Wild ramification and Cotangent bundle

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Ramification of an $\ell$-adic sheaf on a variety over a(n algebraically closed) field $k$

$\text{char } k = p > 0, \; \ell \neq p$
Smooth $\ell$-adic sheaf $\mathcal{F}$
on a smooth connected scheme $U$ over $k$

$\iff$

$\ell$-adic representation

of the algebraic fundamental group $\pi_1(U, \bar{x})$
Ramification along the boundary;

\( X \supset U \) smooth over \( k \),

\( j : U = X - D \rightarrow X \) : open immersion

of the complement of

a divisor \( D \) with simple normal crossings

\( \mathcal{F} : \) smooth on \( U \) \textcolor{red}{\text{ramifies}} along \( D \)
divisor $D \subset X$ with simple normal crossings

Locally, $X = \mathbb{A}^n_k = \text{Spec } k[T_1, \ldots, T_n]

\supset \quad D = (T_1 \ldots T_r) \quad (0 \leq r \leq n)$
Want: Characteristic cycle

$$\text{Char } j_! \mathcal{F}$$

Linear combination of conic subvarieties

of dimension $$d = \dim X$$

of the cotangent bundle $$T^* X = \mathcal{V}(\Omega^1_{X/k})$$

of dimension $$2d$$
conic subvarieties: irreducible components

of support of \( \text{Char } j!\mathcal{F} \)

classified by dimension of the fibers

0: 0-section coefficient = rank \( \mathcal{F} \)

1: subject of this talk (non-degenerate)

\( \geq 2: ??? \)
Analogy with $\mathcal{D}$-modules in char. 0 microlocal analysis
Char $\mathcal{M}$ for holonomic $\mathcal{D}$-module $\mathcal{M}$
wild ramification in char $p > 0$
vs
irregular singularity in char 0
Related works
Deligne (unpublished) : using jet bundle
Laumon : Euler number for surfaces
Kato: rank 1
Abbes-S.: Ramification group of local field with imperfect residue field
Expected properties of Char $j_!\mathcal{F}$:

1. Determined by wild ramification
2. Compute the characteristic class (SGA5) and the Euler number
3. Controls nearby cycles
4. Compatible with the pull-back by non-characteristic morphism
Example 1: $X$ curve (classical)

(1) \[
\text{Char } j! \mathcal{F} = -(\text{rank } \mathcal{F} \cdot [T_X^* X] + \sum_{x \in D} \text{dim tot}_x(\mathcal{F}) \cdot [T_x^* X])
\]

$T_Y^* X$ conormal bundle of subscheme $Y \subset X$

$T_X^* X$ : 0-section, $T_x^* X$ : fiber at $x$

\[
\text{dim tot}_x(\mathcal{F}) : \text{total dimension}
= \text{rank} + \text{Swan conductor } Sw_x(\mathcal{F})
\]

\mid

measure of wild ramification
(2) $C(j_!F)$ (characteristic class) = $[\text{Char } j_!F]$

in $H^2(T^*X, \mathbb{Q}_\ell(1)) = H^2(X, \mathbb{Q}_\ell(1))$

Consequently, if $X$ is proper,

$$\chi_c(U, F) = (\text{Char } j_!F, [T^*_XX])_{T^*X}$$

(Grothendieck-Ogg-Shafarevich)

$$(\chi_c(U, F) = \sum_{q=0}^{2} (-1)^q \dim H^q_c(U, F))$$
(3) Induction formula: \( \pi : X \to Y \)
finite generically étale morphism of curves

\[
\dim \text{tot}_y \pi_* F = \dim \text{tot}_x F \\
+ \text{rank } F \cdot \text{length } \Omega^1_{X/Y,x}
\]
Example 2: $\mathcal{F}$ tamely ramified along $D$ (easy)

(1) $\text{Char } j_! \mathcal{F} = (-1)^d \text{rank } \mathcal{F} \cdot \sum_{I} [T_{X_I}^* X]$

$d = \dim X$, $D = \bigcup D_i$, $X_I = \bigcap_{i \in I} D_i$

$T_{X_I}^* X \subset T^* X \times_X X_I$ : conormal bundle

(2) $C(j_! \mathcal{F}) = [\text{Char } j_! \mathcal{F}]$, 

$\chi_c(U, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U, \mathbb{Q}_\ell)$
Example 3: Artin-Schreier sheaf (typical)

\[ X = \mathbb{A}^2 = \text{Spec } k[x, y] \supset U = \text{Spec } k[x^{\pm 1}, y] \]

(1) (i) \( \mathcal{F} \) defined by \( t^p - t = \frac{1}{x^n} \), \( p \nmid n \)

\[
\text{Char } j!\mathcal{F} = [T^*_XX] + (n + 1) \cdot [T^*_DX]
\]

(rank \( \mathcal{F} = 1 \), \( (-1)^2 = 1 \), \( \dim \text{ tot}_0 = n + 1 \))
Example 3: Artin-Schreier sheaf (typical)

\[ X = \mathbb{A}^2 = \text{Spec } k[x, y] \supset U = \text{Spec } k[x^{\pm 1}, y] \]

(1) (ii) \( F : t^p - t = \frac{y}{x^n}, \quad p \mid n, \)

\( ((p, n) \neq (2, 2) : \text{non-exceptional case}) \)

\[ \text{Char } j_!F = [T_X^*X] + n \cdot [\langle dy \rangle \text{ over } D (\simeq T^*D)] \]
Points in Definition of Char $j!F$:

1. Invariant of wild ramification

(\text{New method: Blow-up at the ram. locus $R \subset X$ in the diagonal $X \to X \times X$})

(More detail at the end, if time permits)

- Ramification of $F$ is \textit{bounded by $R+$}

\[ R = \sum_{i} r_i D_i, \quad r_i \geq 1 \text{ rational} \]

(assume integer for simplicity)
Pts in Def. of Char $j_! \mathcal{F}$ (cnt’d.):

1. Invariant of wild ramification

• characteristic forms \((r = \text{rank } \mathcal{F})\)

\[
\omega^{(i)}_1, \ldots, \omega^{(i)}_r \in \Gamma(D_i, \Omega^1_{X/k}(R) \otimes \mathcal{O}_{D_i})
\]

(precisely speaking, defined over purely inseparable coverings of étale schemes $D_{ij}$ over $D_i$)
Pts in Def. of Char \( j_! \mathcal{F} \) (cnt’d.):

Examples of characteristic forms

(i) \( t^p - t = \frac{1}{x^n}, \ p \nmid n \)

\[
\omega = d \frac{1}{x^n} = \frac{-n dx}{x^{n+1}} \in \Gamma(D, \Omega_{X/k}^1((n+1)D) \otimes \mathcal{O}_D)
\]

(ii) \( t^p - t = \frac{y}{x^n}, \ p \mid n, \ (p, n) \neq (2, 2) \)

\[
\omega = d \frac{y}{x^n} = \frac{dy}{x^n} \in \Gamma(D, \Omega_{X/k}^1(nD) \otimes \mathcal{O}_D)
\]
The exceptional case in (ii) : \[ t^2 - t = \frac{y}{x^2} \]

\[ \omega = \frac{\sqrt{y}dx + dy}{x^2} \in \Gamma(D', \Omega^1_{X/k}(2D) \otimes \mathcal{O}_{D'}) \]

\( D' \rightarrow D \) : inseparable covering of degree 2
Pts in Def. of Char $j_!\mathcal{F}$ (cnt’d.):

2. Non-degenerate (Assumption)

$(\Rightarrow$ component of support of Char $j_!\mathcal{F}$
$(\subset T^*X)$ has fiber dim $\leq 1$ over $X$)

$D_{ij}$ are finite étale over $D_i$ and

$\omega_{1}^{(i)}, \cdots, \omega_{r}^{(i)}$ are nowhere vanishing

(copy : characteristic forms

$$\omega_{1}^{(i)}, \cdots, \omega_{r}^{(i)} \in \Gamma(D_{ij}, \Omega_{X/k}^{1}(R) \otimes \mathcal{O}_{D_{ij}})$$

defined over purely inseparable coverings

of étale schemes $D_{ij}$ over $D_i$, $r = \text{rank}\mathcal{F}$)
Pts in Def. of Char $j_!\mathcal{F}$ (cnt’d.):

characteristic forms

$\omega_j^{(i)} \in \Gamma(D_{ij}, \Omega^1_{X/k}(R) \otimes \mathcal{O}_{D_{ij}})$ define

$$\omega_j^{(i)} : L(R) \times_X D_{ij} \to T^* X \times_X D_{ij} \to T^* X$$

$L(R)$: line bundle defined by the divisor $R$

E.g. $t^p - t = \frac{y}{x^n} (p \mid n)$:

$$L(nD) \times_X D \to T^* X \times_X D : x^n \mapsto dy$$
Definition of Char $j_! \mathcal{F}$:

Char $j_! \mathcal{F}$

$$= (-1)^d \left( (\text{rank } \mathcal{F}) \cdot [T^*_X X] + \sum_i r_i \cdot \sum_{j=1}^{\text{rk } \mathcal{F}} \frac{\text{Im } \omega^{(i)}_j}{[D_{ij} : D_i]} \right)$$

$$\omega^{(i)}_j : L(R) \times_X D_{ij} \to T^*_X X$$

defined by characteristic forms.
Example of Char $j_!\mathcal{F}$:

$$tp - t = \frac{y}{x^n} \quad (p \mid n, \ (p, n) \neq (2, 2))$$

$$L(nD) \times_X D \to T^*X \times_X D : x^n \mapsto dy$$

Char $j_!\mathcal{F}$

$$= (-1)^2 \left( 1 \cdot [T^*_X X] + n \cdot \langle dy \rangle \text{ over } D(\cong T^*_D) \right)$$
Results on Char $j!\mathcal{F}$:

Characteristic class and the Euler number

$$C(j!\mathcal{F}) = [\text{Char } j!\mathcal{F}]$$

in $H^{2d}(X, Q_\ell(d))$ ($d = \dim X$).
Results on Char $j!\mathcal{F}$ (cnt’d.):
If $X$ is proper,

$$\chi_c(U, \mathcal{F}) = (\text{Char } j!\mathcal{F}, [T_X^*X])_{T^*X}$$

(general’n of Grothendieck-Ogg-Shafarevich)

$$(\chi_c(U, \mathcal{F}) = \sum_{q=0}^{2d} (-1)^q \dim H^q_c(U, \mathcal{F}))$$
Results on Char $j_!\mathcal{F}$ (cnt’d.):

Pull-back by non characteristic morphism $f: X' \to X$ of smooth schemes

- $j': U' = f^{-1}(U) = X' - D' \to X'$
- $f^*(\text{Char } j_!\mathcal{F}) \cap \text{Ker}(df: T^*X \times_X X' \to T^*X')$

is in the 0-section

$$T^*X \xleftarrow{f} T^*X \times_X X' \xrightarrow{df} T^*X'$$
Results on Char $j_!\mathcal{F}$ (cnt’d.):
Pull-back by non characteristic morphism $f : X' \to X$ of smooth schemes

\[ \text{Char } j'_! f^* \mathcal{F} = df(f^*(\text{Char } j_! \mathcal{F})) \]

\[ T^*X \xleftarrow{f} T^*X \times_X X' \xrightarrow{df} T^*X' \]

Consequence: Char $j_! \mathcal{F}$ is char’zed by restriction to curves $C \subset X$
Results on $\text{Char } j_!\mathcal{F}$ (cnt’d.):

Local acyclicity for non characteristic smooth morphism $f : X \to Y$ of smooth schemes

- $\text{Char } j_!\mathcal{F} \cap df(T^*Y \times_Y X) \subset T^*X$

is in the 0-section $T^*_X X$

$$T^*Y \times_Y X \xrightarrow{df} T^*X \supset \text{Char } j_!\mathcal{F}$$
Results on \( \text{Char } j_! \mathcal{F} \) (cnt’d.): 

Local acyclicity for non characteristic smooth morphism \( f : X \rightarrow Y \) of smooth schemes

\[ j_! \mathcal{F} \text{ is locally acyclic rel. to } f \]

Local acyclicity:

Vanishing cycles \( \phi^q = 0 \) (\( \dim Y = 1 \)) or \( H^q(\text{Milnor fibers}) = 0 \) (in general) for \( q > 0 \)
Conjecture (Deligne):
morphism \( f : X \to A^1 = \text{Spec } k[t] \)
non characteristic except at a closed pt \( x \)
Then,

\[- \dim \text{tot}_x \phi (j_! F) = (\text{Char } j_! F, dt(X))_{T^*X}\]
Conjecture:

\[-\dim \text{ tot}_x \phi(j! \mathcal{F}) = (\text{Char } j! \mathcal{F}, dt(X))_{T^*X}\]

Deligne-Milnor formula: \( X = U, \mathcal{F} = \mathbb{Q}_\ell, \)
\( x; \) isolated singularity of \( X \to Y \)

Consequence (Hasse-Arf):
Char \( j! \mathcal{F} \) has integral coefficients
New Method to study ramification:

- Partial blow-up $P^{(R)} \to X \times X$ at the ram. locus $R \subset X$ in the diagonal $X \to X \times X$

$\left( N_{X/P^{(R)}} = \Omega^1_{X/k}(R), \ N_X/X \times X = \Omega^1_{X/k} \right)$

- Groupoid structure $P^{(R)} \times_X P^{(R)} \to P^{(R)}$ lifting

\[ (X \times X) \times_X (X \times X) \cong X \times X \times X \]
\[ \text{pr}_{13} \to X \times X \]

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Blow-up to study ramification

\((R\text{ integer coefficients } > 1 \text{ for simplicity}):\)

\[
T(R) \times_X D \subseteq P(R) \supseteq U \times U
\]

\[
D \subseteq X \supseteq U = X - D
\]

Groupoid structure on \(P(R)\) induces addition on a twisted tangent bundle

\[
T(R) \times_X D = V(\Omega^1_{X/k}(R) \otimes \mathcal{O}_D)
\]
Blow-up to study ramification:

\( V: \ G\)-torsor over \( U \), \( G \) finite

\[
\begin{align*}
W^{(R)} & \subseteq V \times V/\Delta G \\
T(R) \times_X D & \subseteq P^{(R)} \subseteq U \times U
\end{align*}
\]

\( W^{(R)} \) largest open in the normalization étale over \( P^{(R)} \)
Blow-up to study ramification:

Ram’n of $G$-torsor $V$ over $U$ bounded by $R+$:

\[
\begin{array}{c}
W^{(R)} \xleftarrow{\cong} V \times V/\Delta G \xleftarrow{} V/G \\
\downarrow \quad \downarrow \quad \| \\
P^{(R)} \xleftarrow{\cong} U \times U \xleftarrow{\Delta U} U
\end{array}
\]

is extended to $X \to W^{(R)}$

(largest open étale over $P^{(R)}$)
Blow-up to study ramification:

Ram’n of $G$-torsor $V$ over $U$ bounded by $R\uparrow$:

$U = V/G \to V \times V/\Delta G$ is ext’d to $X \to W^{(R)}$

$W^{(R)} \overset{\subset}{\rightarrow} V \times V/\Delta G$

$T(R) \times_X D \overset{\subset}{\rightarrow} P^{(R)} \overset{\subset}{ightarrow} U \times U$

$W^{(R)}$ largest open étale over $P^{(R)}$

$\Rightarrow W^{(R)} \to P^{(R)}$ morphism of groupoids
Blow-up to study ramification:

\[ E(R) \xrightarrow{\subset} W(R) \xleftarrow{\supset} V \times V/\Delta G \]

\[ T(R) \times_X D \xrightarrow{\subset} P(R) \xleftarrow{\supset} U \times U \]

Non-degenerate:

étale morphism \( W(R) \to P(R) \) of groupoids induces a finite étale morphism

\[ E(R) \to T(R) \times_X D \]

of smooth group schemes over \( D \)
Blow-up to study ramification:

Non-degenerate:

Finite étale morphism $E^{(R)} \to T(R) \times_X D$

of smooth group schemes over $D$

Classification of étale isogeny to vector bundle by linear forms (defined over inseparable covering)

defines characteristic forms