (\mathcal{M}_r f)(x),$$

and Proposition 2.2 yields the following

Theorem 2.3. *For $t \in \mathbb{R}$,*

$$\begin{aligned} (M_t^* Rf)(x) &= \sigma_{n-2} \int\limits_{|t|}^{\infty} (\mathcal{M}_r f)(x)(r^2 - t^2)^{(n-3)/2} r dr \\ &= \pi^{(n-1)/2}(I_-^{(n-1)/2} f_0(\sqrt{\cdot}))(t^2), \end{aligned} \tag{2.11}$$

and therefore

$$f(x) = \pi^{(1-n)/2} \lim_{r \to 0} (\mathcal{D}_-^{(n-1)/2} [(M_{\sqrt{\cdot}}^* Rf)(x)])(r). \tag{2.12}$$

Now we introduce another important operator. Set $t = 0$ in (2.10), and denote

$$(R^*\varphi)(x) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} \varphi(\theta, x \cdot \theta) d\theta, \quad x \in \mathbb{R}^n, \tag{2.13}$$

where $\varphi(\theta, t)$ is a function on the manifold of all hyperplanes in $\mathbb{R}^n$. The operator (2.13) is called *the dual Radon transform* (this motivates "*" in (2.10)). It represents an integral of φ over the set of all hyperplanes through x. From (2.11) we get

$$\begin{aligned}(R^*Rf)(x) &= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int\limits_0^\infty r^{n-2} dr \int\limits_{S^{n-1}} f(x + r\theta) d\theta \\ &= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|} dy = (Hf)(x)\end{aligned}$$

or

$$R^*Rf = Hf. \tag{2.14}$$

For $n = 3$, Hf represents the well-known Newton potential. Discussion of (2.14) can be found in Radon's paper. It is worth noting that the equality (2.14) was communicated to Radon by Blaschke who discovered a striking connection of R and R^* with the potential theory.

The integral Hf is a member of the analytic family of Riesz potentials

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \quad \gamma_{n,\alpha} = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \tag{2.15}$$

which were introduced in 1935 in the thesis of O. Frostman (M. Riesz' student) and studied thoroughly by M. Riesz. Formally, $I^\alpha = (-\Delta)^{-\alpha/2}$ where Δ is the Laplace operator. More information about Riesz potentials can be found in [R1], [SKM]. In terms of (2.15), (2.14) reads

$$R^*Rf = c_n I^{n-1} f, \quad c_n = 2(2\pi)^{n-1}. \tag{2.16}$$

Thus, operators R and R^* can be inverted formally by

$$\begin{aligned}R^{-1} &= H^{-1}R^* = c_n^{-1}(-\Delta)^{(n-1)/2} R^*, \\ (R^*)^{-1} &= RH^{-1} = c_n^{-1} R(-\Delta)^{(n-1)/2}.\end{aligned} \tag{2.17}$$

These formulas were predicted in Radon's paper. Rigorous justification of (2.17) was given much later.

Fundamental ideas of Funk, Blaschke and Radon were extended to more general settings in further developments.

Resuming this section, we recall the basic tools implemented by Radon:

1. Group of motions.
2. Mean value operators.
3. Riemann-Liouville fractional integrals and Riesz potentials.

One more important thing should be mentioned. I mean harmonic analysis, which was not used by Radon but played a key role in Minkowski's treatment of the similar transform on the sphere.

Consider the Fourier transform

$$\hat{f}(\xi) = \int\limits_{\mathbb{R}^n} f(x) e^{ix\cdot\xi} dx. \tag{2.18}$$

Fix $\xi = \theta\rho$, $\theta \in S^{n-1}$, $\rho > 0$, and integrate (2.18) first over the hyperplane $x \cdot \theta = t$. We get

$$\hat{f}(\theta\rho) = \int\limits_{-\infty}^{\infty} e^{it\rho} dt \int\limits_{x\cdot\theta=t} f(x) dm(x) = \mathcal{F}_{t\to\rho}(Rf)(\theta, t) \tag{2.19}$$

where $\mathcal{F}_{t\to\rho}$ denotes the one-dimensional Fourier transform. By the symmetry (2.2), (2.19) extends to all $\rho \in \mathbb{R}$.

The equality (2.19) is known as the Central Slice Theorem. It enables us to invert the Radon transform using known inversion formulas for the Fourier transform. Conversely, by using inversion formulas for the Radon transform and the one-dimensional Fourier transform, one can invert the Fourier transform in n dimensions. This observation plays a crucial role in developing the Fourier analysis in numerous different settings. For example, the inversion formula for the Fourier transform on the real hyperbolic space results from those for the Mellin transform and the horocycle transform [H2], [H4], [VK]. The latter is an analogue of the euclidean Radon transform adapted to the hyperbolic space.

An excellent account of Radon's contribution to integral geometry is presented in [Gi2], [GGG2]. More information about Radon transforms on the euclidean space and their k-plane generalizations can be found in [GGG1], [GGG2], [GGV], [H2], [H4], [H5], [Ke], [Na]. Inversion formulas for these transforms in the framework of L^p-spaces were obtained in [Ru3]-[Ru5], [Ru8], [Ru9], [Ru14].

3. The Minkowski-Funk transform and related topics

3.1. **Historical notes.** Let $x \in S^n \subset \mathbb{R}^{n+1}$, $x^\perp$ be the central hyperplane orthogonal to x. The Minkowski-Funk transform is defined by

$$(Rf)(x) = \int\limits_{S^n \cap x^\perp} f(y) d_x y = \int\limits_{x \cdot y = 0} f(y) d_x y \tag{3.1}$$

where $d_x y$ denotes the induced Lebesgue measure on the "great circle" $S^n \cap x^\perp$. For $n = 2$, this transform was studied by Minkowski [Min] (1904) and later by Funk Fu1 (1913). Using decomposition in Legendre functions, Minkowski proved injectivity of R on the space of continuous functions. It means that $Rf \equiv 0$ implies $f \equiv 0$ for $f \in C_{even}(S^2)$. Funk reduced the equation $Rf = \varphi$ to Abel's integral equation by making use of a suitable averaging operator.

Radon's paper actually suggests the following two approaches to the inversion problem:

(a) via averaging and fractional differentiation;

(b) via representation of $R^* R$ as a potential operator.

The idea of (a) amounts to Funk. The idea of (b) is due to Blaschke who communicated it to Radon. Both methods were extended to all n as follows:

(1959) S. Helgason (the method (b) for n odd).

(1963) V.I. Semyanistyi (the method (b) for all $n \geq 2$ by lifting to the Fourier transform on $\mathbb{R}^{n+1}$).

(1990) S. Helgason (the method (a) for all $n \geq 2$).

(1998) B. Rubin (the method (b) for all $n \geq 2$ by using harmonic analysis on S^n).

The Minkowski-Funk transform (3.1) is also called *the spherical Radon transform* and *the Funk transform.* Note that there exist "spherical Radon transforms" of different kinds (see, e.g., [A], [Q]). Furthermore, Funk was aware of Minkowski's work (see [Rad], part C).

3.2. **The hemispherical transform.** In 1916 Funk [Fu2] considered the following problem: How to reconstruct a star-shaped body from its "half-volumes"? In polar coordinates the problem reduces to the integral equation

$$(Ff)(x) \equiv \int\limits_{x \cdot y > 0} f(y) dy = \varphi(x), \qquad x \in S^n, \tag{3.2}$$

with integration over the hemisphere $\{y \in S^n : x \cdot y > 0\}$. Funk solved this equation for $n = 2$ and zonal $f \equiv f(y_{n+1})$ by reducing (3.2) to the Abel integral equation. Campi [Ca] studied this equation for arbitrary

$f \in L^2(S^2)$ using decomposition in spherical harmonics. The case of all $n \ge 2$ and $f \in L^p(S^n)$ (instead of f one can take a finite Borel measure on S^n) was investigated by Rubin [Ru7].

As we shall see below, it is convenient to deal with the following modification of Ff:

$$(\tilde{F}f)(x) \equiv \int\limits_{S^n} f(y)\,\mathrm{sign}(x \cdot y)dy = 2(Ff)(x) - \int\limits_{S^n} f(y)dy. \tag{3.3}$$

3.3. **Connection with the euclidean Fourier transform.** Operators (3.1) and (3.2) arising in geometry are particular cases of more general operators which are well known in PDE and harmonic analysis. For $0 < Re\alpha < 1$, let

$$(A^\alpha f)(x) = \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\pi^{(n+1)/2}} \int\limits_{S^n} (-ix \cdot y)^{\alpha-1} f(y)dy, \qquad x \in S^n, \tag{3.4}$$

where the branch of $(...)^{\alpha-1}$ is chosen so that

$$(-ix \cdot y)^{\alpha-1} = |x \cdot y|^{\alpha-1}\big[\sin\frac{\alpha\pi}{2} + i\sin\frac{(1-\alpha)\pi}{2}\mathrm{sgn}(x \cdot y)\big]. \tag{3.5}$$

A direct calculation yields

$$\int\limits_{\mathbb{R}^{n+1}} \frac{f\big(\frac{\xi}{|\xi|}\big)}{|\xi|^{n+\alpha}} e^{i\xi\cdot\eta} d\xi = c_{\alpha,n}|\eta|^{\alpha-1}(A^\alpha f)\big(\frac{\eta}{|\eta|}\big), \quad c_{\alpha,n} = 2^{1-\alpha}\pi^{(n+1)/2}. \tag{3.6}$$

For $f \in C^\infty(S^n)$, the definition (3.4) and the equality (3.6) can be extended to all $\alpha \in \mathbb{C}$ by using analytic continuation. According to (3.5), one can write

$$A^\alpha f = U^\alpha f + iV^\alpha f, \tag{3.7}$$

where

$$(U^\alpha f)(x) = \frac{\Gamma\big((1-\alpha)/2\big)}{2\pi^{n/2}\Gamma(\alpha/2)} \int\limits_{S^n} f(y)|x \cdot y|^{\alpha-1}dy, \quad \alpha \ne 1, 3, 5, \ldots; \tag{3.8}$$

$$(V^\alpha f)(x) = \frac{\Gamma(1-\alpha/2)}{2\pi^{n/2}\Gamma\left((1+\alpha)/2\right)} \int\limits_{S^n} f(y)|x \cdot y|^{\alpha-1} sgn(x \cdot y)dy, \tag{3.9}$$

$$\alpha \ne 2, 4, 6, \ldots.$$

The operator U^α (V^α) annihilates odd (even) functions. Furthermore,

$$\lim_{\alpha\to 0} U^\alpha f = cRf, \qquad V^1 f = c\tilde{F}f, \quad c = \frac{\pi^{(1-n)/2}}{2}. \tag{3.10}$$

Thus the Minkowski-Funk transform and the hemispherical transform can be regarded as members of the analytic family $\{A^\alpha\}$ (or $\{U^\alpha\}$ and $\{V^\alpha\}$ respectively). Integrals $U^\alpha f$ are the Fourier symbols of generalized Riesz potentials [Sa1]. The case $\alpha = 2$ represents the *cosine transform*

$$(\mathfrak{C}f)(x) = \int_{S^n} f(y)|x \cdot y|dy \tag{3.11}$$

playing an important role in convexity [Ga1], [Sch2].

The formula (3.6) was known long ago (cf. [GŠ], [Es]). Connection of U^α with the Fourier transform on $\mathbb{R}^{n+1}$ was studied in detail by Semyanistyi [Se1], [Se2], who established a remarkable equality

$$(U^\alpha)^{-1} = U^{1-n-\alpha}. \tag{3.12}$$

For $\alpha = 0$, this gives an inversion formula for the Minkowski-Funk transform, and the problem is how to represent the right hand side of (3.12) explicitly (we shall return to this question later). Close results were obtained by Koldobsky [Ko1], [Ko3]. His "Blaschke-Levy representation" (or the p-cosine transform)

$$f \to \int_{S^n} f(y)|x \cdot y|^p dy \tag{3.13}$$

(without a normalizing factor) mimics (3.8).

3.4. **Fourier-Laplace multipliers and Sobolev spaces.** All aforementioned operators on S^n are spherical convolutions of the form

$$(Af)(x) = \int_{S^n} f(y)a(x \cdot y)dy. \tag{3.14}$$

Such operators can be investigated using the relevant harmonic analysis on S^n (see, e.g., a survey paper [Sa2]). For operators (3.7)-(3.9) (which include Minkowski-Funk, hemispherical, p-cosine and some other important transforms), this way is much less technical than lifting into $\mathbb{R}^{n+1}$. Moreover, it enables us to obtain a series of deep results. We recall some known facts. Let $\{Y_{j,k}(x)\}$ be an orthonormal basis of spherical harmonics on S^n. Here $j \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$; $k = 1, 2, \dots, d_n(j)$ where $d_n(j)$ is the dimension of the subspace of spherical harmonics of degree j. It is known ([Mü], p. 4) that

$$d_n(j) = (n + 2j - 1)\frac{(n + j - 2)!}{j!\ (n - 1)!}.$$

If $f = \sum\limits_{j,k} f_{j,k} Y_{j,k}$, $f_{j,k} = \int_{S^n} f(x) Y_{j,k}(x) dx$ (the Fourier-Laplace coefficients of f), then

$$Af = \sum_{j,k} \lambda_j f_{j,k} Y_{j,k} \tag{3.15}$$

where λ_j's are evaluated by the Funk-Hecke formula [Mü] as

$$\lambda_j = \sigma_{n-1} \int_{-1}^{1} a(\tau)(1-\tau^2)^{n/2-1} P_j(\tau) d\tau, \tag{3.16}$$

$\sigma_{n-1} = |S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$, $P_j(\tau)$ being (the generalized) Legendre polynomials. The sequence $\{\lambda_j\}$ is called *the Fourier-Laplace multiplier* of A, and we write $A \sim \{\lambda_j\}$. The operator A is bounded on $L^2(S^n)$ if and only if $\sup_j |\lambda_j| < \infty$. If $f \in C^\infty(S^n)$, and $\lambda_j = O(j^m), m > 0$, then the series $\sum\limits_{j,k} \lambda_j f_{j,k} Y_{j,k}(x)$ is absolutely and uniformly convergent and represents a C^∞-function [Ne].

Lemma 3.1. ([Ru1], Lemma 32.1) *Let X be any of the spaces $C(S^n)$, $L^p(S^n)$, $1 \le p < \infty$, or $\mathfrak{M}(S^n)$ (the Banach space of finite Borel measures on S^n). If the multiplier $\{\lambda_j\}$ of the operator A satisfies*

$$\lambda_j = \sum_{j=0}^{N-1} \frac{c_j}{j^{\delta+j}} + O(j^{-\delta-N}), \quad j \to \infty, \tag{3.17}$$

for some $\delta \ge 0$, $\delta + N > n$, then A (initially defined by (3.15) on C^∞-functions) can be extended to a linear bounded operator on X.

We denote by $\mathcal{S}' = \mathcal{S}'(S^n)$ the dual of $C^\infty(S^n)$ (the space of distributions on S^n). Given $\gamma \in \mathbb{R}$ and $p \in (1, \infty)$, the Sobolev space $L^\gamma_p = L^\gamma_p(S^n)$ is defined by

$$L^\gamma_p = \{f \in \mathcal{S}' \colon \; f^{(\gamma)} = \sum_{j,k} (j+1)^\gamma f_{j,k} Y_{j,k} \in L^p\}; \qquad \|f\|_{L^\gamma_p} = \|f^{(\gamma)}\|_p.$$

According to (3.16), by using tables of integrals [PBM], one can obtain the following multiplier representations:

$$A^\alpha \sim \{i^j a_{j,\alpha}\}, \qquad i = \sqrt{-1}, \quad a_{j,\alpha} = \frac{\Gamma(j/2 + (1-\alpha)/2)}{\Gamma(j/2 + (n+\alpha)/2)}, \tag{3.18}$$

$$U^\alpha \sim \{u_{j,\alpha}\}, \qquad u_{j,\alpha} = \begin{cases} (-1)^{j/2} a_{j,\alpha} & \text{if } j \text{ is even}, \\ 0 & \text{if } j \text{ is odd}, \end{cases} \tag{3.19}$$

$$V^\alpha \sim \{v_{j,\alpha}\}, \qquad v_{j,\alpha} = \begin{cases} 0 & \text{if } j \text{ is even}, \\ (-1)^{(j-1)/2} a_{j,\alpha} & \text{if } j \text{ is odd}, \end{cases} \tag{3.20}$$

Since all multipliers have a power behavior for $j \to \infty$ (up to oscillation), the corresponding operators act from $C^\infty(S^{n-1})$ to $C^\infty(S^{n-1})$. From (3.18)-(3.20), one can also obtain exact information about action of A^α, U^α, V^α in the scale L^γ_p of Sobolev spaces. The case $p = 2$ is trivial. In the general case we have the following

Theorem 3.2. *Let* $1 < p < \infty,\ \alpha \in \mathbb{C};\ \alpha \neq 1, 3, 5, \dots$.

(i) *The operator* U^α *can be extended as a linear bounded operator, acting from* L^β_p *into* L^γ_p *provided*

$$Re\ \alpha \geq \gamma - \beta - \frac{n-1}{2} + \Big|\frac{1}{p} - \frac{1}{2}\Big|(n-1). \tag{3.21}$$

(ii) *If* (3.21) *fails, then there is an even function* $f_0 \in L^\beta_p$ *so that* $U^\alpha f_0 \notin L^\gamma_p$.

Corollary 3.3. *The following proper embeddings hold:*

$$L^\delta_{p,even} \subset U^\alpha(L^p_{even}) \subset L^\gamma_{p,even}, \tag{3.22}$$

provided

(3.23)

$$\gamma = Re\ \alpha + \frac{n-1}{2} - \Big|\frac{1}{p} - \frac{1}{2}\Big|(n-1), \qquad \delta = Re\ \alpha + \frac{n-1}{2} + \Big|\frac{1}{p} - \frac{1}{2}\Big|(n-1),$$

$$\alpha \notin \{1, 3, 5, \dots\} \cup \{-n, -n-2, -n-4, \dots\}.$$

We explain the basic idea of how these statements can be proven (see [Ru6], [Ru7]). Let $f = f^+ + f^-$, $f^\pm(x) = (f(x) \pm f(-x))/2$. Then $U^\alpha f = U^\alpha f^+ = A^\alpha f^+$, $\|f^+\|_{L^\beta_p} \leq \|f\|_{L^\beta_p}$. The estimate $\|A^\alpha f\|_{L^\gamma_p} \leq \text{const}\|f\|_{L^\beta_p}$ is equivalent to $\|A^1 f\|_{L^\delta_p} \leq \text{const}\|f\|_p$, $\delta = \gamma - \beta - \text{Re}\ \alpha + 1$. This can be easily checked by using Strichartz' multiplier theorem [Str1] or Lemma 3.1. The operator $A^1 f$ arises as a symbol of the Calderòn-Zygmund singular integral. The above estimate of $A^1 f$ holds if and only if (3.21) is satisfied [G1]-[G3], [Kr]. The counter-exapmle that proves (ii) can be built using the argument from [Kr, Sec. 5].

Note that for $p = 2$, the gap in (3.22) disappears, and we get

$$U^\alpha(L^2_{\text{even}}) = L^\gamma_{2,\text{even}}, \qquad \gamma = \text{Re}\ \alpha + \frac{n-1}{2}.$$

The case of the Minkowski-Funk transform corresponds to $\alpha = 0$. For the cosine transform one should set $\alpha = 2$. Similar statements hold for

V^α. Note also that from (3.19) the inversion formula (3.12) becomes obvious, and in the Minkowski-Funk case we get

$$cU^{1-n}Rf = f, \qquad c = \frac{\pi^{(1-n)/2}}{2}. \tag{3.24}$$

Thus we need a "convenient" representation of the operator U^{1-n}. This problem will be discussed in Section 4 in a more general set-up.

3.5. **Restriction theorem.** Let $(R_{S^n}f)(x)$ be the Minkowski-Funk transform on $S^n \subset \mathbb{R}^{n+1}$, and let S^k, $2 \le k \le n-1$, be the section of S^n by the coordinate plane $\mathbb{R}^{k+1} = \mathbb{R}e_1 + \ldots + \mathbb{R}e_{k+1}$. What can one say about the restriction of $R_{S^n}f$ onto S^k? Suppose, for simplicity, that $f \in C^\infty_{even}(S^n)$. Then, for $x \in S^k$, $(R_{S^n}f)(x) \in C^\infty_{even}(S^k)$, and therefore it is represented as $(R_{S^k}\varphi)(x)$ with some $\varphi \in C^\infty_{even}(S^k)$ which is unique. An interesting result belonging to Fallert, Goodey, and Weil [FGW], gives an elegant explicit formula for φ. A short derivation of this formula is as follows.

Given $u \in S^k$, we denote by $S^{n-k}(u)$ the $(n-k)$-dimensional unit sphere in the subspace $(\mathbb{R}^{k+1})^\perp + \mathbb{R}u$, and set

$$\varphi(u) = \frac{1}{2} \int\limits_{S^{n-k}(u)} f(x)|x \cdot u|^{k-1} dm_u(x), \qquad v \in S^k, \tag{3.25}$$

where $dm_u(x)$ is the induced Lebesgue measure on $S^{n-k}(u)$.

Theorem 3.4. *If $f \in C^\infty_{even}(S^n)$, and $x \in S^k$, then $(R_{S^n}f)(x) = (R_{S^k}\varphi)(x)$, φ being defined by (3.25).*

Proof. Let $x = e_{k+1}$. If γ is a rotation in the 2-plane $\{e_{k+1}, e_{n+1}\}$ so that $\gamma e_{n+1} = e_{k+1}$, then $(R_{S^n}f)(e_{k+1}) = \int_{S^{n-1}} f(\gamma\sigma)d\sigma$. By passing to bispherical coordinates

$$\sigma = \eta\cos\theta + \zeta\sin\theta, \qquad 0 \le \theta \le \pi/2,$$

$$\eta \in S^{k-1} \subset \mathbb{R}e_1 + \ldots + \mathbb{R}e_k, \quad \zeta \in S^{n-k-1} \subset \mathbb{R}e_{k+1} + \ldots + \mathbb{R}e_n,$$

[VK, pp. 12, 22], we have $(R_{S^n}f)(e_{k+1}) = \int_{S^{k-1}} A(\eta)d\eta$,

$$A(\eta) = \int\limits_0^{\pi/2} \sin^{n-k-1}\theta\cos^{k-1}\theta\, d\theta \int\limits_{S^{n-k-1}} f(\gamma(\eta\cos\theta + \zeta\sin\theta)d\zeta. \tag{3.26}$$

Since $\gamma\eta = \eta$ and $\gamma\zeta \in (\mathbb{R}^{k+1})^\perp = \mathbb{R}e_{k+2} + \ldots + \mathbb{R}e_{n+1}$ the inner integral in (3.26) reads

$$\int\limits_{S^n \cap (\mathbb{R}^{k+1})^\perp} f(\eta\cos\theta + \omega\sin\theta)d\omega.$$

Hence $A(\eta) = \int_{S^{n-k}(\eta)} f(w)|w \cdot \eta|^{k-1} dw$, and the result follows by rotation invariance. $\square$

Theorem 3.4 is of independent interest. It will also play an important role in Section 8.

4. Totally geodesic Radon transforms on the sphere

4.1. **Definition and basic properties. Mean value operators.** Let Ξ be the set of k-dimensional totally geodesic submanifolds (k-geodesics) $\xi \subset S^n$, $1 \le k \le n-1$. Each k-geodesic is a section of S^n by the relevant $(k+1)$-dimensional plane in $\mathbb{R}^{n+1}$ through the origin. The case $k = n-1$ corresponds to "great circles". The totally geodesic Radon transform of a sufficiently good function f on S^n is defined by

$$(4.1) \qquad (Rf)(\xi) = \int\limits_{\xi} f(x) dm(x) = \int\limits_{d(x,\xi)=0} f(x) dm(x), \qquad \xi \in \Xi,$$

where $dm(x)$ denotes the induced Lebesgue measure on ξ and $d(x, \xi)$ is the geodesic distance between x and ξ. For $k = n-1$, (4.1) coincides with the Minkowski-Funk transform.

To get a better feeling of how the distance $d(x, \xi)$ can be measured, we denote by $\{\xi\}$ the $(k+1)$-plane containing ξ, and by $\{\xi\}^\perp$ the subspace orthogonal to $\{\xi\}$. If $\mathrm{Pr}_{\{\xi\}^\perp} x$ denotes the orthogonal projecton of x onto $\{\xi\}^\perp$, and $V(x,\xi)$ is the $(k+2)$-dimensional volume of the parallelepiped spanned by x and some orthonormal basis in $\{\xi\}$, then

$$(4.2) \qquad \sin[d(x,\xi)] = |\mathrm{Pr}_{\{\xi\}^\perp} x| = V(x,\xi).$$

In particular, if ξ is a great circle orthogonal to $y \in S^n$, then $\sin[d(x,\xi)] = |x \cdot y|$.

We introduce mean value operators which play a key role in the following. Given $f : S^n \to \mathbb{C}$, $\quad \varphi : \Xi \to \mathbb{C}$, and $\theta \in [0, \pi/2]$, let

$$(4.3) \quad (R_\theta f)(\xi) = \int\limits_{d(x,\xi)=\theta} f(x) dm(x), \quad (R^*_\theta \varphi)(x) = \int\limits_{d(x,\xi)=\theta} \varphi(\xi) d\mu(\xi),$$

where $dm(x)$ and $d\mu(\xi)$ stand for the relevant normalized measures. For $\theta = 0$ we have $R_0 f = \sigma_k^{-1} Rf$, $R^*_0 \varphi = R^* \varphi$, where Rf is the Radon transform (4.1), $\sigma_k = |S^k|$, and R^* is called *the dual geodesic Radon transform.*

In order to represent (4.3) in a rigorous analytic form we need some notation. Let $\mathbb{R}^{n+1} = \mathbb{R}^{k+1} \times \mathbb{R}^{n-k}$, $\quad \mathbb{R}^{k+1} = \mathbb{R}e_1 + \ldots + \mathbb{R}e_{k+1}$, $\mathbb{R}^{n-k} = \mathbb{R}e_{k+2} + \ldots + \mathbb{R}e_{n+1}$, $\quad e_i$ being coordinate unit vectors. In the following $\sigma_n = |S^n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$, $\xi_0 = S^k$ is the unit sphere in $\mathbb{R}^{k+1}$; $G = SO(n+1)$; $K = SO(n)$ and $K' = SO(k+1)\times SO(n-k)$ are the isotropy subgroups of e_{n+1} and ξ_0 respectively. The set Ξ can be identified with the Grassmann manifold G/K' of all $(k+1)$-dimensional subspaces of $\mathbb{R}^{n+1}$. We define an invariant measure $d\xi$ on Ξ by setting $\int_\Xi \varphi(\xi)d\xi = \int_G \varphi(\gamma\xi_0)d\gamma$ where $\int_G d\gamma = 1$.

For $\theta \in [0, \pi/2]$, let $g_{k+1,n+1}(\theta)$ be the rotation in the plane (e_{k+1}, e_{n+1}) with the matrix $\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$, so that $x_\theta = g_{k+1,n+1}(\theta)e_{n+1} = e_{k+1}\cos\theta + e_{n+1}\sin\theta$ and $d(x_\theta, \xi_0) = \theta$. Given $x \in S^n$, $\xi \in \Xi$, we denote by r_x, r_ξ arbitrary rotations satisfying $r_x e_{n+1} = x$, $r_\xi \xi_0 = \xi$, and set $f_\xi(x) = f(r_\xi x)$, $\varphi_x(\xi) = \varphi(r_x\xi)$.

In this notation operators (4.3) are defined by

$$(R_\theta f)(\xi) = \int\limits_{K'} f_\xi(r g_{k+1,n+1}(\theta)e_{n+1})dr, \tag{4.4}$$

$$(R^*_\theta \varphi)(x) = \int\limits_{K} \varphi_x(\rho[g_{k+1,n+1}(\theta)]^{-1}\xi_0)d\rho. \tag{4.5}$$

Using (4.4) and (4.5), one can prove the following

Lemma 4.1. [H5], [Ru11]

(a) *The following duality relation holds:*

$$\int\limits_\Xi (R_\theta f)(\xi)\varphi(\xi)d\xi = \frac{1}{\sigma_n}\int\limits_{S^n} f(x)(R^*_\theta\varphi)(x)dx \tag{4.6}$$

provided that the integral in either side is finite for f and φ replaced by $|f|$ and $|\varphi|$ respectively. In particular, for $\theta = 0$,

$$\frac{1}{\sigma_k}\int\limits_\Xi (Rf)(\xi)\varphi(\xi)d\xi = \frac{1}{\sigma_n}\int\limits_{S^n} f(x)(R^*\varphi)(x)dx. \tag{4.7}$$

(b) *Operators R_θ, R^*_θ are bounded on L^p, $1 \le p \le \infty$. Namely,*

$$\|R_\theta f\|_{(p)} \le \sigma_n^{-1/p}\|f\|_p, \qquad \|R^*_\theta\varphi\|_p \le \sigma_n^{1/p}\|\varphi\|_{(p)}, \tag{4.8}$$

where $\|\cdot\|_{(p)}$ and $\|\cdot\|_p$ are L^p-norms on Ξ and S^n respectively.

As in the previous section, the general picture becomes more clear if we regard the Radon transforms R, R^* as members of a suitable analytic family of *intertwining operators* commuting with rotations. In order to find these operators, we proceed as follows. Following Funk and Radon, we first note that Radon transforms of zonal (or radial) functions admit representation of the Abel type.

Lemma 4.2. [Ru11] *Given $x \in S^n$, $\xi \in \Xi$, and a measurable function $a(t)$ on $(0,1)$, let*

$$\omega = d(e_{n+1}, x), \qquad a_1(x) = a(\cos\omega) = a(x_{n+1}),$$
$$\theta = d(e_{n+1}, \xi), \qquad a_2(\xi) = a(\sin\theta).$$

Then

$$(4.9) \quad (Ra_1)(\xi) = \frac{2\sigma_{k-1}}{\cos^{k-1}\theta} \int\limits_0^{\cos\theta} (\cos^2\theta - t^2)^{k/2-1} a(t)\,dt,$$

$$(4.10) \quad (R^* a_2)(x) = \frac{\sigma_{k-1}\,\sigma_{n-k-1}}{\sigma_{n-1}\,\sin^{n-2}\omega} \int\limits_0^{\sin\omega} (\sin^2\omega - t^2)^{k/2-1}\, t^{n-k-1}\, a(t)\,dt.$$

4.2. **Analytic families of intertwining operators.** Lemma 4.2 shows the way how to introduce analytic families of intertwining operators including R and R^*. Put $a(t) = t^{\alpha+k-n}$, $\alpha > 0$, in (4.10), so that $a_2(\xi)$ becomes

$$a_2(\xi) = (\sin[d(e_{n+1}, \xi)])^{\alpha+k-n}.$$

A simple calculation yields

$$(R^* a_2)(x) = \frac{\pi^{k/2}\,\sigma_{n-k-1}\,\Gamma(\alpha/2)}{\sigma_{n-1}\,\Gamma((k+\alpha)/2)}\,(1 - x_{n+1}^2)^{(\alpha+k-n)/2},$$

and by duality (4.7) we obtain

$$\int\limits_{\Xi} (Rf)(\xi)\,(\sin[d(e_{n+1}, \xi)])^{\alpha+k-n}\,d\xi = c \int\limits_{S^n} f(y)\,(1 - y_{n+1}^2)^{(\alpha+k-n)/2}\,dy,$$

$$c = \frac{\pi^{k/2}\,\sigma_k\,\sigma_{n-k-1}\,\Gamma(\alpha/2)}{\sigma_n\,\sigma_{n-1}\,\Gamma((k+\alpha)/2)}.$$

Owing to $SO(n+1)$-invariance, one can replace e_{n+1} by $x \in S^n$ and get

$$(4.11) \quad \int\limits_{\Xi} (Rf)(\xi)\,(\sin[d(x, \xi)])^{\alpha+k-n}\,d\xi = c \int\limits_{S^n} f(y)\,(1 - |x \cdot y|^2)^{(\alpha+k-n)/2}\,dy.$$

For $k = n-1$, the left hand side of (4.11) resembles operator U^α from Section 3 (see (3.8)). The right hand side serves as a spherical analog of the Riesz potential (2.15). After appropriate normalization we arrive at the following intertwining operators:

$$(4.12) \qquad (R^\alpha f)(\xi) = \gamma_{n,k}(\alpha) \int\limits_{S^n} (\sin[d(x, \xi)])^{\alpha+k-n} f(x)dx,$$

$$(\overset{*}{R}{}^{\alpha}\varphi)(x)=\gamma_{n,k}(\alpha)\int\limits_{\Xi}(\sin[d(x,\xi)])^{\alpha+k-n}\varphi(\xi)d\xi, \tag{4.13}$$

$$\gamma_{n,k}(\alpha)=\frac{\Gamma((n-\alpha-k)/2)}{2\pi^{n/2}\,\Gamma(\alpha/2)},$$

assuming $Re\alpha>0, \quad \alpha+k-n\neq 0,2,4,\dots$.

Looking at the right hand side of (4.11), we also set

$$\begin{aligned}(Q^{\alpha}f)(x) &= c_{n,\alpha}\int\limits_{S^n}(1-|x\cdot y|^2)^{(\alpha-n)/2}f(y)dy\\ &= c_{n,\alpha}\int\limits_{S^n}(\sin[d(x,y)])^{\alpha-n}f(y)dy,\end{aligned} \tag{4.14}$$

$$c_{n,\alpha}=\frac{\Gamma((n-\alpha)/2)}{2\pi^{n/2}\Gamma(\alpha/2)}, \quad Re\alpha>0, \quad \alpha-n\neq 0,2,4,\dots.$$

These definitions can be extended to excluded values of α by inserting the relevant logarithmic factor [Ru11]. One can show that for continuous f and φ,

$$\lim_{\alpha\to 0}R^{\alpha}f=c_1Rf, \qquad \lim_{\alpha\to 0}\overset{*}{R}{}^{\alpha}\varphi=c_2R^{*}\varphi, \tag{4.15}$$

$$c_1=\frac{1}{2\pi^{k/2}}, \qquad c_2=\frac{\sigma_k}{2\pi^{k/2}\sigma_n}.$$

Thus analytic families $\{R^{\alpha}\}$ and $\{\overset{*}{R}{}^{\alpha}\}$ actually include the Radon transform and its dual (in the case $k=n-1$ both families coincide).

By taking into account (4.2), we have

$$R^{n-k+1}f=\gamma_{n,k}(n-k+1)\int\limits_{S^n}f(x)V(x,\xi)dx. \tag{4.16}$$

This provokes us to call operators (4.12) and (4.13) *the generalized cosine transforms*. The potential type operator (4.14) will be called *the generalized sine transform*.

Note that the kernel of the potential Q^{α} has a point singularity $x=y$, whereas the singularity of the kernel of R^{α} and $\overset{*}{R}{}^{\alpha}$ is spread over the k-dimensional circle ξ. Note also that the Fourier-Laplace multiplier of Q^{α} has the form

$$\frac{\Gamma((j+n-\alpha)/2)\,\Gamma((j+1)/2)}{\Gamma((j+\alpha+1)/2)\,\Gamma((j+n)/2)} \qquad (\sim (j/2)^{-\alpha} \quad \text{as} \quad j\to\infty) \tag{4.17}$$

(this can be checked using the Funk-Hecke formula (3.16) and [PBM, 2.21.2(3)]). For sufficiently good f, it follows that

$$Q^0 f \equiv \lim_{\alpha\to 0} Q^\alpha f = f. \tag{4.18}$$

The equality (4.11), which played so far a purely heuristic role, implies

$$\frac{\sigma_n}{\sigma_k} \overset{*}{R}{}^{\alpha} Rf = \frac{\Gamma(n/2)}{\Gamma((n-k)/2)}\, Q^{\alpha+k} f.$$

More generally, the following statement holds.

Theorem 4.3 (Ru11)**.** *Let $f \in L^1(S^n)$, $\alpha \ge 0$. Then*

(4.19)
$$\lambda \overset{*}{R}{}^{\alpha} Rf = \frac{\Gamma((n-k)/2)}{\Gamma(n/2)} R^* R^\alpha f = Q^{\alpha+k} f, \qquad \lambda = \frac{\sigma_n\, \Gamma((n-k)/2)}{\sigma_k\, \Gamma(n/2)}.$$

In the case $\alpha + k - n = 0, 2, 4, \dots$, the last equality in (4.19) holds if and only if all Fourier-Laplace coefficients of f up to order $\alpha + k - n$ are zeros.

This theorem has many remarkable consequences. Below we list some of them.

4.3. **Inversion formulas for C^∞-functions.** By (4.19),

$$\lambda \overset{*}{R}{}^{-k} Rf = f \tag{4.20}$$

(at least formally); cf. (3.24).

Let Δ be the Beltrami-Laplace operator on S^n. The Fourier-Laplace multiplier of Δ is $-j(j+n-1)$. Denote $\Delta_\alpha = [-\Delta + (n-\alpha)(\alpha-1)]/4$. By (4.17), for $f \in C^\infty(S^n)$ we have

$$\Delta_\alpha Q^\alpha f = Q^{\alpha-2} f, \qquad \alpha - n \ne 0, 2, \dots,$$

and therefore if $\alpha = 2m\, (\ne n, n+2, \dots)$ is an even positive integer, then $Q^\alpha f$ can be inverted by a polynomial of Δ. Owing to (4.19), this observation leads to the following

Theorem 4.4. *Let $\varphi = Rf$, $f \in C^\infty_{even}(S^n)$, $1 \le k \le n-1$,*

$$P_m(\Delta) = \Delta_2 \Delta_4 \dots \Delta_{2m} \tag{4.21}$$

(i) If n is odd, then

$$f = \lambda P_m(\Delta) \overset{*}{R}{}^{2m-k} \varphi \quad \forall m \ge k/2. \tag{4.22}$$

(ii) If n is even, then (4.22) is applicable only for $k/2 \le m \le n/2 - 1$, and another inversion formula also holds:
(4.23)
$$f = P_{n/2}(\Delta)\Big[\frac{2^n}{(n-1)!\,\sigma_k}\int_\Xi \varphi(\xi)\log\frac{1}{\sin[d(x,\xi)]}d\xi\Big] + \frac{1}{\sigma_k}\int_\Xi \varphi(\xi)d\xi.$$

The equality (4.19) enables us to obtain explicit inversion formulas for the generalized cosine transform $R^\alpha f$ (see [Ru11] for details).

The formula (4.22) for $m = k/2$, k even, was obtained by Helgason [H1], [H5]. Theorem 4.4 (for any $1 \le k \le n-1$) was proved in [Ru11]. Inversion of operators R^α for $k = n-1$ was studied in detail in [Ru2].

4.4. **Inversion formulas for non-smooth functions.** Suppose we want to invert Rf for $f \in L^p(S^n)$ or $f \in C(S^n)$. In view of (4.20), the problem is how we can realize operator $\overset{*}{R}{}^{-k}$. Below we consider two approaches to this problem.
4.4.1. **Reduction to the Abel integral equation**. This approach was developed by Helgason [H2]. Given $x \in S^n$ and $t \in (-1,1)$, we denote

$$(\mathbb{M}_t f)(x) = \frac{(1-t^2)^{(1-n)/2}}{\sigma_{n-1}}\int_{\{y\in S^n:\, x\cdot y = t\}} f(y)d\sigma(y). \tag{4.24}$$

The integral (4.24) represents the mean value of f on the planar section of S^n by the hyperplane $x \cdot y = t$, and $d\sigma(y)$ stands for the induced Lebesgue measure on this section. We introduce the Riemann-Liouville fractional integrals [SKM]

$$(I^\alpha_{0+}\psi)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi(\tau)(t-\tau)^{\alpha-1}d\tau, \qquad t > 0. \tag{4.25}$$

Lemma 4.5. *Let $f \in L^1_{even}(S^n)$ and R^*_θ be the mean value operator defined by (4.3), (4.5). Then*

$$(R^*_\theta Rf)(x) = 2\pi^{k/2}(\cos\theta)^{1-k}(I^{k/2}_{0+}\psi_x)(\cos^2\theta), \tag{4.26}$$

$$\psi_x(\tau) = \tau^{-1/2}(\mathbb{M}_{\sqrt{\tau}}f)(x).$$

By inverting the fractional integral in (4.26), we obtain the following

Theorem 4.6. ([H3] - [H5], [Ru11]) *Suppose that $\beta > 0$ is chosen so that $k/2 + \beta = m \in \mathbb{N}$. Then*

$$f(x) = \frac{1}{\pi^{k/2}\Gamma(\beta)}\Big[\Big(\frac{d}{dv^2}\Big)^m\int_0^v (v^2-u^2)^{\beta-1}u^k(R^*_{\cos^{-1}(u)}Rf)(x)du\Big]_{v=1}.$$

If k is even, then

$$f(x)=\frac{1}{2\pi^{k/2}}\Big[\Big(\frac{d}{dv^2}\Big)^{k/2}[v^{k-1}(R^*_{\cos^{-1}(v)}Rf)(x)]\Big]_{v=1}.$$

For $f\in L^p(S^n)$, $1\le p<\infty$, all derivatives in these formulas exist in the a.e. sense and in the L^p-norm. If $f\in C(S^n)$ they are understood in the usual sense for each $x\in S^n$.

There is another way to invert the fractional integral in (4.26). This employs so-called Marchaud's fractional derivatives [Ru1], [SKM], and enables us to replace derivatives in Theorem 4.6 by finite differences. For $k=1$, the inversion formula is especially simple.

Theorem 4.7. [Ru12] *Let $\varphi=Rf$, $f\in L^p(S^n)$, $1\le p<\infty$. Then*

$$f=\frac{R^*\varphi}{2\pi}+\frac{1}{2\pi}\int\limits_0^{\pi/2}\frac{R^*\varphi-R^*_\theta\varphi}{\sin^2\theta}\cos\theta\,d\theta,\qquad \int\limits_0^{\pi/2}=\lim_{\varepsilon\to 0}\int\limits_\varepsilon^{\pi/2}, \tag{4.27}$$

where the limit is understood in the L^p-norm and in the a.e. sense. If $f\in C(S^n)$, then $\lim\limits_{\varepsilon\to 0}$ is uniform.

4.4.2. **Direct regularization with the aid of "wavelet transforms"**. The idea is as follows. We start with (4.13) and replace the power function $a^{\alpha+k-n}$ where $a=\sin[d(x,\xi)]$, by the formula

$$a^{\alpha+k-n}=\frac{1}{c_{\alpha,w}}\int\limits_0^\infty\frac{w(a/t)}{t^{n-\alpha-k+1}}dt,\qquad c_{\alpha,w}=\int\limits_0^\infty w(s)\frac{ds}{s^{1+\alpha+k-n}}. \tag{4.28}$$

The auxiliary function w is assumed to be sufficiently good. It will be specified later. Let us replace $(\sin[d(x,\xi)])^{\alpha+k-n}$ in (4.13) according to (4.28) and change the order of integration. We get

$$(\overset{*}{R}{}^{\alpha}\varphi)(x)=\frac{\gamma_{n,k}(\alpha)}{c_{\alpha,w}}\int\limits_0^\infty\frac{(W\varphi)(x,t)}{t^{1-\alpha}}\,dt, \tag{4.29}$$

$$(W\varphi)(x,t)=\frac{1}{t^{n-k}}\int\limits_\Xi\varphi(\xi)w\Big(\frac{\sin[d(x,\xi)]}{t}\Big)d\xi. \tag{4.30}$$

If w decays sufficiently fast at infinity and

$$\int\limits_0^\infty\tau^{j+n-k-1}w(\tau)d\tau=0\quad\forall j=0,2,4,\dots,2[k/2], \tag{4.31}$$

then (4.29) can be extended to $\alpha = -k$, and we have [RR], [Ru11]

$$\int_0^\infty \frac{(WRf)(x,t)}{t^{1+k}} dt \equiv \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \Big(\dots \Big) = cf(x), \tag{4.32}$$

where the limit is understood in the L^p-norm and in the a.e. sense. If $f \in C(S^n)$, then $\lim\limits_{\varepsilon \to 0}$ is uniform. The constant $c = c(\alpha, k, n, w)$ can be evaluated explicitly.

Owing to (4.31), we call $(W\varphi)(x,t)$ a *continuous wavelet transform* associated to the analytic family $\{\overset{*}{R}{}^{\alpha}\}$. Various applications of continuous wavelet transforms in fractional calculus and integral geometry are discussed in [BR], [Ru1], [Ru3], [Ru6], [Ru15] where one can find further references.

5. Radon transforms on Grassmann manifolds

5.1. **Setting of the problem.** Let $G_{n,k}, G_{n,k'}$ be a pair of Grassmann manifolds of linear k-dimensional and k'-dimensional subspaces of $\mathbb{R}^n$ respectively. Suppose that $1 \le k < k' \le n-1$. Each "point" $\eta \in G_{n,k}$ ($\xi \in G_{n,k'}$) is a non-oriented k-plane (k'-plane) in $\mathbb{R}^n$ passing through the origin. According to Example 1.3, the Radon transform of a sufficiently good function $f(\eta)$ on $G_{n,k}$ is defined by

$$(\mathcal{R}f)(\xi) = \int_{\eta \subset \xi} f(\eta) d_\xi \eta, \qquad \xi \in G_{n,k'}, \tag{5.1}$$

$d_\xi \eta$ being a suitable measure on the space of planes η in ξ. Our goal is to find explicit inversion formulas for (5.1).

Let us discuss this problem. The first question is for what triples (k, k', n) the problem is meaningful. Clearly, one should assume

$$\dim G_{n,k'} \ge \dim G_{n,k} \tag{5.2}$$

(otherwise $\mathcal{R}$ has a non-trivial kernel). By taking into account that $\dim G_{n,k} = k(n-k)$, we conclude that (5.2) is equivalent to $k + k' \le n$ (for $k < k'$). Thus the natural framework for the inversion problem is

$$1 \le k < k' \le n-1, \qquad k + k' \le n. \tag{5.3}$$

For $k = 1$, f can be regarded as an even function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and $(\mathcal{R}f)(\xi)$ represents the totally geodesic Radon transform from Section 4. For $k > 1$, $\quad k' + k = n$, some inversion formulas were announced by Petrov [Pe1] in 1967. His method employs modification of the plane waves decomposition. Unfortunately all proofs in Petrov's article are skipped, and his inversion formulas contain a divergent integral that should be understood somehow in a regularized sense. Another approach, which is based on the use of differential forms was suggested by Gel'fand, Graev and Šapiro [GGŠ] in 1970 (see also [GGR]). The third approach was developed by Grinberg [Gr], Gonzalez [Go] and Kakehi [K]. It agrees with the idea of Blaschke-Radon (cf. (2.17)) to apply a certain differential operator to the composition of the Radon transform and its dual. This method relies on harmonic analysis on Grassmannians. The second and the third approaches are applicable only to $k'-k$ even. Note also that all aforementioned methods deal only with C^∞-functions and resulting inversion formulas are rather complicated.

Below we show how the original Funk-Radon approach via Abel integrals can be adapted to (5.1). Following this way, one can obtain explicit inversion formulas in all admissible dimensions (5.3) for functions f, belonging to $C(G_{n,k})$ and $L^p(G_{n,k})$, $1 \le p < \infty$.

5.2. **Some preparations.** Any treatment of functions on Grassmannians reduces inevitably to functions of matrix argument. We recall some basic facts. Let $\mathfrak{M}_{n,k}$, $n \ge k$, be the space of real matrices having n rows and k columns. We identify $\mathfrak{M}_{n,k}$ with the real Euclidean space $\mathbb{R}^{nk}$ so that for $x=(x_{i,j})$ the volume element is $dx = \prod_{i=1}^n \prod_{j=1}^k dx_{i,j}$. In the following x' denotes the transpose of x. Let S_k be the space of $k \times k$ real symmetric matrices $r=(r_{i,j})$, $r_{i,j}=r_{j,i}$. It can be identified with the Euclidean space of $k(k+1)/2$ dimensions with the volume element $dr = \prod_{i \le j} dr_{i,j}$. In S_k we consider a convex cone $\mathcal{P}_k$ of positive definite matrices r. For $r \in \mathcal{P}_k$, we write $r>0$. Given r_1 and r_2 in S_k, the inequality $r_1 > r_2$ means $r_1 - r_2 > 0$.

We denote by $V_{n,n-k}$ the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. This can be identified with the set of matrices $x \in \mathfrak{M}_{n,k}$ so that $x'x = I_k$ (the identity $k \times k$ matrix). For $x \in V_{n,n-k}$, dx denotes a measure on $V_{n,n-k}$ which is $O(n)$ left-invariant, $O(k)$ right-invariant, and normalized by

$$\sigma_{n,k} \equiv \mathrm{vol}(V_{n,n-k}) = \int\limits_{V_{n,n-k}} dx = \frac{2^k \pi^{nk/2}}{\Gamma_k(n/2)}$$

[Muir, p. 70], [J, p. 57]. Here $\Gamma_k(\alpha)$ is the Siegel Gamma function (5.9). If $k=1$ then $V_{n,1}$ is the unit sphere S^{n-1}, and for $n=k$, $V_{n,n} = O(n)$ represents the orthogonal group in $\mathbb{R}^n$. Furthermore, $V_{n,n-k} = O(n)/O(n-k)$, and $G_{n,k} = V_{n,n-k}/O(k)$. Each function on $G_{n,k}$ can be regarded as an $O(k)$ right-invariant function on $V_{n,n-k}$.

It is convenient to define the Radon transform (5.1) in a slightly different form as follows:

$$(5.4)\qquad (\mathcal{R}f)(\xi) = \int\limits_{\xi} f(x)dm(x), \quad \xi \in G_{n,k'}, \quad k' > k.$$

It means that $f(x)$ is integrated over all k-frames x in ξ with respect to the relevant normalized measure. If $k = 1$ (the case of totally geodesic transforms on S^{n-1}), (5.4) can be inverted by Helgason's formula (cf. Theorem 4.6). Namely,

$$(5.5)\quad f(x) = c\Big[\Big(\frac{d}{d(u^2)}\Big)^{k'-1} \int\limits_0^u (M_v^*\mathcal{R}f)(x)v^{k'-1}(u^2-v^2)^{(k'-3)/2}dv\Big]_{u=1}.$$

Here f is an even function on S^{n-1}, $\quad c = 2^{k'-1}/(k'-2)!\sigma_{k'-1}$, $\sigma_{k'-1}$ is the area of the unit sphere $S^{k'-1}$, and M_v^* is the mean value operator

$$(5.6)\qquad (M_v^*\varphi)(x) = \int\limits_{\{\xi:\, d(x,\xi\cap S^{n-1})=\cos^{-1}(v)\}} \varphi(\xi)\, dm(\xi), \quad x \in S^{n-1}.$$

Our goal is to extend (5.5) to the higher rank case $k > 1$. The assumption of evenness of f is replaced by right invariance under the group $O(k)$, so that f becomes a function on the Grassmann manifold $G_{n,k}$. The domain of integration in (5.6) can be characterized as the set of all $\xi \in G_{n,k'}$ so that

$$x \cdot \mathrm{Pr}_\xi x = \mathrm{length}(x) \times \mathrm{length}(\mathrm{Pr}_\xi x) \times v = v^2,$$

$\mathrm{Pr}_\xi x$ being orthogonal projection of the unit vector x onto the plane ξ. A natural generalization of (5.6) reads

$$(5.7)\qquad (M_r^*\varphi)(x) = \int\limits_{\{\xi:\, x'\mathrm{Pr}_\xi x = r\}} \varphi(\xi)dm(\xi), \quad x \in V_{n,n-k}, \quad r \in \mathcal{P}_k,$$

where $dm(\xi)$ denotes the relevant normalized measure.

5.3. **Gårding-Gindikin fractional integrals.** The scalar averaging parameter v in (5.6) is replaced in (5.7) by the matrix-valued parameter $r \in \mathcal{P}_k$. According to this, the one-dimensional Riemann-Liouville integral (4.25), arising in Helgason's scheme and leading to (5.5), should be replaced by its counterpart associated to the cone $\mathcal{P}_k$. To this end, we employ *the Gårding-Gindikin fractional integral* defined by

$$(5.8)\qquad (I_+^\alpha w)(r) = \frac{1}{\Gamma_k(\alpha)} \int\limits_0^r w(s)\,(\det(r-s))^{\alpha-d}ds,$$

$$d=(k+1)/2, \qquad Re\,\alpha>d-1, \qquad r\in\mathcal{P}_k.$$

The integration in (5.8) is performed over the "interval" $(0,r)=\{s: s\in\mathcal{P}_k, r-s\in\mathcal{P}_k\}$, and $\Gamma_k(\alpha)$ is the Siegel Gamma function

$$\Gamma_k(\alpha)=\int\limits_{\mathcal{P}_k} e^{-\mathrm{tr}(r)}|\det(r)|^{\alpha-d}dr, \qquad \mathrm{tr}(r)=\text{trace of } r. \tag{5.9}$$

This integral converges absolutely for $Re\,\alpha>d-1$, and represents a product of usual Γ-functions:

$$\Gamma_k(\alpha)=\pi^{k(k-1)/4}\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})\dots\Gamma(\alpha-\frac{k-1}{2}). \tag{5.10}$$

For $k=1$, (5.8) coincides with (4.25). Integrals (5.8) were introduced by L. Gårding [Gå] in 1947, who was inspired by M. Riesz, C. Siegel, and S. Bochner. Substantial generalizations of (5.8) are due to S. Gindikin [Gi1].

Fractional integrals (5.8) enjoy the semigroup property

$$I^\alpha_+ I^\beta_+ f=I^{\alpha+\beta}_+ f, \qquad f\in L^1_{\mathrm{loc}}(\mathcal{P}_k), \qquad Re\,\alpha,\ Re\,\beta>d-1. \tag{5.11}$$

If f is compactly supported away from the boundary $\mathrm{bd}(\mathcal{P}_k)$ and infinitely smooth (we write $f\in C^\infty_c(\mathcal{P}_k)$), then $\alpha\to I^\alpha_+ f$ is an entire function, and (5.11) extends to all $\alpha,\beta\in\mathbb{C}$ so that $I^0_+ f=f$. Let

$$D\equiv D_s=\det\left(\eta_{i,j}\,\frac{\partial}{\partial s_{i,j}}\right), \quad \eta_{i,j}=\begin{cases}1 & \text{if } i=j\\ 1/2 & \text{if } i\neq j.\end{cases} \tag{5.12}$$

For $m\in\mathbb{N}$ and sufficiently good f,

$$D^m I^\alpha_+ f=I^{\alpha-m}_+ f, \qquad Re\,\alpha>m+d-1 \tag{5.13}$$

(see, e.g., [Gå]). If $f\in L^1_{\mathrm{loc}}(\mathcal{P}_k)$, $\alpha\in\mathbb{C}$, $m\in\mathbb{N}$, then (5.13) is understood in the sense of distributions over the space $C^\infty_c(\mathcal{P}_k)$ of test functions. Note that D^m is a hyperbolic differential operator, and (5.13) leads to solution of the corresponding Cauchy problem with data on $\mathrm{bd}(\mathcal{P}_k)$ [Gå].

5.4. **Main results.** In the previous sections the inversion procedure for Radon transforms starts with radial (or zonal) functions. In the higher-rank case the situation is the same, but the notion of zonality should be specified.

Definition 5.1. *Let I_ℓ denote the $\ell\times\ell$ identity matrix, and $\sigma_\ell=\begin{bmatrix}0\\ I_\ell\end{bmatrix}\in V_{n,\ell}$ be the coordinate ℓ-frame (for $\ell=1$, σ_ℓ is the north pole of S^{n-1}). A function f on $V_{n,n-k}$ is called ℓ-zonal if it is left-invariant under all orthogonal transformations $g\in O(n)$ preserving σ_ℓ.*

Given a square matrix A, we denote $|A| = \det(A)$. If x is an orthogonal frame then $\{x\}$ denotes a subspace spanned by x.

Theorem 5.2. *Suppose that $f \in L^1(V_{n,n-k})$ is ℓ-zonal and $O(k)$ right-invariant, $1 \le k \le n-1$.*

(i) *Let $1 \le \ell \le \min(k, n-k)$ $(= \operatorname{rank} G_{n,k})$. Then for almost all $x \in V_{n,n-k}$, one can write*

$$f(x) = f_0(s), \qquad s = \sigma'_\ell x x' \sigma_\ell \in \mathcal{P}_\ell$$

(note that $xx'\sigma_\ell = \mathrm{Pr}_{\{x\}}\sigma_\ell$ and $s^{1/2} \sim \cos(\{x\}, \{\sigma_\ell\}$, i.e. $s^{1/2}$ serves as an analog of cosine of the "angle" between $\{x\}$ and $\{\sigma_\ell\}$).

Furthermore,

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,n-k}} f(x)dx = \frac{\Gamma_\ell(n/2)}{\Gamma_\ell(k/2)\,\Gamma_\ell((n-k)/2)} \int\limits_0^{I_\ell} f_0(s)d\mu(s), \tag{5.14}$$

$$d\mu(s) = |s|^{(k-\ell-1)/2}|I_l - s|^{(n-k-\ell-1)/2}ds \tag{5.15}$$

(this formula is well known for $k = \ell = 1$).

(ii) *Let $\varphi(\xi) = (\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}, 1 \le k < k' \le n-1$. Suppose that*

$$1 \le \ell \le \min(k, n-k') \quad (= \min(\operatorname{rank} G_{n,k},\, \operatorname{rank} G_{n,k'})).$$

Then

$$(\mathcal{R}f)(\xi) = F_0(S), \qquad S = \sigma'_\ell \mathrm{Pr}_\xi \sigma_\ell \in \mathcal{P}_\ell \quad (\sim \cos^2(\xi, \{\sigma_\ell\})) \tag{5.16}$$

where F_0 is an Abel type integral of f_0 (in the Gårding-Gindikin sense). The function $f_0(s)$ can be recovered by the formula

$$f_0(s) = \frac{\Gamma_\ell(k/2)}{\Gamma_\ell(k'/2)}|s|^{-(k-\ell-1)/2} D^m I_+^{m-\alpha}[|S|^{(k'-\ell-1)/2}F_0(S)](s), \tag{5.17}$$

$$\alpha = (k'-k)/2, \qquad m \in \mathbb{N}, \quad m > (k'-1)/2,$$

where D^m is understood in the sense of distributions.

Theorem 5.2 follows from the relevant Abel type representation of the Radon transform of ℓ-zonal functions in terms of Gårding-Gindikin fractional integrals [GR]. There exists an analog of Theorem 5.2 for the dual Radon transform $(\mathcal{R}^*\varphi)(x)$ that averages $\varphi(\xi)$ over all $\xi \in G_{n,k'}$ containing $x \in V_{n,n-k}$. Note also that the rank-one case, corresponding to the geodesic transform on S^{n-1}, contains in these theorems.

A higher-rank generalization of Helgason's formula (5.5) is given by the following

Theorem 5.3. [GR] *Let $f \in L^p(V_{n,n-k})$, $1 \le p \le \infty$ (we identify $L^\infty(V_{n,n-k})$ with the space $C(V_{n,n-k})$ of continuous functions). Suppose that f is $O(k)$ right-invariant and $\varphi(\xi) = (\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}$ where*

$$1 \le k < k' \le n-1, \qquad k+k' \le n.$$

Then for any integer $m > (k'-1)/2$ and $\alpha = (k'-k)/2$,

$$f = \frac{\Gamma_k(k/2)}{\Gamma_k(k'/2)} \overset{(L^p)}{\lim_{s \to I_k}} (D^m I_+^{m-\alpha}[|r|^{\alpha-1/2} M_r^* \varphi])(s), \tag{5.18}$$

where M_r^* is the mean value operator (5.7), and differentiation is understood in the sense of distributions. In particular, for $k'-k = 2\ell$, $\ell \in \mathbb{N}$,

$$f = \frac{\Gamma_k(k/2)}{\Gamma_k(k'/2)} \overset{(L^p)}{\lim_{s \to I_k}} (D^\ell[|r|^{\ell-1/2} M_r^* \varphi])(s). \tag{5.19}$$

A few words about technical tools are in order. A key role in our argument belongs to the following lemma on bi-Stiefel decomposition which generalizes Lemma 3.7 from [Herz, p. 495] and extends the notion of bi-spherical coordinates [VK, pp. 12, 22] to Stiefel manifolds.

Lemma 5.4. *Let k and ℓ be arbitrary integers satisfying $1 \le k \le \ell \le n-1$, $k+\ell \le n$. Almost all $x \in V_{n,n-k}$ can be represented in the form*

$$x = \left[\begin{array}{c} ur^{1/2} \\ v(I_k - r)^{1/2} \end{array}\right], \qquad u \in V_{\ell,k}, \quad v \in V_{n-\ell,k}, \quad r \in \mathcal{P}_k,$$

so that

$$\int\limits_{V_{n,n-k}} f(x)dx = \int\limits_0^{I_k} d\nu(r) \int\limits_{V_{\ell,k}} du \int\limits_{V_{n-\ell,k}} f\left(\left[\begin{array}{c} ur^{1/2} \\ v(I_k - r)^{1/2} \end{array}\right]\right) dv,$$

$$d\nu(r) = 2^{-k}|r|^\gamma |I_k - r|^\delta dr, \qquad \gamma = \frac{\ell-k-1}{2}, \quad \delta = \frac{n-\ell-k-1}{2}.$$

The proof of this statement, given in [GR], follows the same lines as Lemma 3.7 from [Herz]. A simpler proof was suggested by Genkai Zhang [Zh].

Open problem. Theorems 5.2 and 5.3 contain differentiation in the sense of distributions. This is inevitable in the framework of the method, by taking into account convergence conditions of Gårding-Gindikin fractional integrals. It would be interesting to derive pointwise inversion formulas for these integrals and for the Radon transform (5.1). For the rank-one case such formulas are well-known (cf. Theorem 4.7).

6. The generalized Minkowski-Funk transform and small divisors

6.1. **Setting of the problem.** Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, $n \geq 1$. For fixed $\theta \in (0, \pi/2)$, we consider the following integral operators on S^n defined by

$$\mathcal{B}_\theta f(x) = \int\limits_{\{y:\, x\cdot y > \cos\theta\}} f(y) dy, \tag{6.1}$$

$$\mathcal{S}_\theta f(x) = \int\limits_{\{y:\, x\cdot y = \cos\theta\}} f(y) d\sigma(y). \tag{6.2}$$

In (6.1) f is integrated over the spherical cap (or the geodesic ball) of radius θ centered at $x \in S^n$, and (6.2) represents an integral of f over the corresponding spherical section. We call (6.1) and (6.2) *the spherical cap transform* and *the spherical section transform*, respectively. Our interest to these operators is motivated by the following

Problem A. Let K be a star-shaped body in $\mathbb{R}^{n+1}, n \geq 1$, centered at the origin O, and let $\Gamma_\theta, \theta \in (0, \pi/2)$, be a solid cone of revolution with vertex O and a fixed vertex angle 2θ. Is it possible to recover the shape of K if intersection volumes $vol_{n+1}(K \cap g\Gamma_\theta)$ are known for all rotations $g \in SO(n+1)$?

The case $\theta = \pi/2$ corresponds to the Funk problem for half-volumes and the hemispherical transform (3.2).

Question: Are operators (6.1) and (6.2) injective on $C^\infty(S^n)$ for $\theta = \pi/3$?

If you don't like $\pi/3$, you may take $\pi/4$, $\pi/6$, or any other $\theta \neq \pi/2$. This innocent question still has no answer rather than in some particular cases which will be indicated below.

In fact, the problem is much more general. It would be interesting to study kernels (subspaces of zeros) and boundedness properties of these operators (e.g., in Sobolev spaces). In most of the cases it is not clear whether these kernels have finite or infinite dimension. Moreover, as we shall see below, it can happened that the operator is injective and bounded from one Sobolev space to another, but the inverse operator is unbounded (this observation is due to R.S. Strichartz).

As in Theorem 3.2, one can characterize all pairs (L^γ_p, L^δ_p) of Sobolev spaces so that operators (6.1), (6.2) are bounded from L^γ_p to L^δ_p [Ru10]. We restrict our consideration to L^2- Sobolev spaces

$$\mathcal{H}^s(S^n) = L^s_2(S^n), \qquad -\infty < s < \infty.$$

The following example illustrates main difficulties of the problem.

Example 6.1. *Consider the operator $\mathcal{S}_\theta$ on the unit circle (the case $n = 1$). If f has the Fourier decomposition $f(x) \sim \sum\limits_j e^{ijx}\hat{f}(j)$, then*

$$\mathcal{S}_\theta f(x) = f(x+\theta) + f(x-\theta) \sim \sum_j e^{ijx}\hat{f}(j)\cos(j\theta).$$

This operator is bounded from L^2 to L^2 for all θ. If θ/π is a rational number, then $\dim\ker\mathcal{S}_\theta = \infty$, and the inverse operator is bounded from $\tilde{L}^2$ to $\tilde{L}^2$, where $\tilde{L}^2$ is the quotient space $L^2/\ker\mathcal{S}_\theta$. If θ/π is irrational, then $\ker\mathcal{S}_\theta = \{0\}$, the operator is injective, but the inverse $\mathcal{S}_\theta^{-1}$ is unbounded from L^2 to L^2 because $\inf_j|\cos(j\theta)| = 0$. One can take a "smaller" space $\mathcal{H}^s$, $s > 0$, and ask, for which s the operator $\mathcal{S}_\theta^{-1}$ is bounded from $\mathcal{H}^s$ to L^2. From the small divisors theory (see, e.g., [Ar], [Yo]) it is known that the answer depends on how fast θ/π can be approximated by rationals. In other words, the answer depends on diophantine properties of θ/π. Thus a simple question from geometry leads to deep problems in number theory.

This example shows how complicated the problem is in higher dimensions, where instead of usual Fourier series one deals with spherical harmonic expansions. The theory of small divisors grew up from celestial mechanics. It was developed in works by Poincaré, Arnold, Moser, Herman, Yoccoz and others for the torus. To the best of my knowledge, analogues theory for S^n has not been created so far, and we encounter a series of challenging open problems.

The first breakthrough for the operator $\mathcal{B}_\theta$ on S^2 was made by P. Ungar [Un] in 1954. In his paper entitled "Freak theorem about functions on a sphere", he proved that the set of all θ for which $\mathcal{B}_\theta$ is injective and the set of all θ for which $\mathcal{B}_\theta$ is non-injective are both dense in $(0, \pi/2)$. After Ungar, there was no essential progress in the problem, rather than very important reformulation in terms of multipliers for spherical harmonic expansions.

Now we switch to another problem, that came from PDE.

Problem B. Let $\triangle$ be the Beltrami-Laplace operator on S^n. Given a fixed number $\theta \in (0,\pi)$ and a function φ on S^n, the problem is to find a solution $u = u(x,\omega)$, $(x,\omega) \in S^n \times [0,\pi]$, of the spherical wave equation

$$\triangle u = u_{\omega\omega} + \left(\frac{n-1}{2}\right)^2 u \tag{6.3}$$

satisfying

$$u(x,\theta)=\varphi(x), \quad u_\omega(x,0)=0. \tag{6.4}$$

For $\theta=0$ this is the usual Cauchy problem on S^n which was studied by Lax and Phillips [LP]. We wonder, for which θ and φ a solution of the aforementioned "shifted problem" exists, is unique and stable under small perturbation of θ and φ. Furthermore, how many solutions, and which solutions exactly, satisfy the homogeneous problem corresponding to $\varphi\equiv 0$?

The problem can be regarded as a spherical modification of the celebrated Dirichlet problem for the vibrating string with fixed ends. The latter was rejected by Hadamard as ill-posed and studied later by Fox and Pucci [FP], Arnold [Ar] and others using tools of number theory.

It turns out that operators (6.1), (6.2), and the Problem B can be studied simultaneously in the framework of the following analytic family of "fractional integrals"

$$(M_t^\alpha f)(x)=\frac{c_{n,\alpha}}{(1-t^2)^{\alpha-1+n/2}}\int\limits_{x\cdot y>t}(x\cdot y-t)^{\alpha-1}f(y)dy, \tag{6.5}$$

$$c_{n,\alpha}=2^{\alpha-1}\pi^{-n/2}\Gamma(\alpha+n/2)/\Gamma(\alpha), \quad Re\,\alpha>0, \quad t=\cos\theta\in(-1,1).$$

For $Re\,\alpha\le 0$, $M_t^\alpha f$ is defined by analytic continuation. We call (6.5) *the generalized Minkowski-Funk transform* by taking into account that for $n=2$ and $t=0$, the special cases $\alpha=0$ and $\alpha=1$ were studied by Minkowski and Funk. For $\alpha=1$ and $\alpha=0$, $M_t^\alpha f$ coincides with operators (6.1) and (6.2) respectively. For $\alpha=(1-n)/2$, $u(x,\omega)=(M^\alpha_{\cos\omega}f)(x)$, represents the Lax-Phillips solution to the Cauchy problem for the wave equation (6.3). For arbitrary α, it gives a solution to the Cauchy problem to the more general Euler-Poisson-Darboux equation. The latter was studied by many authors on spaces of constant curvature.

6.2. **Reformulation of the problem in terms of the Fourier-Laplace multipliers.** The Fourier-Laplace multiplier $m_t^\alpha(j)$ of M_t^α can be explicitly evaluated using by the Funk-Hecke formula (3.16). We have

$$m_t^\alpha(j)=\Gamma\left(\alpha+\frac{n}{2}\right)\left(\frac{\sqrt{1-t^2}}{2}\right)^{1-\alpha-n/2}P^{1-\alpha-n/2}_{j-1+n/2}(t) \tag{6.6}$$

where $P^\mu_\nu(t)$ $(-1<t<1)$ is the associated Legendre function [Er].

The following statement is immediate from (6.6).

Theorem 6.2. *Let α and s be real numbers,*

$$\alpha \neq -n/2 - \ell, \qquad \ell \in \mathbb{Z}_+; \qquad \rho = \alpha + (n-1)/2.$$

Then

(i) $m_t^\alpha(j) = O(j^{-\rho})$ *as* $j \to \infty$, *and therefore* M_t^α *is a linear bounded operator from* $\mathcal{H}^s$ *into* $\mathcal{H}^{s+\rho}$*;*

(ii) M_t^α *is injective on* $\mathcal{H}^s$ *if and only if* $P_{j-1+n/2}^{1-\alpha-n/2}(t) \neq 0 \; \forall j \in \mathbb{Z}_+$*;*

The exponent $s + \rho$ in (i) is best possible. An analogue of (i) is known in the more general scale L_p^α [Ru10]. The statement (ii) reduces the injectivity problem to studying zeros of the associated Legendre functions $P_{j-1+n/2}^{1-\alpha-n/2}(t)$ regarding as functions of j with fixed t. For $\alpha = 0, 1$, this statement is due to Schneider [Sch1] (see also [BZ]).

The criterion (ii) looks elegant. However, it does not resolve the problem and represents only the first step. We still cannot retrieve from (ii) any answer about injectivity for specific $t = \cos\theta$, say, $\theta = \pi/3$, and further investigation is needed.

Example 6.3. *Let $n = 3$, $\alpha = 1$. One can show that*

$$m_t^1(j) = \frac{3\Gamma(j)}{\Gamma(j+3)} \, \frac{\cos\theta \, \sin(j+1)\theta - (j+1)\sin\theta\cos(j+1)\theta}{\sin^3\theta}. \tag{6.7}$$

Thus the operator $\mathcal{B}_\theta$ (see (6.1)) on S^3 is injective if and only if

$$\tan(j+1)\theta \neq (j+1)\tan\theta, \quad \text{for all integers } j \geq 1. \tag{6.8}$$

The case $\theta = \pi/4$ can be treated manually, and we get the following

Proposition 6.4. *The operator $\mathcal{B}_{\pi/4}$ is injective on S^3.*

It is not clear, how to manage (6.8) in the general case, especially if θ/π is irrational.

In the next two subsections we present some results from [Ru10] which shed some light to the problem.

6.3. **Some partial results.** Let

$$\theta = \beta\pi, \qquad \beta \in (0,1); \qquad t = \cos(\beta\pi), \qquad \beta \neq 1/2,$$

$\tilde{\mathcal{H}}^s = \mathcal{H}^s / \ker M_t^\alpha$ if $\ker M_t^\alpha \neq \{0\}$, and $\tilde{\mathcal{H}}^s = \mathcal{H}^s$ otherwise. One should discriminate between the two cases: β rational ($\beta \in \mathbb{Q}$), and β irrational ($\beta \notin \mathbb{Q}$).

For simplicity, we restrict to the cases $\alpha = 0$ and $\alpha = 1$ corresponding to operators $\mathcal{B}_\theta$ and $\mathcal{S}_\theta$, and suppose that $n \geq 2$ (other cases can be found in [Ru10]). All theorems presented below are proven by making use of explicit representation of the first two terms in the asymptotic expansion of the multiplier $m_t^\alpha(j)$ as $j \to \infty$.

Theorem 6.5. *Let $n \geq 2$ and θ be a rational multiple of π. If $n \neq 3$, then $\mathcal{B}_\theta$ and $\mathcal{S}_\theta$ have a finite-dimensional kernel. If $n = 3$, then* $\dim\ker \mathcal{S}_\theta < \infty$ *and* $\dim\ker \mathcal{B}_\theta = \infty$.

Dimensions of the kernels can be estimated from above. This enables us to treat some special cases manually. For example, the following statement holds.

Proposition 6.6. *Operators $\mathcal{B}_\theta$ and $\mathcal{S}_\theta$ are injective on S^2 for $\theta = \pi/3$ and $\theta = \pi/4$.*

Let us characterize the action of inverses of $\mathcal{B}_\theta$ and $\mathcal{S}_\theta$ in the scale of quotient spaces $\tilde{\mathcal{H}}^s$, assuming that θ is a rational multiple of π. We set $\beta = \theta/\pi = a/b$ where a and b are relatively prime positive integers. Denote

$$\rho = (n+1)/2, \qquad \psi_n(\theta) = b(n-1)(1-2\beta)/4$$

for the operator $\mathcal{B}_\theta$, and

$$\rho = (n-1)/2, \qquad \psi_n(\theta) = b(n-1)(1-2\beta)/4 - b/2$$

for the operator $\mathcal{S}_\theta$. For the sake of simplicity, in the following theorem we write M_θ instead of $\mathcal{B}_\theta$ and $\mathcal{S}_\theta$.

Theorem 6.7. *If $\psi_n(\theta) \notin \mathbb{Z}$, then $(M_\theta)^{-1}$ is bounded from $\tilde{\mathcal{H}}^{s+\rho}$ to $\tilde{\mathcal{H}}^s$, and therefore*

$$M_\theta(\tilde{\mathcal{H}}^s) = \tilde{\mathcal{H}}^{s+\rho}.$$

If $\psi_n(\theta) \in \mathbb{Z}$, then $(M_\theta)^{-1}$ is unbounded from $\tilde{\mathcal{H}}^{s+\rho+\mu}$ to $\tilde{\mathcal{H}}^s$ for all $\mu \in [0,1)$, but it is bounded from $\tilde{\mathcal{H}}^{s+\rho+1}$ to $\tilde{\mathcal{H}}^s$. In this case the following proper embeddings hold:

$$\tilde{\mathcal{H}}^{s+\rho+1} \subset M_\theta(\tilde{\mathcal{H}}^s) \subset \tilde{\mathcal{H}}^{s+\rho}.$$

Example 6.8. *Let $\theta = \pi/3$. In this case $\psi_n(\theta) = (n-1)/4$ for $\mathcal{B}_\theta$, and $\psi_n(\theta) = (n-7)/4$ for $\mathcal{S}_\theta$. Hence*

$$\mathcal{B}_\theta(\tilde{\mathcal{H}}^s) = \tilde{\mathcal{H}}^{s+(n+1)/2} \quad \text{if} \quad n \neq 1 (\mathrm{mod}\, 4), \quad \text{i.e.} \quad n \neq 1, 5, 9, 13, \ldots.$$

Otherwise $\tilde{\mathcal{H}}^{s+(n+3)/2} \subset B_\theta(\tilde{\mathcal{H}}^s) \subset \tilde{\mathcal{H}}^{s+(n+1)/2}$. Similarly,

$$\mathcal{S}_\theta(\tilde{\mathcal{H}}^s) = \tilde{\mathcal{H}}^{s+(n-1)/2} \quad \text{if} \quad n \neq 3 (\mathrm{mod}\, 4), \quad \text{i.e.} \quad n \neq 3, 7, 11, 15, \ldots.$$

Otherwise $\tilde{\mathcal{H}}^{s+(n+1)/2} \subset S_\theta(\tilde{\mathcal{H}}^s) \subset \tilde{\mathcal{H}}^{s+(n-1)/2}$.

In the irrational case we have the following general statement.

Theorem 6.9. *If $t = \cos\beta\pi$, $\rho = \alpha + (n-1)/2$, and β is irrational, then M_t^α is bounded from $\mathcal{H}^s$ into $\mathcal{H}^{s+\rho}$, but $(M_t^\alpha)^{-1}$ is unbounded from $\mathcal{H}^{s+\rho+\mu}$ into $\mathcal{H}^s$ for any $\mu \in [0,1)$.*

Proof. By (6.6),

$$|m_t^\alpha(j)| \le c\frac{\Gamma(j+(n+1)/2-\rho)}{\Gamma(j+(n+1)/2)}[|| j\beta - r|| + O(j^{-1})]$$

where $|| a|| = \inf_{k\in\mathbb{Z}} |a-k|$ is the distance from r to the nearest integer, and $r = (\rho - 1 - \beta(n-1))/2$. By the Tchebychef-Kronecker theorem [HW, Theorem 440, p. 365], there exist infinitely many j's satisfying $\|j\beta - r\| < 3/j$. Hence $\inf_{j\ge 1} j^{\rho+\mu}|m_t^\alpha(j)| = 0 \quad \forall \mu \in [0,1)$, and we are done. □

Remark 6.10. An analogue of this theorem for $\mu \ge 1$ represents an open problem.

6.4. **The cases** $\alpha = (1-n)/2$ **and** $\alpha = (3-n)/2$**.** In these cases $m_t^\alpha(j)$ has an especially simple form:

$$\text{(6.9)} \quad m_t^{(1-n)/2}(j) = \cos(j+\lambda)\beta\pi, \qquad m_t^{(3-n)/2}(j) = \frac{\sin(j+\lambda)\beta\pi}{(j+\lambda)\sin\beta\pi},$$

where $\lambda = (n-1)/2$. For irrational β, the operator M_t^α, $t = \cos(\beta\pi)$, acting from $\mathcal{H}^s$ to $\mathcal{H}^{s+\rho}$, $\rho = \alpha+(n-1)/2$, is injective, and the boundedness of $(M_t^\alpha)^{-1}$ is determined by the following diophantine inequalities:

$$\|q\beta + 1/2\| < cq^{-\mu} \qquad \text{if } \alpha = (1-n)/2, \quad n \text{ is odd;}$$

$$\|(q-1/2)\beta + 1/2\| < cq^{-\mu} \qquad \text{if } \alpha = (1-n)/2, \quad n \text{ is even;}$$

$$\|q\beta\| < cq^{-\mu} \qquad \text{if } \alpha = (3-n)/2, \quad n \text{ is odd;}$$

$$\|(q-1/2)\beta\| < cq^{-\mu} \qquad \text{if } \alpha = (3-n)/2, \quad n \text{ is even.}$$

Theorem 6.11. *For* $\mu \in [0,1)$, $(M_t^\alpha)^{-1}$ *is unbounded from* $\mathcal{H}^{s+\rho+\mu}$ *to* $\mathcal{H}^s$*. If* $\mu \ge 1$*, then* $(M_t^\alpha)^{-1}$ *is bounded from* $\mathcal{H}^{s+\rho+\mu}$ *to* $\mathcal{H}^s$ *if and only if there is a constant* $c > 0$ *such that the corresponding diophantine inequality has only finitely many solutions* $q \in \mathbb{N}$*.*

In the case "$\alpha = 0$, $n = 3$", related to the spherical section transform $\mathcal{S}_\theta$ on S^3, the corresponding diophantine inequality is the simplest, and more information can be obtained. To this end, we recall some facts from number theory.

An algebraic number of degree d is a number which satisfies an algebraic equation $a_0x^d + a_1x^{d-1} + \ldots + a_d = 0$ with integer coefficients, and does not satisfy any similar equation of lower degree.

Roth's theorem([Schm2], p. 116). *If* β *is a real algebraic number of degree* $d \ge 2$*, then for each* $\mu > 1$*, the inequality* $\|q\beta\| < 1/q^\mu$ *has only finitely many solutions* $q \in \mathbb{N}$*.*

A real number β is called a *Liouville number* if it can be rapidly approximated by rationals in the sense that for every $m \in \mathbb{N}$ there exist integers p and $q > 1$ such that $|\beta - p/q| < q^{-m}$. The set $\mathbb{L}$ of all Liouville numbers has Lebesgue measure zero. It includes all rationals and infinitely many (actually c) transcendentals, but no algebraic irrationals [Schm1], [PM].

The following theorem resumes our results for the spherical section transform $\mathcal{S}_\theta$ on S^3 if θ is an irrational multiple of π (the rational case was considered in Theorem 6.7).

Theorem 6.12. *Let* $\theta = \beta\pi, \quad \beta \in I \setminus \mathbb{Q}, \quad I = (0, 1)$.

(a) $\mathcal{S}_\theta : \mathcal{H}^s \to \mathcal{H}^{s+1}$ *is injective.*

(b) $(\mathcal{S}_\theta)^{-1}$ *is unbounded from* $\mathcal{H}^{s+1+\mu}$ *to* $\mathcal{H}^s$ *for all* $\mu \in [0, 1)$ *and all* $\beta \in I \setminus \mathbb{Q}$.

(c) $(\mathcal{S}_\theta)^{-1}$ *is unbounded from* $\mathcal{H}^{s+2}$ *to* $\mathcal{H}^s$ *for almost all* $\beta \in I$.

(d) *If* $\mu > 1$, *then* $(\mathcal{S}_\theta)^{-1}$ *is bounded from* $\mathcal{H}^{s+1+\mu}$ *to* $\mathcal{H}^s$ *for almost all* $\beta \in I$ *(by Roth's theorem this is true for all algebraic numbers of degree* ≥ 2*).*

(e) *If* β *is a Liouville number, then* $(\mathcal{S}_\theta)^{-1}$ *is unbounded from* $\mathcal{H}^{s_1}$ *to* $\mathcal{H}^{s_2}$ *for any pair of Sobolev spaces.*

Open problem. It would be interesting to obtain similar results in other dimensions, and also for the spherical cap transform $\mathcal{B}_\theta$.

7. The Busemann-Petty problem

The "classical" theory of Radon transforms focuses on such problems as injectivity, inversion, action in function spaces, characterization of the range, and related problems of harmonic analysis. Interesting problems of another kind arise if one applies these transforms (and their fractional modifications) to characteristic functions of bounded domains. Then geometric properties of domains come into play, and the researcher finds himself in the realm of geometry with its specific problems, methods, language, and the way of thinking. As an example, we consider the following problem which was posed by Busemann and Petty in 1956 [BP].

Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and $u^\perp$ the central hyperplane orthogonal to the unit vector u. Suppose that

$$vol_{n-1}(K \cap u^\perp) \leq vol_{n-1}(L \cap u^\perp) \quad \forall u \in S^{n-1}. \tag{7.1}$$

Does it follow that

$$vol_n(K) \leq vol_n(L) \quad ? \tag{7.2}$$

Many authors contributed to this problem; see, e.g., [BZh], [BFM], [Ga1]- [Ga3], [GKS], [Ko4], [RZ], [Z2] and references therein.

Theorem 7.1. *The Busemann-Petty problem has an affirmative answer if and only if* $n \le 4$.

For $n = 2$ the answer is obvious. Different proofs of the "if" and "only if" parts were suggested. By making use of results from Sections 2 and 3, these proofs can be essentially simplified as follows.

Each origin-symmetric star body K can be identified with its radial function

$$\rho_K(u) = \sup\{\lambda \ge 0 : \lambda u \in K, \, u \in S^{n-1}\} \qquad (\in C_{even}(S^{n-1})),$$

so that $vol_n(K) = n^{-1} \int_{S^{n-1}} \rho_K^n(u) du$.

Definition 7.2. *We denote by* $\mathcal{K}_+^\infty$ *the class of origin-symmetric convex bodies* K *in* $\mathbb{R}^n$ *such that* $\rho_K \in C^\infty(S^{n-1})$ *and the boundary of* K *has a positive Gaussian curvature at each point.*

Lemma 7.3. *The Busemann-Petty problem has an affirmative answer if (7.1) implies (7.2) for any* $K, L \in \mathcal{K}_+^\infty$.

Proof. Let K' and L' be origin-symmetric convex bodies so that (7.1) holds, but (7.2) fails. There exist approximating bodies $K, L \in \mathcal{K}_+^\infty$ satisfying $K \subset K'$, $L' \subset L$, and $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$ (see, e.g., [Ga1], p. 438, and [Hö], Lemma 2.3.2). This contradicts to the assumption of the lemma. $\square$

Owing to Lemma 7.3, it suffices to prove Theorem 7.1 for $K, L \in \mathcal{K}_+^\infty$. Let us consider the hyperplane Radon transform

$$(Pf)(u,t) = \int\limits_{\mathbb{R}^n} f(x)\, \delta(t - x \cdot u)\, dx, \quad t \in \mathbb{R}, \quad u \in S^{n-1},$$

and its fractional analog [Ru3], [Se1]

$$(P^\alpha f)(u,t) = \frac{1}{\gamma_1(\alpha)} \int\limits_{\mathbb{R}^n} f(x) |t - x \cdot u|^{\alpha - 1} dx, \tag{7.3}$$

$$\gamma_1(\alpha) = \frac{\pi^{1/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((1-\alpha)/2)}, \qquad Re\, \alpha > 0, \quad \alpha \ne 1, 3, 5, \dots .$$

This is the Riesz potential of $(Pf)(u,t)$ in the t-variable (cf. (2.15)). Suppose that $f(x) = \chi_K(x)$ is the characteristic function of a body $K \in \mathcal{K}_+^\infty$, and denote

$$F_K^\alpha(u) = (P^\alpha \chi_K)(u, 0), \qquad (\varphi, \psi) = \int\limits_{S^{n-1}} \varphi(u) \psi(u) du.$$

By passing to polar coordinates, we have

$$F_K^\alpha(u) = \frac{1}{\gamma_1(\alpha)} \int\limits_K |x \cdot u|^{\alpha-1} dx = c_\alpha (U^\alpha \rho_K^{n+\alpha-1})(u), \tag{7.4}$$

where $c_\alpha = 2^{1-\alpha}\pi^{n/2-1}(n+\alpha-1)^{-1}$, and U^α is the operator (3.8) corresponding to the Fourier-Laplace multiplier (3.19). Since $\rho_K \in C^\infty_{even}(S^{n-1})$, and $\rho_K > 0$, then $(U^\alpha \rho_K^{n+\alpha-1})(u)$ extends by analyticity (as the Fourier-Laplace series) to all complex $\alpha \neq 1, 3, 5, \dots$, and this extension belongs to $C^\infty_{even}(S^{n-1})$. Owing to (7.4), the same holds for $F_K^\alpha(u)$ if we exclude $\alpha = 1-n$.

The function $F_K^\alpha(u)$ and the operator U^α play a key role in the sequel. We recall (see (3.10), (3.19)) that $U^0 = 2^{-1}\pi^{(1-n)/2}R$, R being the Minkowski-Funk transform, and the Fourier-Laplace multiplier of U^α obeys

$$u_{j,\alpha} u_{j,2-n-\alpha} = 1, \qquad j \quad \text{even}, \qquad (\text{i.e. } (U^\alpha)^{-1} = U^{2-n-\alpha}), \tag{7.5}$$

provided

$$\alpha \neq 1, 3, 5, \dots, \qquad \alpha \neq 1-n, -n-1, -n-3, \dots. \tag{7.6}$$

By the Parseval equality, (7.5) implies the following

Lemma 7.4. *If $K, L \in \mathcal{K}^\infty_+$, then for $\alpha \in \mathbb{C}$ satisfying (7.6),*

$$(F_K^\alpha, F_L^{2-n-\alpha}) = c \sum_{j \text{ } even} \sum_k (\rho_K^{n+\alpha-1})_{j,k} (\rho_L^{1-\alpha})_{j,k} = c(\rho_K^{n+\alpha-1}, \rho_L^{1-\alpha}),$$

$$c = c_\alpha c_{2-n-\alpha} = \frac{2^n \pi^{n-2}}{(n+\alpha-1)(1-\alpha)}.$$

In particular, for $\alpha = 0$,

$$(F_K^0, F_L^{2-n}) = c_n(\rho_K^{n-1}, \rho_L), \qquad c_n = \frac{2^n \pi^{n-2}}{n-1}. \tag{7.7}$$

Lemma 7.5. *For $K, L \in \mathcal{K}^\infty_+$, the following statements are equivalent:*
(i) $F_K^{2-n} \geq 0$.
(ii) *(7.1) implies (7.2).*
(iii) $\rho_K = Rg$, *where $g \in C^\infty_{even}(S^{n-1})$, $g \geq 0$.*

Proof. By (7.4), $F_K^{2-n} = c_{2-n} U^{2-n} \rho_K$. Owing to (7.5), it follows that $U^0 F_K^{2-n} = c_{2-n}\rho_K$ or $\rho_K = Rg$, $g = 2^{-n}\pi^{3/2-n} F_K^{2-n}$. The latter implies equivalence of (i) and (iii).

Now we prove equivalence of (i) and (ii). Given a hyperplane $H = \{x : x \cdot u = t\}$, let

$$A_K(u,t) \equiv (P\chi_K)(u,t) = vol_{n-1}(K \cap H) \tag{7.8}$$

be a parallel section function of K in the direction u. For each u, the function $t \to A_K(u,t)$ is even and infinitely differentiable in a sufficiently small interval $(-\varepsilon, \varepsilon)$. Since

$$(7.9)\quad F_K^\alpha(u) = \frac{2}{\gamma_1(\alpha)} \int\limits_0^\infty t^{\alpha-1} A_K(u,t)dt, \quad Re\,\alpha > 0, \quad \alpha \neq 1, 3, 5, \dots,$$

then $F_K^0(u) = A_K(u,0)$. Suppose that $F_K^{2-n} \geq 0$ and (7.1) holds. Then $F_K^0(u) \leq F_L^0(u)$, and (7.7) yields

$$\begin{aligned} \int \rho_K^n &= (\rho_K^{n-1}, \rho_K) = c_n^{-1}(F_K^0, F_K^{2-n}) \leq c_n^{-1}(F_L^0, F_K^{2-n}) \\ &= (\rho_L^{n-1}, \rho_K) \leq \Big(\int \rho_L^n \Big)^{1-1/n} \Big(\int \rho_K^n \Big)^{1/n}, \end{aligned}$$

$\int = \int_{S^{n-1}}$. Thus (i) implies (ii).

To prove the converse, we follow some ideas of Lutwak [Lu] and Gardner [Ga1]. Suppose that $F_K^{2-n}(u)$ is negative for some u, and introduce a C^∞_{even}-function $F(u)$ which is non-negative if $F_K^{2-n} < 0$ and 0 otherwise. Let $h = U^{2-n}F$ $(\in C^\infty_{even})$, and define a star body L by $\rho_L^{n-1} = \rho_K^{n-1} - \varepsilon h$, $\varepsilon > 0$. If ε is sufficiently small, then $L \in \mathcal{K}^\infty_+$ (see [Ga1], p. 439). Furthermore, by (7.4) and (7.5),

$$(7.10)\qquad F_K^0 = c_0 U^0 \rho_K^{n-1} = c_0 U^0 [\rho_L^{n-1} + \varepsilon h] = F_L^0 + \varepsilon c_0 F.$$

On the one hand, we have $U^0 \rho_L^{n-1} = U^0 \rho_K^{n-1} - \varepsilon F$, and therefore, $vol_{n-1}(L \cap u^\perp) \leq vol_{n-1}(K \cap u^\perp)$. On the other hand, by (7.10),

$$\begin{aligned} (\rho_K^{n-1}, \rho_K) &= c_n^{-1}(F_K^0, F_K^{2-n}) = c_n^{-1}(F_L^0, F_K^{2-n}) + \varepsilon c_0 c_n^{-1}(F, F_K^{2-n}) \\ &< c_n^{-1}(F_L^0, F_K^{2-n}) = (\rho_L^{n-1}, \rho_K). \end{aligned}$$

This implies $vol_n(K) < vol_n(L)$, that contradicts (ii). □

Let us investigate for which n the inequality $F_K^{2-n}(u) \geq 0$ does hold. By taking into account that $F_K^{2-n}(u)$ is the analytic continuation $(a.c.)$ of the integral (7.9), we have the following

Lemma 7.6. *Let $K \in \mathcal{K}^\infty_+$. If n is odd then*

$$(7.11)\qquad F_K^{2-n}(u) = \lambda_n \int\limits_0^\infty t^{1-n} \Big[A_K(u,t) - \sum_{j=0}^{(n-3)/2} \frac{t^{2j}}{(2j)!} A_K^{(2j)}(u,0) \Big] dt,$$

$$\lambda_n = 2^{n-1} \pi^{-1/2} \Gamma((n-1)/2)/\Gamma(1-n/2).$$

If n is even then

$$(7.12)\qquad F_K^{2-n}(u) = (-1)^{1+n/2} A_K^{(n-2)}(u,0).$$

Proof. A similar statement can be found in [GKS], [BFM], [Ko4]. A simple proof is as follows. By the well-known formula from [GS, Chapter 1, Sec. 3], for $-\ell < Re\,\alpha < -\ell + 1, \quad \ell \in \mathbb{N}$, we have

$$\text{(7.13)} \quad a.c. \int_0^\infty t^{\alpha-1} A_K(u,t)dt = \int_0^\infty t^{\alpha-1} \Big[A_K(u,t) - \sum_{j=0}^{\ell-1} \frac{t^j}{j!} A_K^{(j)}(u,0)\Big] dt.$$

Since all derivatives of $A_K(u,t)$ of odd order are zero at $t = 0$, then for ℓ odd, the sum $\sum_{j=0}^{\ell-1}$ can be replaced by $\sum_{j=0}^{\ell}$, and (7.13) holds for $-\ell - 1 < Re\,\alpha < -\ell + 1$. It follows that for n odd one can set $\ell = n$ in (7.13) and obtain (7.11) with

$$\lambda_n = \lim_{\alpha \to 2-n} \frac{2}{\gamma_1(\alpha)} = \lim_{\alpha \to 2-n} \frac{\Gamma((1-\alpha)/2)}{2^{\alpha-1}\pi^{1/2}\Gamma(\alpha/2)} = \frac{\Gamma((n-1)/2)}{2^{1-n}\pi^{1/2}\Gamma(1-n/2)}.$$

On the other hand, the duplication formula for Γ-functions yields

$$F_K^\alpha(u) = \frac{1}{\cos(\alpha\pi/2)\,\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} A_K(u,t)dt,$$

and for n even we have

$$\begin{aligned} F_K^{2-n}(u) &= (-1)^{1+n/2} a.c. \Big\{ \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} A_K(u,t)dt \Big\}_{\alpha=2-n} \\ &= (-1)^{1+n/2} A_K^{(n-2)}(u,0). \end{aligned}$$

□

Corollary 7.7. *If $K \in \mathcal{K}_+^\infty$ then*

$$\text{(7.14)} \qquad F_K^{2-n}(u) = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^\infty \frac{A_K(u,0) - A_K(u,t)}{t^2}\, dt & \text{if } n = 3, \\ \\ -A_K''(u,0) & \text{if } n = 4. \end{cases}$$

Now we can complete the proof of the positive part of Theorem 7.1. Since $K \in \mathcal{K}_+^\infty$ is convex, then by (7.14), $F_K^{2-n}(u) \geq 0$ for $n = 3, 4$. It remains to apply Lemmas 7.5 and 7.3.

The negative result for $n \geq 5$ can be obtained by making use of the idea belonging to Fallert, Goodey, and Weil [FGW]. The argument is as follows. For $n = 5$, a nice counter-example was given in [GKS] (see also [BZh], [Ga1], [Ko4], [Pa]). For $n > 5$, suppose the contrary, that the problem has an affirmative answer. Let $K \in \mathcal{K}_+^\infty$ be a convex body in $\mathbb{R}^5$. Then $\rho_K = R_{S^4}\varphi$ (see notation in Sec. 3.5), where, by Lemma

7.5 and due to the negative result for $n = 5$, φ is non-positive. Let $K_0 \in \mathcal{K}_+^\infty$ be a convex body in $\mathbb{R}^n$, $n > 5$, so that the restriction of ρ_{K_0} onto S^4 coincides with ρ_K. According to our assumption and Lemma 7.5, $\rho_{K_0} = R_{S^{n-1}} f$ for some non-negative $f \in C^\infty_{even}(S^{n-1})$. Then, by Theorem 3.4, φ is positive, and we arrive at contradiction.

7.1. **The lower-dimensional Busemann-Petty problem.** It is natural to generalize the Busemann-Petty problem (7.1)-(7.2) to sections of dimension less than $n-1$. Let $G_{n,i}$, $1 \le i \le n-1$, be the Grassmann manifold of i-dimensional subspaces of $\mathbb{R}^n$ and let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$. Suppose that

$$vol_i(K \cap \xi) \le vol_i(L \cap \xi) \quad \forall \xi \in G_{n,i}. \tag{7.15}$$

Does it follow that

$$vol_n(K) \le vol_n(L) \quad ? \tag{7.16}$$

This problem was posed in [Z1], [BZh]. For $i = n-1$, this is the usual Busemann-Petty problem. If $i = 2$ and $n = 4$, an affirmative answer follows from that in the case $i = n-1$. Bourgain and Zhang [BZh] proved that for $i > 3$ the answer is negative. This proof was corrected in [RZ]. Another proof was given in [Ko2]. For $i = 2$, or 3, the answer is generally unknown. However, in the special case, when K is a body of revolution, the answer for $i = 2$ and 3 is affirmative [GrZ], [RZ], [Z1].

In the last decade, a series of attempts were made to attack the cases $i = 2$ and $i = 3$; see [Mi], [Ru17], [Ru18], [RZ], [Y] for recent results in this direction and some generalizations.

8. Radon transforms on matrix spaces

The present section deals with Radon transforms of functions of matrix argument. These transforms were introduced by Petrov [Pe1] in 1967. After first publications on this subject [Pe1, Pe2, Sh1, Sh2], it became clear that these transforms have a number of striking distinctive features which do not happen in the classical theory of similar transforms over planes in $\mathbb{R}^n$. Some of these features are still mysterious, therefore the interest to Radon transforms on matrix spaces was renewed in recent years; see [GK, Gra, OR1, OR2, Ru16].

Let us describe the essence of the matter. Let $M_{n,m}$, $n \ge m$, be the space of $n \times m$ real matrices $x = (x_{i,j})$, We fix an integer k, $1 \le k < n$, and let $V_{n,n-k} = \{\xi \in M_{n,m} : \ \xi'\xi = I_{n-k}\}$ be the Stiefel manifold of orthonormal $(n-k)$-frames in $\mathbb{R}^n$. Here ξ' denotes the transpose of ξ, and I_{n-k} is the identity matrix. Let $\mathfrak{T}$ be the manifold of matrix planes τ in $M_{n,m}$ defined by

$$\tau \equiv \tau(\xi, t) = \{x \in M_{n,m} : \xi' x = t\}, \qquad \xi \in V_{n,n-k}, \quad t \in \mathfrak{M}_{n-k,m}. \tag{8.1}$$

The Radon transform associated to planes (8.1) and the relevant dual transform are defined by

$$\hat{f}(\tau) = \int_{x \in \tau} f(x), \qquad \check{\varphi}(x) = \int_{\tau \ni x} \varphi(\tau), \tag{8.2}$$

the integration being performed against the corresponding canonical measures. We call (8.2) the *rank-one Radon transforms* if $m = 1$, and the *higher-rank Radon transforms* if $m > 1$.

To avoid possible confusion, we note that notation $\hat{f}$ for the Radon transform in this section differs from Rf in the previous sections. We denote by $\mathcal{S}(M_{n,m})$ the Schwartz space of infinitely differentiable rapidly decreasing functions on $M_{n,m} \sim \mathbb{R}^{nm}$. The Fourier transform of a function $f \in L^1(M_{n,m})$ is defined by

$$(\mathcal{F}f)(y) = \int\limits_{M_{n,m}} \exp(\mathrm{tr}(iy'x)) f(x) dx, \qquad y \in M_{n,m} \,. \tag{8.3}$$

8.1. **The k-plane transform in $\mathbb{R}^n$.** The case $m = 1$ is well-investigated; see [GGG2, H5, Ru14] and references therein. In this case, $\mathfrak{T} = \mathcal{G}_{n,k}$ is the manifold of non-oriented k-planes in $\mathbb{R}^n$, and integrals in (8.2) represent the usual k-plane transform and its dual. Specifically, if $x \in \mathbb{R}^n$, $\xi \in V_{n,n-k}$, $t \in \mathbb{R}^{n-k}$, and $\tau \equiv \tau(\xi, t) \in \mathcal{G}_{n,k}$, then

$$\hat{f}(\tau) = \int\limits_{\{y \in \mathbb{R}^n : \, \xi' y = 0\}} f(y + \xi t) \, d_\xi y, \qquad \check{\varphi}(x) = \int\limits_{SO(n)} \varphi(\gamma \tau_0 + x) d\gamma,$$

where τ_0 is an arbitrary fixed k-plane through the origin and $d_\xi y$ stands for the Lebesgue measure on the plane $\{y \in \mathbb{R}^n : \xi' y = 0\}$.

We equip $\mathcal{G}_{n,k}$ with the measure $d\tau = d\xi dt$. The dual k-plane transform $\check{\varphi}(x)$ is well defined for any locally integrable function φ. Existence of the k-plane transform is characterized by the following theorem.

Theorem 8.1. [So]
(i) *Let f be a continuous function on $\mathbb{R}^n$, satisfying $f(x) = O(|x|^{-a})$. If $a > k$, then the k-plane transform $\hat{f}(\tau)$ is finite for all $\tau \in \mathcal{G}_{n,k}$.*
(ii) *If $f \in L^p$, $1 \le p < n/k$, then $\hat{f}(\tau)$ is finite for almost all $\tau \in \mathcal{G}_{n,k}$.*

Conditions for a and p in (i) and (ii) are sharp.

Inversion problem for the k-plane transform is overdetermined if $k < n-1$, because in this case, the dimension $(n-k)(k+1)$ of the target space is greater than the dimension n of the source space. Regarding characterization of the range of the map $f \to \hat{f}$ on the Schwartz space of rapidly decreasing functions, see [Gonz1], [Gonz1], [Pe3], [Ri].

The basic inversion methods for the k-plane transform are the following [GGG1], [H5], [Ke], [Ru14]:

1. The Fourier transform method (F. John, I.M. Gelfand, S. Helgason).
2. The method of mean value operators (J. Radon, S. Helgason).
3. The method of Riesz potentials (J. Radon, B. Fuglede, S. Helgason).
4. Decomposition in plane waves (I.M. Gelfand)

In these notes, we focus on the second method. As in Section 2, we average $\varphi = \hat{f}$ over all k-planes at fixed distance $r > 0$ from x by setting

$$\check{\varphi}_r(x) = \int\limits_{SO(n)} \varphi(\gamma\tau_r + x)\, d\gamma. \tag{8.4}$$

Here τ_r is an arbitrary fixed k-plane with the property $\mathrm{dist}(\tau_r, 0) = r$. The map $\varphi \to \check{\varphi}_r$ is also known as the*shifted dual k-plane transform.*

Theorem 8.2. *Let $f \in L^p$, $1 \le p < n/k$. If*

$$g_x(r) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} f(x + \sqrt{r}\theta)\, d\theta$$

is the spherical mean of f, then

$$(\hat{f}\,)^{\vee}_{\sqrt{r}}(x) = \pi^{k/2} (I_-^{k/2} g_x)(r),$$

where $I_-^{k/2}$ is the Riemann-Liouville fractional integration operator (2.4).

This theorem implies the following inversion formulas for the k-plane transform $\varphi = \hat{f}$:

For k even:

$$f(x) = \pi^{-k/2} \Big(-\frac{1}{2r}\frac{d}{dr}\Big)^{k/2} \check{\varphi}_r(x)\Big|_{r=0} \tag{8.5}$$

(a local formula);

For k odd:

$$f(x) = \pi^{-k/2} \Big(-\frac{d}{ds}\Big)^{m} (I_-^{m-k/2} \psi_x)(r)\Big|_{r=0}, \qquad \forall m > k/2 \tag{8.6}$$

(a non-local formula).

8.2. **The higher-rank Radon transform on matrix spaces. Existence and injectivity.** Let us consider the general case of Radon transforms over matrix planes (8.1) in the space $M_{n,m}$ of real rectangular matrices. For details and additional information, the reader is addressed to [OR1, OR2]. Precise meaning of integrals (8.2) is the following:
(8.7)
$$\hat{f}(\xi,t)=\int\limits_{\{y\in M_{n,m}:\,\xi' y=0\}} f(y+\xi t)\,d_\xi y, \quad \check{\varphi}(x)=\frac{1}{\sigma_{n,n-k}}\int\limits_{V_{n,n-k}} \varphi(\xi,\xi' x)\,d\xi,$$
where $\xi\in V_{n,n-k}$, $t\in\mathfrak{M}_{n-k,m}$, $\sigma_{n,n-k}=\mathrm{vol}(V_{n,n-k})$ is defined in Section 5.2, and $d_\xi y$ stands for the Lebesgue measure on the plane $\{y\in M_{n,m}: \xi' y=0\}$.

Remark 8.3. Each $\tau\in\mathfrak{T}$ is an ordinary km-dimensional plane in $\mathbb{R}^{nm}$, but the set $\mathfrak{T}$ has measure zero in the manifold $\mathfrak{T}'$ of *all* km-dimensional planes in $\mathbb{R}^{nm}$. Specifically, by taking into account that $\dim V_{n,m}=m(2n-m-1)/2$ [Muir, p. 67], we have
$$\begin{aligned}\dim\mathfrak{T} &= \dim(V_{n,n-k}\times\mathfrak{M}_{n-k,m})/O(n-k)\\ &= (n-k)(n+k-1)/2+m(n-k)-(n-k)(n-k-1)/2\\ &= (n-k)(k+m).\end{aligned}$$
Hence
$$\begin{aligned}\dim\mathfrak{T}'-\dim\mathfrak{T} &= (nm-km)(km+1)-(n-k)(k+m)\\ &= k(n-k)(m^2-1)>0 \quad \text{if} \quad m>1.\end{aligned}$$

As in the previous section, the first question is for which functions f on $M_{n,m}$ the Radon transform $\hat{f}$ does exist.

Theorem 8.4. (i) *Let f be a continuous function on $M_{n,m}$, satisfying $f(x)=O(\det(I_m+x'x)^{-a/2})$. If $a>k+m-1$, then the Radon transform $\hat{f}(\tau)$ is finite for all $\tau\in\mathfrak{T}$.*
(ii) *If $f\in L^p(M_{n,m})$, and $p_0=(n+m-1)/(k+m-1)$, then $\hat{f}(\tau)$ is finite for almost all $\tau\in\mathfrak{T}$ provided* $1\le p<p_0$.

Conditions for a and p in this theorem cannot be improved.

Our main focus is inversion formulas for the Radon transform $\hat{f}$. First we have to figure out what triples (n,m,k) are admissible for this purpose. It is natural to conjecture that dimension of the target space $\mathfrak{T}$ must be greater than or equal to the dimension of the source space $M_{n,m}$. This is equivalent to $1\le k\le n-m$. The validity of this conjecture can be derived from the following theorem.

Projection-slice theorem. *Let* $f \in \mathcal{S}(M_{n,m})$, $1 \le k \le n-m$. *If* $\xi \in V_{n,n-k}$ *and* $b \in M_{n-k,m}$, *then*

$$(\mathcal{F}f)(\xi b) = \mathcal{F}[\hat{f}(\xi, \cdot)](b). \tag{8.8}$$

Theorem 8.5. (i) *If* $1 \le k \le n-m$, *then the Radon transform* $f \to \hat{f}$ *is injective on* $\mathcal{S}(M_{n,m})$.
(ii) *For* $k > n-m$, *the Radon transform is non-injective on* $\mathcal{S}(M_{n,m})$.

Formula (8.8) can also be used for inversion of the Radon transform on $\mathcal{S}(M_{n,m})$ in terms of the Fourier transform.

8.3. **The Gårding-Gindikin fractional integrals and the method of mean value operators.** In this subsection we demonstrate how the method of mean value operators and the relevant fractional-calculus technique (cf. Section 2, 4.4, 8.1) can be generalized to functions of matrix argument.

Let $\mathcal{P}_m$ be the cone of positive definite symmetric $m \times m$ real matrices,

$$\Gamma_m(\alpha) = \int\limits_{\mathcal{P}_m} \det(r)^{\alpha-(m+1)/2} \exp(-\text{tr}(r)) dr$$

the generalized gamma function associated to $\mathcal{P}_m$; cf. (5.9). The Gårding-Gindikin fractional integrals of a function f on $\mathcal{P}_m$ are defined by

$$(I_+^\alpha f)(s) = \frac{1}{\Gamma_m(\alpha)} \int\limits_{\mathcal{P}_m \cap (s-\mathcal{P}_m)} f(r) \det(s-r)^{\alpha-(m+1)/2} dr,$$

$$(I_-^\alpha f)(s) = \frac{1}{\Gamma_m(\alpha)} \int\limits_{s+\mathcal{P}_m} f(r) \det(r-s)^{\alpha-(m+1)/2} dr, \quad s \in \mathcal{P}_m,$$

[Gå, Gi1]. If f is good enough (e.g., infinitely differentiable and compactly supported away from the boundary of $\mathcal{P}_m$, then both integrals are absolutely convergent if and only if $Re\,\alpha > (m-1)/2$ and extend analytically to all $\alpha \in \mathbb{C}$ as entire functions. Explicit representations of these analytic continuations for the values $\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{m-1}{2}$ are of particular importance.

Theorem 8.6. (i) *The integrals* $I_\pm^\alpha f$ *are convolutions with positive measures if and only if* α *is real and belongs to the "Wallach set"*

$$\mathcal{W} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{m-1}{2}\right\} \cup \left\{\alpha : \alpha > \frac{m-1}{2}\right\}.$$

(ii) *For all $k \in \mathbb{N}$,*

$$(I_+^{k/2} f)(s) = \pi^{-km/2} \int\limits_{\{\omega \in M_{k,m} \colon \omega'\omega < s\}} f(s - \omega'\omega) d\omega,$$

$$(I_-^{k/2} f)(s) = \pi^{-km/2} \int\limits_{M_{k,m}} f(s + \omega'\omega) d\omega,$$

Moreover,

$$(I_\pm^0 f)(s) = f(s).$$

The Gårding-Gindikin fractional integrals were originally introduced in [Gå, Gi1] for the purposes which lie far away from the scope of our consideration. It was striking that they also arise in the theory of Radon transforms if we consider these transforms on the so-called radial functions.

Definition 8.7. *A function f on $M_{n,m}$ is said to be radial if $f(\gamma x) = f(x)$, $\forall \gamma \in O(n)$. Similarly, a function $\varphi(\tau) \equiv \varphi(\xi, t)$ on $\mathfrak{T}$ is radial if $\varphi(\gamma\xi, t) = \varphi(\xi, t)$, $\forall \gamma \in O(n)$.*

One can show that each radial function f on $M_{n,m}$ has the form $f(x) = f_0(x'x)$ and each radial function φ on $\mathfrak{T}$ can be written as $\varphi(\xi, t) = \varphi_0(t't)$.

Theorem 8.8. *If $f(x) = f_0(x'x)$ and $\varphi(\xi, t) = \varphi_0(t't)$, then*

$$\hat{f}(\xi, t) = \pi^{km/2} \, (I_-^{k/2} f_0)(t't),$$

$$\check{\varphi}(x) = c \det(x'x)^{(m+1-n)/2} (I_+^{k/2} \Phi_0)(x'x), \quad c = \frac{\pi^{km/2} \sigma_{n-k,m}}{\sigma_{n,m}},$$

$$\Phi_0(s) = \det(s)^{(n-k-m-1)/2} \varphi_0(s).$$

If $m = 1$, these expressions coincide with the similar ones for the k-plane transforms in $\mathbb{R}^n$ [Ru14].

Theorem 8.8 paves the way to the general (not necessarily radial) case, when the inversion method of mean value operators can be applied. To implement this method, we need in $M_{n,m}$ a certain equivalent of the Euclidean distance.

Definition 8.9. *A matrix-valued distance between points x and y in $M_{n,m}$ is a positive definite matrix defined by*

$$d(x, y) = [(x - y)'(x - y)]^{1/2}. \tag{8.9}$$

A matrix-valued distance between $x \in M_{n,m}$ *and* $\tau = \tau(\xi, t) \in \mathfrak{T}$ *is defined accordingly as*

$$d(x,\tau) = [(\xi' x - t)'(\xi' x - t)]^{1/2}. \tag{8.10}$$

For $m = 1$, both notions coincide with their prototypes in $\mathbb{R}^n$.

We also introduce the *shifted dual Radon transform* of a function $\varphi(\tau) \equiv \varphi(\xi, t)$ on $\mathfrak{T}$. Given a point $z \in M_{n-k,m}$ at matrix distance $s^{1/2}$ from the origin (i.e. $z'z = s$), we define

$$\check{\varphi}_s(x) = \frac{1}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} \varphi(\xi, \xi' x + z) d\xi. \tag{8.11}$$

This is a mean value operator that can be formally written as

$$\check{\varphi}_s(x) = \int\limits_{d(x,\tau)=s^{1/2}} \varphi(\tau)$$

and coincides with the usual dual Radon transform (cf. (8.7)) if $s = 0$.

Theorem 8.10. *Let*

$$f \in L^p(M_{n,m}), \quad 1 \le p < \frac{n+m-1}{k+m-1}.$$

Then

$$(\hat{f})^{\vee}_s(x) = \pi^{km/2} (I^{k/2}_{-} F_x)(s) \tag{8.12}$$

where

$$F_x(r) = \frac{1}{\sigma_{n,m}} \int\limits_{V_{n,m}} f(x + v r^{1/2}) dv$$

is the mean value of f *at* x.

Owing to Theorem 8.10, in order to reconstruct f from $\hat{f}$, it suffices to invert fractional integral on the right-hand side of (8.12) and then pass to the limit as $r \to 0$. To accomplish this procedure, we introduce the differential operator

$$D = (-1)^m \text{det} \left(\eta_{i,j} \frac{\partial}{\partial r_{i,j}} \right), \; \eta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j, \end{cases}$$

acting in the r-variable, $r = (r_{i,j}) \in \mathcal{P}_m$. Let $\mathcal{D}(\mathcal{P}_m)$ be the space of C^∞ functions which are compactly supported away from the boundary of $\mathcal{P}_m$. If $f \in \mathcal{D}(\mathcal{P}_m)$, then $D^j I^\alpha_- f = I^{j-\alpha}_- f$ for all $\alpha \in \mathbb{C}$.

Theorem 8.11. *Let $f \in L^p(M_{n,m})$,*

$$1 \le k \le n-m, \qquad 1 \le p < \frac{n+m-1}{k+m-1}.$$

If $\varphi(\tau) = \hat{f}(\tau)$ and $\Phi_x(s) = \check{\varphi}_s(x)$, then

$$f(x) = \pi^{-km/2} \overset{(L^p)}{\lim_{r \to 0}} (D^{k/2}\Phi_x)(r),$$

where $D^{k/2}$ is understood in the sense of $\mathcal{D}'(\mathcal{P}_m)$-distributions.

The reader is referred to [OOR, OR2] for details and alternative inversion methods, based on implementation of Riesz potentials and wavelet transforms.

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