Before the talk

Motivations of this talk

For Oda, the reaosn is the following:

Some time ago, maybe more than one decade ago, Arakawa and Oda talked about the book of Meyer:
Die Berechnung der Klassenzahl abelscher Körper über quadratischer Zahlkörper. Akademie Verlag, Berlin, 1957.

And Oda promised him to give an expository talk, which was never realized until now.

For our generation, we cannot talk about this kind thing without existence of Takuro Shitani, and actually the importance of this book was pointed out further long time ago, when I was in Sapporo.

So I have some feeling of a debt, from which I should be free.

The 2-nd motivation

We want to have a chance to entre this interesting world of "special values" of zeta abd L functions.

We have to find some "niche", an enough room to survive in this world, anlyising the past and the history of the research.

The 3-rd motivation

In Number Theory, there are some specific themes for which many people make "re-discoveries". Bernoulli numbers, Dedekind sums are such themes.

The senior author is asked sometimes to write referee' reports about the papers discussing such themes.

To know the history will save his time.

Moreover, as many members of Japan Mathematical Society wrote in the quetionaires (*F. enquête*) of Gender-Equality Problem, one of the most discourage chances for them is to find that what he or she discovered newly was showen to be already known.

So probably, we can save the mental peace of some people....,
it would be a very kind work to do.

A short history on investigation of the special values of zeta and L-functions of totally real number fields

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Very Preliminary Version, September 2004

With Appedix: Whittaker functions of the class one degenerate principal series representations of $SL(3, \mathbf{R})$.

Introduction

Even the senior author of this article considers that it seems too early for him to talk about the history of mathematics. This would be more case for the younger. But this workshop is for the memory of Arakawa, so we consider it makes sense to talk about some theme which is a favorite one for him (*cf.* [1, 2, 3, 4, 5, 6]). Probaboly Masanobu Kaneko will talk about these papers of Arawaka directly, we are going to talk about the "circumstances" of these paper.

We can start our talk from the well-known formula of Euler in 1734 on the values of the Riemann zeta function at even positive integers:

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k!)} B_{2k},$$

where B_m is the m-th Bernoulli number. Probably one can find some analogous formulae proved after this result, say, for the Gaussian number field $\mathbf{Q}\sqrt{-1})$. But we do not dig out possible special results here.

The early history of modern investigation of this problem is shortly reviewed in the introductions of Klingen's paper [22] or of Barner's paper [20]. But there is a most extensive history from the dawn of the study of algebraic number field is found in the introduction of Meyer's book [27].

In 20's E. Hecke ([10], p. 219) conjectured the corresponding statement for arbitrary totally real algebraic number fields K, namely

$$\zeta_K(2k) = \pi^{2kn} \Delta^{1/2} r \qquad (k = 1, \ 2, \cdots),$$

where Δ the discriminant, n the degree of K and r a rational number depending on K and k. For quadratic fields there is an elementary proof via Dirichlet L-function shown by Siegel [12]. This was later generalized by Leopoldt [25] for abelian totally real number fields, introducing generalized Bernoulli numbers.

Meyer [27, 28] and Siegel [33] developed the idea of Hecke for real quadratic fields K to have certain "analytic class number field" for abelian extensions L

of K which has "isobaric" ramification at archimedean places. After that this method was extended to the evaluation of certain Hecke L-functions $L(s, K, \chi)$ at positive integers $s = n$ by Meyer [30], Lange [24], Barner [19] and Siegel [34]. Among others, Meyer always pursued *elementary method* to compute these arithmetic invariants by investigating the properties of generalzied Dedekind sums.

The construction of p-adic L-functions, initiated by Kubota-Leopldt in 60's (RIGHT?), become a tredy theme of 70's together with the theory of modular symbols (*cf,* Manin [**?**], Mazur and Swinnerton-Dyer [**?**]), and the people's interest is oriented toward the gemetric intepretation of the known analytic results than to get new analytic expression. However in mid-70's, Shintani's paper [48] brought a paradigm change in the analytic aspect. This method works for arbitrary totally real number fields K of degree n, and gives an effective method of computations of the class numbers of certain abelian extensions L of K and the values of the Hecke $L(s, K, \chi)$ at positive integers s for certain characters χ, by linear combinations of the n products of the values of Bernoulli polynomials at rational numbers. Note that Here appeared the proto-type of higher Dedekind sums. This results is also applied to the construction of p-adic L-functions by Cassou-Noguès [38]. There are some papers of "revisionism" in 70's and 80's.

In 90's new development appeared. In [55, 56], Scech found an ingenous way to generalize the method of Hecke-Meyer-Siegl to arbitary totally real number fields by using conditionally converging Eisenstein series on $GL(n, \mathbf{Z})\backslash GL(n, \mathbf{R})$, whcih is named Eisenstein cocycles. Meanwhile Solomon [**?**, 60] begin to study certain ditribution (RIGHT??), which is called Shintani cocycle, whcih is believed ultimately to give "cohomologous" to Eisenstein cocyles.

Talking about the special values of L-functions, there is a important theme like Lichtenbaum conjecture which almost solved. But we do not touch this kind of aspect of the problem.

1 Before 1950: Hecke, Siegel, and others

It would be wise to leave aside the question whether the unrealized "plan" of Eisenstein or the "dream" of Kroncker covered more general subjects than just the theory of classical complex multiplication, to a serious historian or to an eternal mystery. We begin with the work of Hecke here.

1.1 Hecke

To discuss the invariants of abelian extensions of a real quadratic field K, we should start from Hecke's paper [8], [9] (1917, 1921). In the first paper, he gave a "plan" of this research. It consists of 3 sections: (1) Kronecker's limit formula plus the so-called Hecke's integration formula for real quadratic fields, (2) to write (partial) zeta functions of general algebraic number fields K as pull-back integrations of the Epstein zeta function of degree $[K : \mathbf{Q}]$, (3) the "annoucement of the relative class number formulae for abelian extensions of number fields.

Apart from the arrangement of the original paper, we start from Epstein's zeta function.

1.1.1 *The integral expression of Hecke*

(A): The Epstein zeta functions

Given $g \in GL(n, \mathbf{R})$ or $g \in SL(n, \mathbf{R})$, we associate a $n \times n$ symmteric positive-definite matrix $Y_g = g \cdot {}^t g$ of size n. The whole set of such matrices are denoted by $\mathcal{P}$ or by $\mathcal{P}_1$, respectively, which are isomorphic to a symmetric spaces

$$GL(n, \mathbf{R})/O(n), \qquad \text{or} \qquad SL(n, \mathbf{R})/SO(n).$$

Given a non-zero row vector $\mathbf{m} \in \mathbf{Z}$ of size n, then the value of the real quadratic form $\mathbf{m}Y^t\mathbf{m}$ ($Y \in \mathcal{P}, \in \mathcal{P}_1$) is non-zero, and we can define a series

$$Z(s, Y) := \sum_{\mathbf{m} \in \mathbf{Z}^n - \{0\}} (\mathbf{m}Y^t\mathbf{m})^{-s}$$

for $s \in \mathbf{C}$, which is caledl an Epstein zeta function. It converges uniformly for $Re(s) > \frac{n}{2}$) and invariant under $GL(n, \mathbf{Z})$ or $SL(n, \mathbf{Z})$ with respect to the natural action of $GL(n, \mathbf{Z})$ (resp. $SL(n, \mathbf{Z})$) on $\mathcal{P}$ (resp. $\mathcal{P}_1$) given by

$$Y \mapsto \gamma Y^t\gamma \qquad (\gamma \in GL(n, \mathbf{Z}) \text{ or } SL(n, \mathbf{Z})).$$

Epstein showed the analytic continuation in s and the functional equation of this function. It is a kind of Eisenstein series on $GL(n, \mathbf{Z})\backslash GL(n, \mathbf{R})/O(n)$ or on $SL(n, \mathbf{Z})\backslash SL(n, \mathbf{R})/SO(n)$, belonging to a very small degenerate principal series represenation of $GL(n, \mathbf{R})$ or $SL(n, \mathbf{R})$.

(B): The pull-back to maximal tori

Let K be an algebraic extension of degree n over $\mathbf{Q}$, then we have a natural embedding of the algebraic torus $T_K := Res_{K/\mathbf{Q}}\mathbb{G}_m$ to $GL(n)$, where $Res_{K/\mathbf{Q}}$ is the restriction of scalars of Weil.

In the down-to-earth way, this is defined as follows. Fix a integral basis of the integer ring O_K of K:

$$O_K = \mathbf{Z}\omega_1 + \cdots + \mathbf{Z}\omega_n.$$

Consider the norm form of the linear form $\sum_{i=1}^n x_i\omega_i$ with n variables $x_1, \cdots, x_n$:

$$N(x;\omega) := \prod_{j=1}^n (\sum_{i=1}^n x_i \omega_i^{(j)}).$$

Here

$$\alpha \in O_K \mapsto (\alpha^{(1)}, \cdots, \alpha^{(n)})$$

is the image of the canonical ring homomorphisms :

$$O_K \mapsto O_K \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^n.$$

The action of the multiplicative group $O_K^\times$ on O_K is extended to the action of $T_K(\mathbf{R}) = (O_K \otimes \mathbf{R})^\times$ on $O_K \otimes \mathbf{R} \cong \mathbf{R}^n$. Thus we have compatible groups homomorphisms

$$i_{\mathbf{Z}} : O_K^\times \mapsto GL(n, \mathbf{Z}), \quad i_{\mathbf{R}} : T_K(\mathbf{R}) \mapsto GL(n, \mathbf{R}).$$

Take the norm 1 part $T_K^{(1)}$ in T_K:

$$T_K^{(1)} := \{(x_1, \cdots, x_n) \mid N(x;\omega) = 1\}.$$

Then we have $i_{\mathbf{Z}} : O_K^{(1)} \to SL(n, \mathbf{Z})$.

Choose a point Y_0 in $\mathcal{P}_1$, and restrict $Z(s,Y)$ to the $T_K^{(1)}(\mathbf{R})$-orbit $\mathcal{Q}_{K,\omega}$ of Y_0. Then the function $Z(s,Y)$ $(Y \in \mathcal{Q}_{K,\omega})$ is periodic with respect to $O_K^{(1)}$, and define a function on the compact double coset:

$$O_K^{(1)} \backslash T_K^{(1)}(\mathbf{R})/(T_K^{(1)}(\mathbf{R}) \cap SO(Y_0))$$

of real dimension $r = r_1 + r_2 - 1$, which is a finite extension of a compact real torus of dimension r. Here $SO(Y_0)$ is the stabilizer of Y_0 in $SL(n, \mathbf{R})$ which is isomorphic to $SO(n)$, and $r_1,\ r_2$ are the numbers of real places and complex places of K respectively.

Now we can consider the Fourier expansion of the pull-back $Z(s,*)|_{\mathcal{Q}_{K,\omega}}$:

$$Z(s,Y) = \sum_{\psi \in \widehat{O_K^{(1)} \backslash T_K^{(1)}}(\mathbf{R})} a_\psi(s)\psi(Y).$$

Choose a fundamental domain $D(Y_0)$ in $T_K^{(1)}(\mathbf{R})/(T_K^{(1)}(\mathbf{R}) \cap SO(Y_0))$ with respect to $O_K^{(1)}$. Then the constant term "a_0" with respect to the trivial character "$\psi = 0$", is the average

$$a_0 = \int_{D(Y_0)} Z(s,Y) dv(Y)$$

with an adequate normalization of the invaraint integral dv of $v \in T_K^{(1)}(\mathbf{R})/(T_K^{(1)}(\mathbf{R})\cap SO(Y_0))$. Meanswhile it is given by the Dedekind zeta function :

$$a_0 = w\frac{2^{-r_2 s}\Gamma(\frac{s}{2})^{r_1}\Gamma(s)^{r_2}}{2^{r_1-1}nR\cdot\Gamma(\frac{ns}{2})}\cdot\zeta_K(s),$$

where R is the regulator of K. Similarly other terms a_ψ, which are twisted integral of $Z(s,Y)$ in one side, are also expressed by appropriate zeta functions with Grössencharacter, as shown later. This is the content of §2 of [8].

(C): The case of real quadratic fields

In §1 this is specialized as follows. For a non-zero real number a, we have

$$|a|^{-s} = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty e^{-a^2t}t^{s/2}d\log t.$$

Apply this formula for $a=\mu\in K$ and its conjugate $a=\mu'$, the form the product of the two integrals to obtain a double integral

$$\Gamma(\frac{s}{2})^2|\mu\mu'|^{-s} = \int_0^\infty\int_0^\infty e^{-(\mu^2t+\mu'^2t')}(tt')^{\frac{s}{2}-1}dtdt'.$$

This is, in turn, rewritten by the change of variables

$$t=uv^2,\quad t'=uv^{-2}$$

to have

$$\begin{aligned}\Gamma(\tfrac{s}{2})^2|\mu\mu'|^{-s} &= 4\int_{v=0}^{+\infty}\int_{u=0}^{\infty}e^{-u(\mu^2v^2+\mu'^2v^{-2})}u^s d\log u d\log v.\\ &= 4\Gamma(s)\int_0^{+\infty}(\mu^2v^2+{\mu'}^2v^{-2})^{-s}d\log v.\end{aligned}$$

Choose an (absolute) ideal class A of K, then the partial zeta function $\zeta(s,K,A)$ associated with A is defined by

$$\zeta(s,K,A) := N(\mathfrak{b})^s\sum_{\mu\in\mathfrak{b}/E_K}{}'|N(\mu)|^{-s},$$

if one picks up a fixed ideal $\mathfrak{b}$ belonging to the inverse class A^{-1}. Here E_K is the group of units in O_K. Let $d_K>0$ be the discriminant of K, and let ε be the fundamental unit satisfying $\varepsilon>1$. Then

$$\begin{aligned}\zeta(s,A) &= \tfrac{1}{2}N(\mathfrak{b})^s{\sum}'_{\mu\in\mathfrak{b}/<\varepsilon>}|\mu\mu'|^{-s}\\ &= 2\tfrac{\Gamma(s)}{\Gamma(\frac{s}{2})^2}N(\mathfrak{b})^s\int_0^\infty\sum_{\mu\in\mathfrak{b}/<\varepsilon>}(\mu^2v^2+{\mu'}^2v^{-2})^{-s}d\log v\\ &= 2\tfrac{\Gamma(s)}{\Gamma(\frac{s}{2})^2}N(\mathfrak{b})\int_1^\varepsilon\sum_{\mu\in\mathfrak{b}}(\mu^2v^2+{\mu'}^2v^{-2})^{-s}d\log v,\end{aligned}$$

where the last integrad is an Epstein zeta function for $n=2$.

Recall here that for $n=2$ the Epstein zeta $Z(s,Y)$ is identified with the usual real analytic Eisenstein series

$$f(\tau,s) := Z(s,Y_\tau) = \sum_{(m,n)\in\mathbf{Z}^2-\{0\}}\frac{y^s}{|m\tau+n|^{2s}}$$

on the complex upper half plane $\mathfrak{H}$ by associating $Y_\tau = \frac{1}{y}\begin{pmatrix}1, & x\\ x, & x^2+y^2\end{pmatrix}\in\mathcal{P}_1$ for $\tau = x+\sqrt{-1}y\in\mathfrak{H}$.

Thus summing up the above equalities, we have

Theorem 1.1 *(Hecke's integral expression)*

$$\zeta(s,A) = d_K^{-s/2}\frac{\Gamma(s)}{\Gamma(\frac{s}{2})}\int_1^{\varepsilon^2} f(\tau,s)d\log v$$

in the case of the absolute class in the wide sense.

Remark. We formulated the above formula only for absolute ideal class (in the wide sense). For the cases of ring class or ray classes, good references are Barner [19], Korollar (3.21), or Siegel [33], Proposition**.

1.1.2 *Kronecker's limit formula*

When $n = 2$ we have the following.

Theorem 1.2 *(Kronecer's limit formula) $Z(s,Y)$ has a simple pole at $s = 1$ with residue π and has the Laurent expansion :*

$$f(\tau,s) = \frac{\pi}{s-1} + 2\pi(\gamma_E - \log 2 - \log(\sqrt{y}|\eta(\tau)|^2)) + O(s-1).$$

Here γ_E is the Euler constant, η is the Dedekind eta function, and $O(s-1)$ is the Landau symbol.

Input this to the formula for a_0 (Theorem 1.1). Then we have the following.

Theorem 1.3 *(Hecke's integral expression + Kronecker limit formula) Let ε be the fundamental unit of O_K, and let $\varphi(\mathfrak{b})$ be the constant term of the Laurent expansion at $s = 1$:*

$$\zeta(s,K,A) = \frac{\log\varepsilon^2}{\sqrt{d_K}}\frac{1}{s-1} + \frac{2}{\sqrt{d_K}}\varphi(\mathfrak{b}) + O(s-1).$$

Then it has an integral expression:

$$\varphi(\mathfrak{b}) = \gamma_E\log\varepsilon^2 - \int_1^{\varepsilon^2}\log(d_K^{\frac{1}{4}}(\frac{\omega-\bar{\omega}}{2i})^{\frac{1}{2}}\eta(\omega)\eta(-\bar{\omega}))d\log v.$$

Here the $\mathbf{Z}$-basis $\{\beta_1,\beta_2\}$ of $\mathfrak{b}$ satisfies

$$\beta_1\beta_2' - \beta_2\beta_1' = N(\mathfrak{b})\sqrt{d_K} > 0,$$

and the two number $\omega,\omega' \in \mathbf{C}$ are given by

$$\omega = -\frac{\beta_2 v - i\beta_2'}{\beta_1 v - i\beta_1'}, \quad \bar{\omega} = -\frac{\beta_2 v + i\beta_2'}{\beta_1 v + i\beta_1'}.$$

This, together with its variations, is fundamental to invetigate the L-functions on a real quadratic field K associated with ring class groups or ray class groups. This formula was the starting point of Meyer's research.

1.1.3 *Holomorphic Eisenstein series in the Hilbert modular case*

In the paper [10], Hecke consider Eisenstein serie of two complex variables belonging to the Hilbert modular group associated with a real quadratic field $\mathbf{Q}(\sqrt{d})$. For a fixed character χ_k of the (absolute) ideal class group such that for the principal ideal (λ) its value is given by the signature $[\chi_k((\lambda)) = (sgn\lambda\lambda')$, he consider an Eisenstein series

$$G_k(\tau,\tau';\mathfrak{K},\chi_k) = \sum_{(c,d)_1, c\equiv d\equiv 0\ (\mathfrak{a})}' \frac{|N(\mathfrak{a})|^k \chi_k(\mathfrak{a})}{(c\tau+d)(c'\tau'+d')},$$

which defines a (holomorphic) Hilbert modular form of dimension $-k$ (i.e., of weight k) if $k \geq 3$. And when $k = 2$ we need the method of regularization (Hecke's trick). This Eisenstein series has the Fourier expnasion:

$$G_k(\tau,\tau';\mathfrak{K},\chi_k) = A_k(\mathfrak{K},\chi_k) + B_k \sum_{\nu>0} c_k(\nu,\mathfrak{K},\chi_k) q^{\nu} q'^{\nu'}$$

with

$$A_k(\mathfrak{K},\chi_k) = |N(\mathfrak{a})|^k \chi_k(\mathfrak{a}) \textstyle\sum'_{(\mu)_1,\mu\equiv 0(\mathfrak{a})} \frac{1}{N\mu}^k = e(1)\zeta(k;\mathfrak{K},\chi_k),$$
$$B_k = \tfrac{(2\pi)^{2k}}{\Gamma(k)^2} \tfrac{1}{\sqrt{d}^{2k-1}},$$

Since, at least in retrospect, it is not difficult to see that the coefficients $c_k(\nu,\mathfrak{K},\chi_k)$ are rational numbers, it is natural to expect that the ratio $A_k(\mathfrak{K},\chi_k)/B_k$ is also a rational number. (And this is the case for quadratic field by using Dirichlet L-function.)

Anway, Heck did not seem to publish any proof of "Satz 3" in [10]. However this paper contains two ideas: (1) to investigate the values of Dedekind zeta functions or their varaints at positive integers, one should investigate the constant terms of appropriate Eisenstein series, and (2) write the Dedekind zeta functions and related L-functions (or their values at natural numbers) as the Fourier coefficients of the "hyperbolic Fourier expansion of Eisenstein series along tori in $SL(2)$ or $GL(2)$.

Note that these are both Fourier expansions: one is parabplic, the other hyperbolic. These appear repeatedly in the later papers by other mathematicians.

1.2 Siegel

Some authors quoted Siegel's paper [12] (1922) as another source of the problem. But the theme of this paper is to consider Waring's problem for real quadratic fields, and his interest in this paper is concentrated into "singular series" in the sense of Hardy-Littlewood. One find a comments in the last few pages (pp. 152–153). Here he wanted to describe the asymptotic behaviour of the singular series: to show that the leading coefficient is a rational number. To have this he needed a finite sum expression of the values of the Dirichlet L-function associated with the real quadratic character at the positive integers, in terms of the values of Bernoulli polynomials at rational numbers. This should be considered as a proto-type of generalized Bernoulli numbers later developed by Leopoldt [25].

This kind of results appears sometimes in the later papers on quadratic forms by Siegel. As far as we can see from his publications, there seems to be no systematic discussion about this kind of problem up to the time of Tata Lecture Note [33].

1.3 Herglotz

The first responce to the paper [9] of Hecke came from Gustav Herglotz [11] (1923), whose main concerns were analysis and differential geomtery but also wrote several interesting paper on number theory. He tried to rewrite Hecke's integral (Theorem 1.1) in a elementary forms. The Dedekind sums show up already in his paper as a summand in the sum expression of Hecke's integral. His method is to approach the two ends of the Hecke integral to the cusps of $SL(2,\mathbf{Z})$, which was later used by Siegel [33]. Other important summand is written in terms of Gauss function $\psi(z) = \frac{d}{dz}\log\Gamma(z)$. Almost the same result together with a certain function $F(x)$ (by the same symbol as that of Herglotz by chance! or by mediocre?) was reproduced by Zagier as a part of his paper [53] (see the added in proofs).

Herglotz's fomulae:

$$\varphi(\mathfrak{A}) = \frac{\pi^2}{\sqrt{d_K}} + \frac{2E\log\varepsilon^2}{\sqrt{d_K}} + \frac{2(a\ n)-u}{6n}\frac{\pi^2}{\sqrt{d_K}} - \frac{4}{\sqrt{d_K}}\mathrm{Lim}_{\tau=0}RJ(\frac{a}{b}+i\tau).$$

$$\varphi(\mathfrak{A}) = C_0 + \frac{v\pi^2}{3n} + \frac{2(a\ n)-u}{6n}\frac{\pi^2}{\sqrt{d_K}} + \frac{\log n}{\sqrt{d_K}} + \frac{2}{\sqrt{d_K}}(\Psi(\frac{s}{n}|\frac{a}{n}) - \Psi(\frac{s'}{n}|\frac{a}{n})).$$

d_K	: the discriminant of K;
ε	: the fundamental unit of K;
E	: some rational number, hopefully not so important
$(a\ n)$	: **Dedekind sum** ;
$a,\ b,\ c,\ d$	: rational numbers determined by $SL(2,\mathbf{Z})$-realization of ε;
$n = \|b\|, u = a + d, v$	:;
$RJ(z)$	: a part of Hecke's integral;
$\Psi(x)$	: $d\log\Gamma(x+1)/dx$, Gauss's function

1.4 Dedekind sums by Rademacher and others

Richard Dedekind (1831-1916) began the study of *Dedekind sums* which appears in the transformation formula of the Dedekind eta function $\eta(\tau)$ with respect to $SL(2,\mathbf{Z})$. Hans Rademacher (1892–1969) developed the investigation of Dedekind sums, as found in his Collected Papers [14] (the papers 26–29, 31, 44, 53, 56, 57, 59, 67) and in the monograph [15] of Rademacher-Grosswald. The first motivation for him to investigate this seemed to be the study of the partition numbers $p(n)$ (or its generationg function).

For our purpose, it suffices to review the part which is closely related to the transformation formula of $\log\eta$:

$$\begin{aligned}\log\eta(\tau) &= \tfrac{\pi i\tau}{12} + \textstyle\sum_{m=1}^{\infty}\log(1-q^m)\\ &= \tfrac{\pi i\tau}{12} - \textstyle\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}q^{mr} \quad (q = e^{2\pi i\tau} \text{ with } \tau\in\mathbf{C}, \mathrm{Im}\tau > 0)\end{aligned}$$

of $\eta(\tau) = e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1-e^{2\pi im\tau})$. The classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{\mu=1}^{k-1}((\frac{h\mu}{k}))\cdot((\frac{\mu}{k})) = \sum_{\mu=1}^{k-1}\frac{\mu}{k}((\frac{h\mu}{k}))$$

with

$$((x)) := \begin{cases} x-[x]-1/2, & \text{if } x \notin \mathbf{Z} \\ 0, & \text{if } x \in \mathbf{Z} \end{cases}.$$

For an element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$, we define

$$\Phi(M) := \begin{cases} b/d & \text{for } c=0 \\ \frac{a+d}{c} - 12(\text{sign}c)s(d,|c|) & \text{for } c \neq 0. \end{cases}$$

Then we can write the transformation formula of $\log\eta$ as

$$\begin{aligned}\log\eta(\frac{a\tau+b}{c\tau+d}) &= \log\eta(\tau) \\ &\quad + \frac{1}{2}(\text{sign}c)^2\log(\frac{c\tau+d}{i\text{sign}c}) + \frac{\pi i}{12}\Phi(M).\end{aligned}$$

(*cf.* p. 49, Formula (60) of [15]).

An important result is the composition law, given as follows (p. 51, Formula (62) of [15]).

Theorem 1.4 *([13] (1931)) If $M'' = M'M \in SL(2,\mathbf{Z})$, i.e., if*

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have

$$\Phi(M'') = \Phi(M) + \Phi(M) - 3\text{sign}(cc'c'').$$

This propeprty of Φ, together with the reciprocity law of the Dedekind sum:

$$s(k,h) + s(h,k) = -\frac{1}{4} + \frac{1}{12}(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}),$$

is considered essential in *elementary* computation of the Dedekind sum, and this is generalized by Meyer [28] later.

A start of this kind generalization is found in the papaer of Apostol [16] (1950). Here the function $\log\eta$ is replaced by a Lambert series :

$$G_p(\tau) := \sum_{n=1}^{\infty} n^{-p}q^n/(1-q^n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{-p}q^{mn} \quad (p \geq 1).$$

Then $p=1$ this is $-\log$ of the generating function of the partition $p(n)$. Here a generalization of Dedekind sum :

$$s_p(h,k) := \sum_{mu=1}^{k-1}\frac{mu}{k}\bar{B}_p(h\mu/k)$$

is introduced, where $\bar{B}(x)$ is the p-the Bernoulli function, i.e., $B_p(x - [x])$. The reciprocity law of $s_p(h, k)$ is Theorem 1 (p. 149) and the transformation formula of $G_p(\tau)$ is given as Theorem 2 (p. 153). There was a missing term in the transformation formula, and this was corrected by Iseki [17] (1957).

Remark. We try to pin-point the papers of Leonard Carlitz, which is related to the Dedekind sums. His papers are sometimes quoted more often than the papers of the initiative authors on the same particular themes. We are at a loss before the numbers of his papers, and are forced to leave the question for the readers themselves to find the adequate references among his publications.

2 From 1951 untill 1969

In 50's there was a development to have class number fomula of abelian extension of real quadratic fields, mainly by Meyer. He evaluated the values at $s = 1$ of the Hecke L-series $L(s, K, \chi)$ for real quadratic fields K, starting from Hecke's integration formula. The final result is to have an expression of $L(1, K, \chi)$ as a product of power of π and a rational number which is effectively computable by finite steps utlizing Dedekind sums. A similar result is discussed by Siegel in Chapter 2 of his Tata Lecture Note [33].

In 60's this method is extended to the problem of the evaluation of $L(s, K, \chi)$ at positve integer $s = m$ with appropriate parity. This also was initiated by Meyer [30] for the case of $s = 2$, and later extended to general m by Lange [24], Barner [20] and Siegel [34].

2.1 Class number formula by Meyer and Siegel

2.1.1 *Meyer's monograph, 1957*

The study of Curt Meyer on the class-number formula is found mainly in [27] and [28]. The former one, the book [27] is an improved vesion of his dissertation at Berlin in 1950, written under the guidance of Helmut Hasse. The second one [28] is his "Habilitationsschrift" at Hamburg in 1955. These are very elaborated and important papers. This book has an extensitive introduction on the history of the problem, but it is not so kind about the explanation of the organization of its contents.

It is consists three chapters, but the main body of the book is the chapter 2 (§4 - §13) in the total 15 sections of the title : "Kroneckersche Grenzformeln für die L-Funktionen der Ringklassen und der Strahlklassen in quadratischen Zahlkörpern und ihre Anwendung auf die Summation für L-Reihen".

Though it is true that he prove a number of applications of the Kronecker limit formula to express the values of L-functions $L(s, \Omega, \psi)$ or $L(s, \Omega, \chi)$ at $s = 1$, associated with an imaginary or real quadratic field Ω and for a ringclass character ψ or a rayclass character χ, a bit confusing point is that all the titles of the sections §4-§13 has the same word "Kronecker Grenzformeln" in their titles, and do not indicate the contents properly.

In the first two section (§4, §5), he considers the case of imaginary quadratic fields. Here are variants of the Kronecker limit formula. The Laurent expansion modulo $O(s-1)$ of the partial zeta functions associated with ring classes and ray classes are described by using the "singular values" of certain elliptic modular invaraints introduced by Hasse. This is a part of the classical theory of complex multiplication. This is known from the time of Kronecker, more or less.
Remark. A most substantial result from slightly different view point is found in the paper Ramachandra [31] (1964), which is the source of *elliptic units.*

From §6 to §13, the case of real quadratic fields is investigated. But as we see soon §6 and §7 have no substantial results.

To explain this it is better to use the table of the (infinite types) of the ray class characters in Siegel [33], Chap. II, §5, p. 115 :

Type	(i)	(ii)	(iii)	(iv)
Conductor at infinity	1	p_∞	$\mathfrak{p}_\infty$	$\mathfrak{p}'_\infty$
The associated sign character	1	$\frac{N(\lambda)}{\lvert N(\lambda)\rvert}$	$\frac{\lambda}{\lvert\lambda\rvert}$	$\frac{\lambda'}{\lvert\lambda'\rvert}$

The characters of type (i) is discussed in §6 and §7 in Meyer's book. But he had just introduced the notations of the integrations of elliptic modular forms and pointed out that they have nice formal properties as "arithmetic invaraints" (p. 48–50 in §6 and p. 54–56 in §7). His original and substantial results are §9–§13, the last 5 sections in Chapter II of [27]. This is the characters of type (ii) in the terminology of Siegel [33].

Meyer's finial end is to write the results in "elementary ways". His investigation proceeds in 3 steps:

(i) Apply the integration formula of Hecke, to write the difference $\zeta(1, \Omega, A) - \zeta(1, \Omega, A^*)$ as a finite sum terms of the form $\log\{f(M(\tau))/f(\tau)\}$ with some modular form f and with some element $M \in SL(2, \mathbf{Z})$ representing units in O_Ω. Here A and A^* defines the same ideal class in the wide sense, but distinct in the narrow sense (impizite Grenzformel, §9, §10).

(ii) The term are written in terms of some (generalized) Dedekind sums (explizite Grenzeformel, §12, §13).

(iii) These Dedekind sums are investigetd deeply, among other their composition rule are studied ([28]).

Siegel never discussed the last step (the (generalized) Dedekind sums) in his writings. He also could not get substantial results for the characters of type (i), for which Meyer had no results.

To handle the ray classes, he need the so-called "second Kronecker limit formula" formulated by using σ functions of Weierstrass or Fricke-Klein. The transformation formula of these σ functions was discussed in §11, and the multipliers of the tranformation are generalized Dedekind sums.

Probably we cannot get much by looking the formula in his book. Anyway substantially equivalent results are written in a more readable way by Siegel [33].

Remark 0. We said nothing about §8 of Meyer [27]. This is the cases of characters of types (iii) and (iv). We shall remark again in the subsubsection after the next.

Remark 1. Thus the essence of Meyer's results is to give the relative class number of the extension L/L_0 where L/Ω is an abelian extension and L_0 totally real and L its CM extension. Probably it is not unexpected, such problem is decribed by generalized Dedekind sums.

Remark 2. The paper of Zagier [53] is quite instructive to understand the essence of Meyer's work (*cf.* §4 of [53]). But the part of the continued fraction (§5) will probably strongly related to the composition rule of the generalized Dedekind sums of [28].

Remark 3. The book [27] of Meyer is quite unreadable. Other people around us (*e. g.* Shintani, Arakawa) had the same opinion. There are a few reasons of this difficulty: say, it uses the symbols in the papers of Hasse; and as a whole the organization of book is not good, becaused all the substancial results are collected to the single chapter; but the most serious fact is that there is no statement written as Theorem (Satz), Proposition, and Lemma (Hilfassatz). This is also the case for other papers of him and his student Lange. The readers have to find the important statements by themselves. And if they want to quote them, they have the challenging jobs to point out the important statements by the formula numbers.

2.1.2 *Generalized Dedekind sums by Meyer, 1957*

To handle the L-functions associated with the characters of the ray class group, Meyer has to handle the transfomation formula of elliptic modular forms of higher level. Then we have a new multiplier in the transfromation formula different from Dedekind sums: this is *generalized Dedekind sums* and the reciprocity law and *the composition law* is the theme of Meyer's paper [28]).

This paper has never been quoted by any one, except for by the book of Rademacher and Grosswald on Dedekind sums. It is in the reference of the book of Hirzebruch-Zaiger, but I am not sure whether they rellay remark to this paper.

2.1.3 *Class number formula by Siegel, 1961*

This is the theme of Chapter 2, §5 of the famous Tata Lecture Note [33] by him.

In Chapter I, he discussed not only the 1-st Kronecker formula for Eisenstein series $f(\tau, s)$ of level 1, but also the 2-nd Kronecker limit formula for the Eisenstein series of higher level belonging to the principal congruence subgroups of level $f > 1$. Here in place of Dedekind η function, there appears the f-division values of the (odd) theta function $\vartheta_1(\tau, u)$, which is essentially the same function as Weierstrass σ (*cf.* Whittaker-Watson, *Modern Aanalysis*, §*, p. ****).

After recalling the classical result on Pell equation by Kronecker in §1 of Chapter II, Siegel's lecture proceeds in an almost pararell way to that of the book of Meyer [27]: application of Kronecker limit formula to imaginary quadratic fields (including the comments to the results of generalized Gauss sums by Hasse) in §2, an review of Hecke's integral expression in §3, real quadaric fields and ray class characters χ of type (i) in §4, and real quadratic fields and ray class characters χ of type (ii) in §5; and gives essetially the same results or "non-results", as Meyer's. But his presentation and the arguments of the proofs are very lucid.

The case of ray class characters χ of infinite type (i)
The main "result" of this case is Theorem 11 (Type (i)) in p. 124:

Theorem 2.1 *For a ray class character with conductor $\mathfrak{f} \neq (1)$ and infinite type $v(\lambda) \equiv 1$ in K, the value of $L(s, \chi)$ at $s = 1$ is given by*

$$L(1, \chi) = \frac{1}{2\varepsilon(\mathfrak{f})\sqrt{N(\mathfrak{f})}} \sum_B \bar{\chi}(\mathfrak{b}_B) \int_{\tau_0}^{\tau_0^*} \log |\varphi(v_B, u_B, \tau)|^2 \frac{dz}{F_B(z)},$$

where B runs over all the ray classes modulo $\mathfrak{f}$ and $F_B(z)$ is given as

$$F_B(\tau) = \frac{\sqrt{d_K}}{\omega - \omega'}(tau - \omega)(\tau - \omega') = a_1\tau - 2 + b_1\tau + c_1$$

with a primitive form satisfying $a_1 > 0$ $b_1^2 - 4a_1c_1 = d_K$.

We do not explain the function $\varphi(v_B, u_B, \tau)$, but it is a quotient of the divion value of ϑ by $\eta(\tau)$ which appears in the Kroneker's second limit formula. The point here is that there still remain the integration of $\log |\varphi(v_B, u_B, \tau)|$ which is still a transcendent which is not evaluated in "elementary way".

But this gives no essential new results, beacause this is simply a paraphrase of Hecke's formula (the same as Meyer at §6 and §7 in his book [27]. The difference is that Siegel mentioned the method of Herglotz.
Remark. This direction seems to be still a dead-end until the present, as far as we know. The point $s = 1$ is not critical value in the sense of Deligne (Corvalis, 1979).

The case of ray class characters χ of infite type (ii) In this case the main result is Theorem 12 (Type (ii)) in p. 133. This is substantially the same case as Meyer. We write some formula to make the talk substancial.

Theorem 2.2 *For a ray character χ of conductor $\mathfrak{f}$ with infinite type $v(\lambda) = N(\lambda)/|N(\lambda)|$ in a real quadratic field K of discriminat d_K, we have*

$$L(1,\chi) = \frac{\pi^2}{\varepsilon(\mathfrak{f})\sqrt{N(\mathfrak{f})d_K}} \sum_B \bar{\chi}(B)G(B)$$

where the summation is over all ray class B modulo $\mathfrak{f}$ and

$$G(B) = \frac{v(\beta_1)}{2\pi i} \times \begin{cases} [\log \varphi(v_B, u_B, \tau)]_{\tau_0}^{\tau_0^*}, & \text{for } \mathfrak{f} \neq (1) \\ [\log(\sqrt{(\tau-\omega)(\tau-\omega')})\eta^2(\tau))]_{\tau_0}^{\tau^*}, & \text{for } \mathfrak{f} = (1). \end{cases}$$

WHAT IS β_1 I HAVE TO EXPLAIN.

The next step is to compute the value $\frac{1}{2\pi i}[\log \varphi(v_B, u_B, \tau)]_{\tau_0}^{\tau_0^*}$ whihc is a rational number $\in \frac{1}{12f}\mathbf{Z}$ with $f = N(\mathfrak{f})$.

Theorem 2.3 *Under the same hypothesis and notation,*

$$G(B) = v(\beta_1)\{\mathcal{P}_2(u_B)\frac{a+d}{c} - \sum_{k=0}^{c-1} \mathcal{P}_1(\frac{k+u_B}{c})\mathcal{P}_1(\frac{k+u_B}{c} - v_B) - \nu(\mathfrak{f})\}$$

where $u_B = v_B = 0$ for $\mathfrak{f} = (1)$. Here $\mathcal{P}_1(x), \mathcal{P}_2$ are the Bernoulli functions periodic modulo $\mathbf{Z}$ such that $\mathcal{P}_1(x) = x - [x] - \frac{1}{2}$ if $0 \leq x < 1$,

$$\mathcal{P}_2(x) = \frac{1}{2}\{(x^2 - [x])^2 - (x - [x])\} + \frac{1}{12}$$

, and the constant $\nu(\mathfrak{f})$ equals to $\frac{1}{4}$ if $\mathfrak{f} = (1)$ or equals to 0 otheriwese.

The method of proof basically use the idea to Herglotz, to approach z_0 to ∞ and z_0^* to another cusp.

We do not write here the exact formlua, but his computation is used again in [34] in the computation of the special values, and here appears a "Lambert series" at least implicitly, which is an iterated indefnite integral of a holomorphic Eisenstei series.

2.1.4 *The case of the ray characters of infinite type (iii), (iv): An unsloved problem by Meyer and Siegel*

If one reads the arguments of Meyer and Siegel, one finds there is one difficult case that is not completely solved either by Meyer or by Siegel. This this the cases of character type (iii) and (iv). In Meyer [27], this case is handle in §8 (p. 56–66) of his book. However the formula involves the terms which are expressed by modified Bessel functions. As far as we know no one gave any algebraic expression of $L(1, K, \chi)$ in this case, and probably it may not have such expression. There is a paper by Shintani [49] to write the special values $L(m, K, \chi)$ for this type of χ by using Barnes double gamma function Γ_2.

2.2 Leopoldt, 1958, 1962

We cannot say much about the work of Leopoldt here. But in his survey article [**?**], he discussed *generalized Bernoulli numbers*, *p-adic L-functions and their values at* $s = 1$, and integral basis of the integer ring of the abelian extensions over $\mathbf{Q}$ in terms the Gaussian sums. These becaome the proto-type of subsequent generalization for other algbraic number fields. In the introduction of this paper, the author, quoting the paper of Hasse at 1952, mentioned the so-called "Hasse's program", which seems to mean the attempt and effort to have effectively computable way to have arithmetic invariants of algebraic numbers fields.

2.3 Klingen's papers, 1962

There are two papers [22] and [23] related to our theme. The main theorem of the first paper [22] is the following.

Theorem 2.4 *Let K be a totally real algebraic number field of degree n and disciminant d_K. For a natural number k let χ_k be a character of the ideal class group of K in the narrow sense such that its infinite type is given by $\chi((\alpha)) = N(\alpha)^k$ for principal ideals (α). Then for the partial Dedekind zeta function of any given ideal class A, we have*

$$\zeta_K(k, A, \chi_k) = \pi^{kn}\sqrt{d_K}\chi_k(\mathfrak{a})r \qquad (\mathfrak{a} \in A)$$

with a rational number r, if either of the following two conditions is satisfied:

(i) k is an even natural number,

(ii) k is an odd natural number > 1, and any unit of K has positive norm.

He used the Eisenstein series which are Hilbert modular forms over K defined as follows. For the given totally real number field K of degree n, let $K = K^{(1)}, \cdots, K^{(n)}$ be its conjugations and $\tau = (\tau^{(1)}, \cdots, \tau^{(n)})$ be n independent complex numbers in the upper half plane. For given element $a \in K$, let $N(a)$ be its norm, and let χ_k be the character of the narrow ideal class group given in the above theorem. Then we define an Eisenstein series

$$G_k(\tau, A, \chi_k) = \sum_{(c,d),\ c\equiv d\equiv 0(\mathfrak{a})}{}' \frac{|N(\mathfrak{a})|^k \chi_k(\mathfrak{a})}{N(c\tau + d)^k} \qquad (\mathfrak{a} \in A)$$

Here the accent of the summation symbol means that the term correponding to the pair $c = d = 0$ is deleted, and (c, d) runs over a complete system of representatives of non-associated pairs.

This series converges uniformly if $k > 2$ in any compact subset of τ's, and defines a holomorphic Hilbert modular form of weight k (or of dimension $-k$ in the old terminology). When $k = 2$ we can use the regularization procedure of Hecke [10] to start with

$$G_k(\tau, A, \chi_k, s) = \sum_{(c,d),\ c\equiv d\equiv 0(\mathfrak{a})}{}' \frac{|N(\mathfrak{a})|^{k+2s}\chi_k(\mathfrak{a})}{N(c\tau+d)^k|N(c\tau+d)|^{2s}} \qquad (\mathfrak{a} \in A)$$

for $Re(s) > 1 - \frac{1}{2}k$. Then the constant term of this is given by $e\zeta_K(k, A, \chi_k)$ with the index of the subgroup of totally positive units in K in the whole unit group, and other Fourier coefficients are the common constant

$$\frac{(-2\pi i)^{kn}|N(\mathfrak{a})|^{k-1}\chi_k(\mathfrak{a})}{\sqrt{d_K}((k-1)!)^n}$$

times natural numbers which are generalized sum of divisors. If we normalize $G_k(\tau, A, \chi_k, s)$ by the last constant, the new series has rational Fourier coefficients except for the constant coefficient. Klingen showed this remaining coefficient is also rational by elimination method, since the graded ring of Hilbert modular forms is finitely generated by Hans Maass's result.

Later Siegel [35] give a modified proof, reducing the problem to the struture of the graded ring of elliptic modular forms over **Z**, by pulling-back Hilbert modular forms to elliptic modular forms utlizing the diagonal modular embedding.
Remark. Klingen suggested yet another argument to use the volume formula of the fundamental domain of Siegel modular groups and the Gauss-Bonnet formula for V-manifolds in the sense of Satake, to settele the case of Dedekind zeta functions. This method gives a shaper result to control the denominator of the rational factor r.

2.4 The papers on the values of L-functions by Meyer, Lange, Barner, and Siegel

The values $L(2, K, \chi)$ were obtained for characters of the (absolute) ideal class group of K by Meyer [30]. The values of $L(s, K, \chi)$ at s being a postive integer are obtained in the three papers Heinrich Lange [24], Klaus Barner [20], and Siegel [34]. The strategy of the proofs are almost the same as that of the proofs of the class number formula, and the results are described by similar invariants.

Here we do not discuss the so much details of the proofs, but we want to remark that in the proofs of Meyer, Lange, and Barner, the explicit calculation of the iterated primitives (i.e., the iterated indefinite integrals) plays the key role techinically. These are given the name *Lambert series*. The method of Siegel is slightly different.

2.4.1 *Meyer, Lange, and Barner*

Meyer [30] handled the values at $s = 2$, and his student Lange [24] the case of absolute class group in the narrow sense, and Barner [20] the case of ring class group. The method is in common with these three papers. But the paper of Barner is written in the natural order to make the pass-way of the logic very clear, so that it is most readable, in spite that it is quite computational. The latter half of Lange's paper is devoted to given many examples of computation of $L(2, \Omega, \chi)$.

Remark. In these papers, there appear new kinds of generalized Dedekind sums. As far as we can see, there is no discussion about "elementary way" to compute them. Because we cannot check all the subsequent papers, especially when they do not quote these three papers, we are ignorant whether this problem is treated in the later literature.

We do not write formulae in this subsection. They are given in the next subsection.

Remark The result of Barner [20] was extended for ray class characters by Katayama [21].

2.4.2 *Siegel, again*

Let us recall the main result of Siegel [34].

Let K be a real quadratic field. Let $\mathfrak{a} := \mathfrak{b}(\mathfrak{D}\mathfrak{f})^{-1}$, where $\mathfrak{D}$ is the different of K. Choose an integral basis w_1, w_2 of $\mathfrak{a}$ such that $w_1 w_2' - w_1' w_2 > 0$ (the existence of such a basis is clear). Let $\varepsilon_\mathfrak{f}$ be the generator of $E_\mathfrak{f}^+$ with $\varepsilon_\mathfrak{f} > 1$ and since $\varepsilon_\mathfrak{f}\mathfrak{a} \subset \mathfrak{a}$, there exists a $\sigma \in SL(2, \mathbf{Z})$ such that $\begin{pmatrix} \varepsilon_\mathfrak{f} w_1 \\ \varepsilon_\mathfrak{f} w_2 \end{pmatrix} = \sigma \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

Then the Hecke's integral formula implies that

$$\zeta_\mathfrak{f}(1-k, \mathfrak{b}) = L_k \int_{z_0}^{\sigma(z_0)} E_k(z|\mathrm{Tr}(w_1), \mathrm{Tr}(w_2)) Q(z)^{k-1} dz,$$

$(k = 1, 2, \dots)$ where z_0 is any point in $\mathfrak{H}$,

$$E_k(z|u, v) = \sum_{m,n=-\infty}^{\infty}{}' \frac{\exp 2\pi i(mu + nv)}{(nz - m)^{2k}}$$

is the Eisenstein series of weight $2k$ for the principal congruence subgroup $\Gamma(N\mathfrak{f})$,

$$Q(z) = (w_1 z + w_2)(w_1' z + w_2').$$

and L_k is some constant depending on k.

Theorem 2.5 *For $k = 1, 2, \dots,$*

$$\zeta_\mathfrak{f}(\mathfrak{b}, 1-k) = (-1)^k \Big(\frac{N\mathfrak{f}}{N\mathfrak{a}}\Big)^{k-1} \mathrm{sgn}(w_1 w_1')$$
$$\cdot \sum_{i=0}^{2k-1} (-1)^i \frac{c^{2k-i-1}}{i!(2k-i)} R_k^{(i)}\Big(-\frac{d}{c}\Big) S_i(\sigma|\mathrm{Tr}(w_1), \mathrm{Tr}(w_2)),$$

where

$$R_k(z) = \int_{a/c}^{z} Q(z)^{k-1} dz$$

and

$$S_i(\sigma|u, v) = \sum_{r (\mathrm{mod}\ c)} P_i\Big(v - \frac{u+r}{c} d\Big) P_{2k-i}\Big(\frac{u+r}{c}\Big)$$

is the generalized Dedekind sum. When $k = 1$ and $\mathfrak{f} = (1)$, the correction term $-1/4$ should be added.

Siegel's proof ([34]) starts from an "Eichler integral"

$$\int_{\tau_0}^{\tau_0^*} Q(\tau)^{k-1}\psi_s(\tau)d\tau$$

of an Eisenstein series $\psi_k(\tau) = E_k(z|u,v)$ $(2 \le k \in \mathbf{Z})$ of weight k belonging to a principal congruence subgroup $\Gamma(f)$ of $SL(2,\mathbf{Z})$, along a path corresponding to a hyperbolic element $\sigma \in SL(2,\mathbf{Z})$, though he do not use the word "Eichler integral" (Hilfssatz 1, p. 14 of [34]). The fact that this gives a twisted sum of partial zeta functions associated with ray classes of the real quadratic field K is proved, just by a paraphrase of the proof of the Hecke integral expression. Meanwhile, since $Q(\tau)$ is a quadratic polynomial in τ, by the integration by part, the same integral is written in terms of $2s-1$ times interated indefinite integral (i.e., the Lambert series of Meyer, Lange, and Barner). It become a Dirichlet series $c_0 + \sum_{n=1}^{\infty} \frac{c_n}{n^{2k-1}}$ with coefficients c_n, each of which is a finite sum

$$\sum_{m=1}^{2k-1} (-1)^m [D^{2k-m} R(z) D^{m-1} q_n(\tau)]_{\tau_0}^{\tau_o^*}.$$

Here $q_n(\tau)$ is given by

$$q_n(\tau) = \frac{\exp\{2\pi in(v+(u-\frac{1}{2})\tau\} + \exp\{-2\pi in(v+(u-\frac{1}{2})\tau\}}{\exp(-\frac{n}{2}\tau) - \exp(\frac{n}{2}\tau)} \quad (n = 1, 2, \cdots).$$

THE END OF EXPLANATION OF PROOF.

These two papers who were young (Lange, Barner) at that time give rather exhausive results if one is concered on real quadratic fields as the base fileld and if one's interest is the expression of the special values in terms of Dedekind sums. This is also the case about the papers of Siegel. From now on, we see only the history of rediscovery and reproduction about these two points. But there are also definitely new results for higher degree fields.

3 From 70's to 80'hs: the time of paradigm change or just a itnterlude?

3.1 Shintani

After the work of Siegel ([34],[35]), the next breakthrough was brought by T. Shintani ([48]). He obtained a remarkable explicit formula for the special values at non-positive integers of the partial zeta function of totally real number field of any degree and gave a new proof of their rationality.

His method is complex-analytic and with no use of modular forms. He first introduced a new zeta function which can be regarded as a generalization of Hurwitz zeta function and expressed its special values in terms of certain generating function which is generalization of generating function of Bernoulli numbers. In the next, along the cone decomposition of $(\mathbf{R}_{>0})^n$ ($[F:\mathbf{Q}]=n$), he wrote the partial zeta function $\zeta_{\mathfrak{f}}(s,\mathfrak{b})$ as a finite sum of his (sector) zeta function, and thus arrived at the explicit formula.

We recall his results more precisely. Let $A=(a_{ij})$ be an $r\times m$ matrix ($r\leq m$) with positive entries. For $x=(x_1,\dots,x_r)$ and $s\in\mathbf{C}$, Shintani defined a zeta function:

$$\zeta(s,A,x):=\sum_{n_1,\dots,n_r=0}^{\infty}\prod_{j=1}^{m}\Big\{\sum_{i=1}^{r}a_{ij}(n_i+x_i)\Big\}^{-s}.$$

When $r=m=1$ and $A=1$, it coincides with the Hurwitz zeta function $\sum_{n=0}^{\infty}(n+x)^{-s}$, equivalently a partial zeta function of $\mathbf{Q}$. The zeta function $\zeta(s,A,x)$ is continued to a meromorphic function on the whole s-plane and the special value at $s=1-m$ ($m=1,2,\dots$) is evaluated as

$$\zeta(1-k,A,x)=(-1)^{m(k-1)}k^{-m}\sum_{l=1}^{m}\frac{B_k(A,1-x)^{(l)}}{m},$$

where $(k!)^{-m}B_k(A,y)^{(l)}$ is the coefficient of $u^{(k-1)m}(t_1\cdots t_{l-1}t_{l+1}\cdots t_m)^{k-1}$ in the Laurent expansion at the origin of the generating function

$$\prod_{i=1}^{r}\frac{\exp(uy_i\sum_{j=1}^{m}a_{ij}t_j)}{\exp(u\sum_{j=1}^{m}a_{ij}t_j)-1}\Bigg|_{t_l=1}.$$

Let us relate $\zeta_{\mathfrak{f}}(s,\mathfrak{b})$ and $\zeta(s,A,x)$. From the definition,

$$\zeta_{\mathfrak{f}}(s,\mathfrak{b})=(N\mathfrak{b})^{-s}\sum_{\mu}N(\mu)^{-s},$$

where the summation is over all totally positive number in F which satisfy $\mu-1\in\mathfrak{b}^{-1}\mathfrak{f}$ and are not associated with each other under the action of the group $E_{\mathfrak{f}}^{+}$, totally positive units in F congruent to 1 modulo $\mathfrak{f}$. Keep this in mind, we decompose the set of totally positive elements V_+ of F which can be seen the subset $(\mathbf{R}_{>0})^n$ under the action of $E_{\mathfrak{f}}^{+}$. Then we have

$$V_+=\bigsqcup_{u\in E_{\mathfrak{f}}^{+}}\bigsqcup_{j\in J}u\,C_j$$

where $C_j = C(v_{j_1}, \dots, v_{j_{r(j)}}) = \{\sum_{k=1}^{r(j)} t_j v_{j_k} \mid t_j > 0\}$ is a open simplicial cone with generators $v_{j_1}, \dots, v_{j_{r(j)}} \in \mathfrak{f}$ and $\sharp(J) < \infty$. According as the above decomposition, we obtain the expression

$$\zeta_{\mathfrak{f}}(s, \mathfrak{b}) = (N\mathfrak{b})^{-s} \sum_{j \in J} \sum_{x \in R(j, \mathfrak{b}^{-1}\mathfrak{f}+1)} \zeta(s, A_j, x)$$

and therefore reach an effective explicit formula. Here $A_j \in M(r(j), n)$ with (l, m)-th entry is m-th conjugate $v_{j_l}^{(m)}$ and for each subset S of F, $R(j, S)$ is the set of $x = (x_1, \dots, x_{r(j)}) \in \mathbf{Q}^{r(j)}$ satisfying the conditions (i) $0 < x_k \le 1$ and (ii) $\sum_{k=1}^{r(j)} x_k v_{j_k} \in S$.

Shintani's formula, when applied to a real quadratic fields, recovers the Siegel's formula ([34]). He also obtained a relative class number formula for a totally imaginary quadratic extension K of a totally real number field F and computed the class number of $K = \mathbf{Q}(\exp(2\pi\sqrt{-1}/7)$ $(F = \mathbf{Q}(2\cos 2\pi/7),\ [F : Q] = 3)$.

APPLICATION TO THE CASE $n > 2$ **** %

3.2 Modular symbols by Manin, and by Mazur, Swinnerton-Dyer

3.3 p-adic L-functions

Barsky, Cassou-Nagues, Ribet, Deligne-Ribet

3.4 Dedekind sums and geometry: Hirzebruch-Zagier

3.5 Revision in terms of Eichler integrals

L. Goldstein ([42]) re-proved the formula of Siegel ([34]) for real quadratic fields in the framework of Eichler integral which is studied by Eichler ([40]) and Shimura ([47]). The technique used in Goldstein's paper are familiar ones, however, the method is neat and conceptual.

We first prepare some basic notation. Let $\mathfrak{a} := \mathfrak{b}(\mathfrak{d}\mathfrak{f})^{-1} = \mathbf{Z}w_1 + \mathbf{Z}w_2$, where $\mathfrak{d}$ is the different of F and put $W = \begin{pmatrix} w_1 & w_1' \\ w_2 & w_2' \end{pmatrix}$. Let $\varepsilon_{\mathfrak{f}}$ be the generator of $E_{\mathfrak{f}}^{+}$ with $\varepsilon_{\mathfrak{f}} > 1$ and since $\varepsilon_{\mathfrak{f}}\mathfrak{a} \subset \mathfrak{a}$, there exists a $\sigma \in SL(2, \mathbf{Z})$ such that $\begin{pmatrix} \varepsilon_{\mathfrak{f}} w_1 \\ \varepsilon_{\mathfrak{f}} w_2 \end{pmatrix} = \sigma \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

Then the Hecke's integral formula implies that

$$\zeta_{\mathfrak{f}}(1-k, \mathfrak{b}) = L_k \int_{z_0}^{\sigma(z_0)} E_k(z|\mathrm{tr}(w_1), \mathrm{tr}(w_2)) Q(z)^{k-1} dz,$$

$(k = 1, 2, \dots)$ where z_0 is any point in $\mathfrak{H}$,

$$E_k(z|u, v) = \sum_{m,n=-\infty}^{\infty}{}' \frac{\exp 2\pi i(mu + nv)}{(mz + n)^{2k}}$$

is the Eisenstein series of weight $2k$ for $\Gamma = \Gamma(f)$ with $f = N\mathfrak{f}$, $Q(z) = (w_2 z - w_1)(w_2' z - w_1')$ and $L_k = (-\mathrm{sgn}(\det W)) \frac{(2k-1)!}{(2\pi)^{2k}} (\frac{N(\mathfrak{f})}{N(\mathfrak{a})})^{k-1}$.

Let us recall some fundamental facts about Eichler integrals ([40], [47], [42]). Let h be an automorphic form on $\mathfrak{H}$ with weight $n+2$ for a Fuchsian group Γ (assumed to have the cusp $i\infty$). The *Eichler integral* $H(z)$ of h is $(n+1)$-fold iterated integral of h. There are many choices for such integral, in particular, we may choose

$$H_0(z) = \frac{1}{(n+1)!} a_0 z^{n+1} + \Big(\frac{\lambda}{2\pi i}\Big)^{n+1} \sum_{m=1}^{\infty} \frac{a_m}{m^{n+1}} e^{2\pi i m z/\lambda},$$

where $h(\tau) = \sum_{m=0}^{\infty} a_m e^{2\pi i m\tau/\lambda}$ is the Fourier expansion at the cusp $i\infty$.

For any Eichler integral $H(z)$ and $\sigma \in \Gamma$, set

$$S_\sigma(z) := H(\sigma(z)) j(\sigma, z)^n - H(z).$$

where $j(\sigma, z) = (cz+d)$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is known that $S_\sigma(z) \in \mathbf{C}_n[z]$ (=polynomials in z of degree $\leq n$), which is called a period polynomial. Moreover, if we denote by $S_\sigma(z) = (z^n, z^{n-1}, \dots, 1) S(\sigma)$ with $S(\sigma) \in M(n+1, 1, \mathbf{C})$, the cocycle relation holds:

$$S(\sigma\tau) = {}^t M_n(\tau) S(\sigma) + S(\tau) \quad \sigma, \tau \in \Gamma.$$

Here $M_n(\sigma) \in SL(n+1, \mathbf{R})$ is the matrix of n-th symmetric tensor representation of $SL(2, \mathbf{R})$. This relation is essentially the consequence of the automorphy of h.

The target in the paper [42] is (transformation formula of) 'generalized Eichler integral'

$$G(z, p) = \int_{z_0}^{z} h(\tau) p(\tau) d\tau$$

with $p(z) \in \mathbf{C}_n[z]$.

Proposition ([42]) *If $p(\sigma(z)) j(\sigma, z)^n = p(z)$, the quantity ${}^t P P_n^{-1} S(\sigma)$ does not depend on the choice of Eichler integral and*

$$G(\sigma(z), p) - G(z, p) = n!\, {}^t P P_n^{-1} S(\sigma).$$

Here $p(z) = (z^n, z^{n-1}, \dots, 1) P$ *and* $P_n = \begin{pmatrix} & & p_1 \\ & \cdot^{\cdot^{\cdot}} & \\ p_{n+1} & & \end{pmatrix}$ *with* $p_j = (-1)^{j-1} \begin{pmatrix} n \\ j-1 \end{pmatrix}$.

Then the task is reduced to compute the period polynomial for some Eichler integral of $E_k(z|u, v)$, that is,

$$H_0(\sigma(z)) j(\sigma, z)^{2k-2} - H_0(z).$$

The method used here is quite standard one. He expressed the constant term of $H_0(z)$ by inverse Mellin transformation of (finite sum of) certain zeta function (the product of partial L-function of $\mathbf{Q}$). By shifting the line of integration and applying the functional equation of the above zeta function, the period polynomial is computed by the residue calculation.

In [42], the exceptional case $k = 1$ with $\mathfrak{f} = \mathfrak{O}_F$ is seemed to be excluded. But, because of the connection between the special value at $s = 0$ of Dedekind zeta function and the transformation formula of $\log(\eta(z))$ (see also §** Sczech of this note), it has already (implicitely ?) covered by the previous work of Goldstein and Torré ([41]), in which the same technique is employed.

The argument of Goldstein seems to be essentially the same as the previous work of Meyer, Barner (Lange?). In their papers, starting from the Hecke's integration formula, the transformation formula of Lambert series (certain integration of an Eisenstein series) plays a central role (see §1.3) and the proof of Apostol ([16]) used "Mellin transform technique" as mentioned above.

4 After 1990 -cocycles on $GL(n, \mathbf{Q})$-

Around 1990, new approach begins by G. Stevens ([50]) and R. Sczech ([55], [56]). Both of them and lately D. Solomon ([60], [61]) construct certain 'universal' 1-cocycles on the group $GL(2, \mathbf{Q})$ by different method and show that the special value $\zeta_{\mathfrak{f}}(s, \mathfrak{b})$ (at integers) admits a cohomological interpretation. Roughly speaking, these author's work consists of the following contents:

(i) Construction of a cocycle.

(ii) Description of the link between the cocycle and the special value of partial zeta function.

(iii) Evaluation of the cocycle by means of generalized Dedekind sums.

We remark that, when (i) and (iii) are established, the cocycle property implies the "reciprocity law" of the resulting generalized Dedekind sum, for example, the classical reciprocity law or its generalization by Rademacher and others (cf. [15]), and more generalized ones are re-proved in these papers.

Moreover, with the aid of the cocycle relation, one can design an efficient algorithm for the computation of special value and numerical examples are given in [57], [58].

4.1 Stevens

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4.2 Sczech

Sczech's idea is to focus the period

$$\int_{\tau}^{A\tau} \sum_{m,n}{}'(mz+n)^{-2}dz, \qquad A \in SL(2, \mathbf{Z}),\ \tau \in \mathfrak{H} \tag{$\sharp$}$$

of an Eisenstein series of weight 2. The integrand of ($\sharp$) does not converge absolutely, and the handling of this kind sums is attributed to Eisenstein by Weil ([52]). We are ignorant who justified firmly the argument to evaluate ($\sharp$) as $\log \eta(\tau) - \log \eta(A\tau)$, which is heuristically natural, that can be expressed by (classical) Dedekind sum. On the other hand, according to "Hecke trick," ($\sharp$) is related to $\zeta_{\mathfrak{f}}(1, \mathfrak{b})$.

To avoid some difficulties in treating the period (♯), Sczech considered the series

$$\sum_{m,n}{}' \frac{A\tau - \tau}{(mA\tau + n)(m\tau + n)} \tag{♮}$$

which arises from (♯) by termwise integration.

This series is more elementary one and can be discussed by (real) analytic method, though still the series converges conditionally, then we should specify a limit process, called "Q-limit." Here Q is some fixed nondegenerate binary form. In the case of (♮), it means that the summation is taken so that $\lim_{t\to\infty}\sum_{|Q(m,n)|<t}$.

The "cocycle" (♮) is already well-known (see Schoenberg's book [] ?). But Sczech noticed the advantage of extention to $n > 2$ and in his paper [56], which is ingeniously achieved while cocycles of Stevens and Solomon are not at hand now.

Following [56] (see also [58]), we briefly recall the definition of Eisenstein cocycle for $GL(n,\mathbf{Q})$. Let $\mathcal{A} = (A_1, \dots, A_n)$ be an n-tuple of matrices $A_i \in GL(n,\mathbf{R})$ and A_{ij} the j-th column of A_i. Then for nonzero $x \in \mathbf{R}^n$ and each A_i, there exists at least one A_{ij} such that $\langle x, A_{ij}\rangle \neq 0$ and denoted by A_{ij_i} the first column in A_i with this property. Define

$$\psi(\mathcal{A})(x) := \frac{\det(A_{1,j_1}, \dots, A_{n,j_n})}{\langle x, A_{1,j_1}\rangle \cdots \langle x, A_{n,j_n}\rangle}.$$

To treat the values of $\zeta_{\mathfrak{f}}(s,\mathfrak{b})$ at $s = -1, -2, \dots$, we act on $\psi(\mathcal{A})(x)$ a differential operator $P(-\partial/\partial x_1, \dots, -\partial/\partial x_n)$ with a homogeneous polynomial $P(X_1, \dots, X_n)$:

$$\psi(\mathcal{A})(P,x) := P\Big(-\frac{\partial}{\partial x_1}, \dots, -\frac{\partial}{\partial x_n}\Big)\psi(\mathcal{A})(x).$$

Now we can define the *Eisenstein cocycle* Ψ by averaging over the lattice $\mathbf{Z}^n$:

$$\Psi(\mathcal{A})(P,Q,v) := (2\pi\sqrt{-1})^{-n-\deg P} \sum_{x\in\mathbf{Z}^n} \exp(2\pi\sqrt{-1}\langle x, v\rangle)\psi(\mathcal{A})(P,x)\Big|_Q.$$

Here $v \in \mathbf{R}^n$ and $|_Q$ means the Q-limit.

Since we are considering the special values at non-positive integers, the exponential function is multiplied, of course, it dose not appear and the sum is taken over $\mathbf{Z}^n + u$ to treat the values at positive integers and the resulting cocycle is called *trigonometric cocycle* which is introduced in [] (when $P = 1$?) and discussed in [55] for $n = 2$. The results of [55] and [56] are summarized as follows.

(i) Ψ represents a nontrivial cohomology class in $H^{n-1}(GL(n,\mathbf{Z}), M)$ where M is the set of all $\mathbf{C}$-valued functions $f(P,Q,v)$. Here the action of $X \in GL(n,\mathbf{Z})$ is defined by $Xf(P,Q,v) = \det(X)f({}^tXP, X^{-1}Q, X^{-1}v)$.

(ii) For suitable choices of $(\mathcal{A}, P, Q, v)$, the Eisenstein cocycle Ψ represents the special values of partial zeta function at non-positive integers. More precisely, we denote $\mathfrak{f}\mathfrak{b}^{-1} = \sum \mathbf{Z}w_j$ with the dual basis $\{w_j^*\}$ determined by $\mathrm{Tr}(w_i^* w_j) = 1$ and define $P(X) = N\mathfrak{b}\cdot N(\sum_j X_j w_j)$, $Q(X) = N(\sum_j X_j w_j^*)$

and $v=(v_j)\in\mathbf{Q}^n$ with $v_j=\mathrm{Tr}(w_j^*)$. Let $\rho: E_\mathfrak{f}^+\to SL(n,\mathbf{Z})$ be the map defined by $\rho(\varepsilon)=W^*\mathrm{diag}(\varepsilon^{(1)},\dots,\varepsilon^{(n)})(W^*)^{-1}$ with $W=((w_i^*)^{(j)})$ and set $A_i=\rho(\varepsilon_i)$ with $E_\mathfrak{f}^+=\langle\varepsilon_1,\dots,\varepsilon_{n-1}\rangle$. Then

$$\zeta_\mathfrak{f}(1-s,\mathfrak{b})=\eta\sum_{\varepsilon\in E^+/E_\mathfrak{f}^+}\sum_{\pi\in\mathfrak{S}_{n-1}}\mathrm{sgn}(\pi)$$
$$\cdot\,\Psi((1,A_1,A_1A_2,\dots,A_1\cdots A_{\pi(n-1)}))(P^{s-1},Q,\rho(\varepsilon)v).$$

Here $\eta\in\{\pm1\}$ is determined by $\eta=(-1)^{n-1}\mathrm{sgn}(\det W)\mathrm{sgn}(\det(\log\varepsilon_i^{(j)}))$.

(iii) Ψ can be expressed by finite sum of generalized Dedekind sum, which can be explicitly given, if the datum $(\mathcal{A},P,Q,v)$ is specified.

Roughly speaking (i) and (ii) follows from the way of construction of Ψ. In the proof of (iii), the most important and difficult part is to control the Q-limit which is necessary (only) for the special value at $s=0$. The "Q-limit formula" is studied in Sczech's dissertation [] for $n=2$ and the latter part of [56] for general n. For example,

$$\lim_{t\to\infty}\sum_{\substack{(p_1,p_2)\in\mathbf{Z}^2\\|Q(p_1,p_2)|<t}}\frac{\exp 2\pi\sqrt{-1}(p_1v_1+p_2v_2)}{p_1^k\,p_2^l}=\frac{(2\pi\sqrt{-1})^{k+l}}{k!\,l!}P_k(v_1)P_l(v_2)+S(Q)$$

where $P_k(y)=-\frac{k!}{(2\pi\sqrt{-1})^k}\sum_m'\frac{e^{2\pi\sqrt{-1}my}}{m^k}$ is the periodical Bernoulli function and the error term

$$S(Q)=\begin{cases}\pi^2-\frac{2\pi}{m}\sum_{i=1}^m|\arg\frac{\alpha_i}{\beta_i}| & \text{if } (v_1,v_2)\in\mathbf{Z}^2 \text{ and } k=l=1,\\ 0 & \text{otherwise.}\end{cases}$$

with $Q(p_1,p_2)=\prod_{i=1}^m(\alpha_ip_1-\beta_ip_2)$

The explicit formulas given in [56, Theorem 6,7] are certainly effective, though, are rather complicated to perform practical computations for given datum. He emphasizes that the reason is the cocycle property is not used to find them.

Let us observe how the cocycle relation reduce the task in the case of $\zeta_\mathfrak{f}(0,\mathfrak{b})$ for a real quadratic field. WHO FIRST NOTICED ?? "continuous fractional algorithm" For the purpose, we should compute

$$\Psi\left(1,\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)(1,Q,v)$$
$$=\frac{1}{2c}P_2(v_2)+\frac{d}{2c}P_2(cv_1-av_2)-\sum_{j(\mathrm{mod}\ c)}P_1\Big(\frac{j+v_2}{|c|}\Big)P_1\Big(a\frac{j+v_2}{c}-v_1\Big).$$

If $|c|$ is not small, the number of the last sum is not suitable to evaluate. Then we apply the Euclidean algorithm to the first column of $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and obtain a decomposition

$$A=B_1\cdots B_N,\qquad B_i=\begin{pmatrix}b_i&-1\\1&0\end{pmatrix}$$

Thus the cocycle relation $\Psi(AB) = \Psi(A) + A\Psi(B)$ leads to

$$\Psi(1, A) = \sum_{i=0}^{N-1} (B_1 \cdots B_i)\Psi(1, B_{i+1}).$$

At a rough estimate, the order of N is $\log |c|$, therfore, the number of terms is reduced from $|c|$ to $\log |c|$.

In the recent paper [58], Gunnells and Sczech introduce a higher dimensional Dedekind sum D inspired by the definition of the Eisenstein cocycle. For a lattice L of rank l with $L^* = \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$, $\sigma = (\sigma_1, \ldots, \sigma_r) \in (L^*)^r$ $(r \geq l)$, a tuple $e = (e_1, \ldots, e_r)$ of positive integers and $v \in L^* \otimes \mathbf{R}$, define

$$D(L, \sigma, e, v) := (2\pi i)^{-\sum e_i} \sum_{x \in L}{}' \frac{\exp(2\pi i \langle x, v\rangle)}{\langle x, \sigma_1\rangle^{e_1} \cdots \langle x, \sigma_r\rangle^{e_r}},$$

where $\langle\ ,\ \rangle : L \times L^* \to \mathbf{Z}$ is the paring If $e_j = 1$ for some j, the above series converges conditionally, then the Q-limit is applied to define D. The reciprocity law of the new Dedekind sum is derived and combined with a refinement of the modular algorithm of Ash and Rudolph ([]), they estimate the number of terms. As an application of their effective algorithm, some of numerical examples of $\zeta_{\mathfrak{f}}(0, \mathfrak{b})$ are computed in [57], where $\mathfrak{f} = N\mathfrak{O}_F$ with various integers N and $\mathfrak{b} = \mathfrak{O}_F$ for cubic or quartic fields

4.3 Solomon

Solomon defined another cocycle on $(P)GL(2, \mathbf{Q})$ which he called Shintani cocycle. His method is algebro-combinatorial and is inspired by the work of Shintani ([48]) as the name indicates. The starting point of Solomon's work is the modification of the generating functions which appear in the Shintani's explicit formula.

Let Λ be a rank 2 lattice in $\mathbf{R}^2$ and $\mathbf{x} \in \mathbf{R}^2/\Lambda$. The symbols $\mathfrak{r}$ and $\mathfrak{s}$ denote Λ-rational rays emananting from the origin in $\mathbf{R}^2$, that is, equivalence classes for the multiplicative action of $\mathbf{Q}_+^\times$ on $\mathbf{Q}\,\Lambda\backslash\{\mathbf{0}\}$. For the above datum with $\mathfrak{r} \neq \pm\mathfrak{s}$ and $\mathbf{z} = (z_1, z_2)$, define

$$\widetilde{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z}) := \frac{\sum_{\mathbf{a} \in \mathbf{x} \cap P(\mathbf{r}, \mathbf{s})} e^{\mathbf{z}.\mathbf{a}}}{(1 - e^{\mathbf{z}.\mathbf{r}})(1 - e^{\mathbf{z}.\mathbf{s}})}.$$

Here we have chosen $\mathbf{r} \in \mathfrak{r} \cap \Lambda$ and $\mathbf{s} \in \mathfrak{s} \cap \Lambda$ and $P(\mathbf{r}, \mathbf{s})$ denotes the half-open parallelogram

$$P(\mathbf{r}, \mathbf{s}) := \{\mu\mathbf{r} + \nu\mathbf{s} \mid \mu, \nu \in \mathbf{R}, 0 < \mu \leq 1, 0 \leq \nu < 1\}.$$

The 'Shintani function' P is defined to be the mean of $\widetilde{P}$:

$$P(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z}) := \frac{1}{2}\mathrm{sgn}(r_1 s_2 - r_2 s_1)(\tilde{P}(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z}) + \tilde{P}(\Lambda, \mathbf{x}, \mathfrak{s}, \mathfrak{r}; \mathbf{z})).$$

Here $(r_1, r_2) \in \mathfrak{r}$ and $(s_1, s_2) \in \mathfrak{s}$. Let $P(\Lambda, \mathfrak{r}, \mathfrak{s})$ be a map $\mathbf{R}^2/\Lambda \ni \mathbf{x} \mapsto P(\Lambda, \mathbf{x}, \mathfrak{r}, \mathfrak{s}; \mathbf{z})$ and define the *Shintani cocycle* $\Psi_{\mathfrak{r}}$ by

$$\Psi_{\mathfrak{r}}(M) := P(\mathbf{Z}^2, \mathfrak{r}, M\mathfrak{r})$$

for $M \in GL(2, \mathbf{Q})$ and the fixed ray $\mathfrak{r}$.

The cocycle relation follows from the "Juxtaposition Lemma" and the connection between the Shintani cocycle $\Psi_{\mathfrak{r}}(M)$ and the special values of zeta function is essentially guaranteed by the work of Shintani.

The procedure of giving an explicit form of Shintani function in terms of generalized Dedekind sum is relatively easier, for one reason, thanks to the way of definition of the Shintani function, we need no difficulity such as the Q-limit argument in the paper of Sczech.

As an application, essentially due to the cocycle properties of Shintani functions, Solomon discusses a new proof of Halbritter's reciprocity law for generalized Dedekind sum ([]).

Extension to general n is attempted by Hu and Solomon []. They introduce a new parametrization of simplicial cones instead of "generating rays" in 2-dimensional case, and succeed the constructions of cocycles in case of $n = 3$, and $n \geq 4$ for 'generic' cones. They also evaluated the cocyles in terms of generalized Dedekind sums, though, degenerate cones for $n \geq 4$ is not also achieved obstructed by the combinatorial difficulties.

5 Postscript: Is there any lacune in the state of arts of the present studies ?

We have the table of the achievement on the effective computation of arithmetic invariants:

$n = [K : \mathbf{Q}]$	Class number formula	Special values at $k \geq 2$
$n = 2$	Meyer '57('50), Siegel '61('57/'58)	Meyer '66, Lange '68, Siegel, Barner '69
$n \geq 3$	XXX	Sczech 1994, Solomon (?)

There seems still to a big problem at XXX.

Problem For arbitrary totally real number field K of degree $n \geq 3$, and for pair of abelian extensions L/L_0 over K such that L_0 is totally real and L a CM-extension of L_0, find *effectively computable formula* of the relative class number h_L/h_{L_0} in terms of something like
newline hyper Dedekind sums or ultra Dedekind sums.

The method of Shintani [49] is universal and quite powerful even for numeraical computation. But the multiple gamma functions would not be elementary functions.

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6 Appendix

We have no time!

But there should be discussion about the Fourier expansion of Epstein zeta functions more seriously, we believe. We have some quite preparatory result to extend our former result on Whittaker functions on $SL(3, \mathbf{R})$ in Japanse Journal of Math., (2004), :