Ramification theory of schemes
in mixed characteristic case

(joint work with K. Kato)

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Abstract

We define generalizations of classical invariants of ramification, for coverings on a variety of arbitrary dimension over a local field of mixed characteristic. For an ℓ-adic sheaf, we define its Swan class as a 0-cycle class supported on the closed fiber. We present a formula for the Swan conductor of cohomology and its relative version.

Let $K$ be a complete discrete valuation field of characteristic 0 and $F$ be the residue field of $K$. We assume $F$ is a perfect field of characteristic $p > 0$.

Let $U$ be a separated smooth scheme purely of dimension $d$ of finite type over $K$. Let $f : V \to U$ be a finite étale morphism. The goal of this talk is to introduce a map

\[(\log, \Delta_v) : Z_d(V \times_U V) \to F_0 G(\overline{F}_F) \otimes_{\mathbb{Z}} \mathbb{Q}\]

and to show that this map gives generalizations of classical invariants of ramification.

0.1 source

Since $V \to U$ is assumed finite étale, the fiber product $V \times_U V$ is also finite étale over $U$ and hence is smooth of dimension $d$ over $K$. Thus, $Z_d(V \times_U V)$ is the free abelian group generated by the classes of irreducible components of $V \times_U V$. In particular, if $U$ is connected and $V$ is a Galois covering of Galois group $G$, it is identified with the free abelian group $\mathbb{Z}[G]$.

0.2 target

For a noetherian scheme $X$, the Grothendieck group of the category of coherent $O_X$-modules is denoted by $G(X)$. Let $F_n G(X) \subset G(X)$ denote the topological filtration generated by the classes of modules of dimension of support at most $n$. 
For $V$ as above, we define $\mathcal{C}_V$ to be the category, whose objects are proper flat schemes $Y$ over $O_K$ containing $V$ as a dense open subscheme. A morphism $Y' \to Y$ in $\mathcal{C}_V$ is a morphism $Y' \to Y$ over $O_K$ inducing the identity on $V$. We put

\[(0.2)\quad F_0G(\overline{V}_F) = \lim_{\leftarrow} \mathcal{C}_V F_0G(Y_F).\]

The transitions maps are proper push-forwards.

For a map $f : V \to U$ of separated smooth schemes of finite type over $K$, the push-forward map

\[(0.3)\quad f_* : F_0G(\overline{V}_F) \longrightarrow F_0G(\overline{U}_F)\]

is defined. In particular, taking $U = \text{Spec } K$, the degree map

\[(0.4)\quad \text{deg} : F_0G(\overline{V}_F) \longrightarrow \mathbb{Z}\]

is defined. For a finite flat map $f : V \to U$ of separated smooth schemes of finite type over $K$, the flat pull-back map

\[(0.5)\quad f^* : F_0G(\overline{U}_F) \longrightarrow F_0G(\overline{V}_F)\]

is defined by the flattening theorem of Raynaud-Gruson.

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1 Classical case

Let $K$ and $F$ be as above. Let $L$ be a finite separable extension and consider $f : V = \text{Spec } L \to U = \text{Spec } K$. Then, the target group $F_0\text{G}(\overline{V}_F)$ of the map (0.1) is $F_0\text{G}(\text{Spec } O_L \otimes_{O_K} F) = \mathbb{Z}$. The map $f^* : F_0\text{G}(\overline{U}_F) = \mathbb{Z} \to F_0\text{G}(\overline{V}_F) = \mathbb{Z}$ is the multiplication by the ramification index $e_{L/K}$ and the map $f_* : F_0\text{G}(\overline{U}_F) = \mathbb{Z} \to F_0\text{G}(\overline{V}_F) = \mathbb{Z}$ is the multiplication by the residual degree $f_{L/K}$. If $L$ is a Galois extension of Galois group $G$, the source group $\mathbb{Z}_d(V \times_U V)$ is $\mathbb{Z}[G]$.

1.1 different

For a finite separable extension $L$ of $K$, the different and its wild part are defined by

\begin{align}
D_{L/K} &= \text{length}_{O_L} \Omega_{O_L/O_K}^1, \\
D_{L/K}^{\log} &= D_{L/K} - (e_{L/K} - 1).
\end{align}

We have $D_{L/K}^{\log} \geq 0$. The equality is equivalent to $p \nmid e_{L/K}$. For an intermediate extension $K \subset M \subset L$, we have chain rules

\begin{align}
D_{L/K} &= D_{L/M} + e_{L/M} D_{M/K}, \\
D_{L/K}^{\log} &= D_{L/M}^{\log} + e_{L/M} D_{M/K}^{\log}.
\end{align}

1.2 Artin and Swan characters

If $L$ is a Galois extension of Galois group $G$, the Artin character and the Swan character are defined by

\begin{align}
a_{L/K}(\sigma) &= a_G(\sigma) \\
&= \begin{cases} D_{L/K} & \text{if } \sigma = 1 \\
-\text{length}_{O_L} O_L/(\sigma(x) - x; x \in O_L) & \text{if } \sigma \neq 1,
\end{cases}
\end{align}

\begin{align}s_{L/K}(\sigma) &= s_G(\sigma) \\
&= \begin{cases} D_{L/K}^{\log} & \text{if } \sigma = 1 \\
-\text{length}_{O_L} O_L/(\sigma(x)/x - 1; x \in O_L \setminus \{0\}) & \text{if } \sigma \neq 1.
\end{cases}
\end{align}

We have $s_G(\sigma) = 0$ unless the order of $\sigma$ is a power of $p$. We also have $s_G(\sigma) = s_G(\sigma')$ if $\langle \sigma \rangle = \langle \sigma' \rangle$.

For a subgroup $H \subset G$ with corresponding intermediate extension $M$, we have

\begin{align}s_H(\sigma) &= \begin{cases} s_G(1) - e_{L/M} D_{M/K}^{\log} & \text{if } \sigma = 1 \\
s_G(\sigma) & \text{if } \sigma \neq 1.
\end{cases}
\end{align}
and a similar equality for the Artin character, for \( \sigma \in H \). For a quotient group \( G \rightarrow \overline{G} \) with corresponding intermediate extension \( M \), we have

\[
(1.8) \quad e_{L/M} \cdot s_{\overline{G}}(\sigma) = \sum_{\tau \in G, \sigma = \overline{\tau}} s_G(\tau)
\]

and a similar equality for the Artin character, for \( \sigma \in G \). In particular, putting \( \overline{G} = 1 \), we obtain \( \sum_{\sigma \in G} a_G(\sigma) = \sum_{\sigma \in G} s_G(\sigma) = 0 \).

### 1.3 Swan conductor

Let \( M \) be an \( \ell \)-adic representation of the absolute Galois group \( G_K = \text{Gal}(\overline{K}/K) \). Let \( L \) be a finite Galois extension of \( K \) of Galois group \( G \) such that the reduction modulo \( \ell \) of the restriction to \( G_L \subset G_K \) is trivial. Then, the Swan conductor of \( M \) is defined by

\[
(1.9) \quad \text{Sw}(M) = \frac{f_{L/K}}{|G|} \sum_{\sigma \in G} s_{L/K}(\sigma) \text{Tr}(\sigma : M)
\]

\[
(1.10) \quad = \frac{f_{L/K}}{|G|} \sum_{\sigma \in G} s_{L/K}(\sigma) (\dim M^\sigma - \frac{\dim M^\sigma / M^p}{p - 1}).
\]

It is independent of \( L \) by (1.8). We can use the second equality as the definition for a mod-\( \ell \) representation.

The Hasse-Arf theorem asserts that \( \text{Sw}(M) \) is an integer. We have \( \text{Sw}(M) \geq 0 \) and the equality holds if and only if the restriction to the \( p \)-Sylow subgroup \( P_K \) of the inertia subgroup \( I_K \subset G_K \) is trivial. The equality (1.7) implies the induction formula

\[
(1.11) \quad \text{Sw}(\text{Ind}_{G_L}^{G_K} M) = f_{L/K}(\dim M \cdot D_{L/K}^\log + \text{Sw}(M)).
\]

### 2 Definition of the map (0.1)

#### 2.1 Logarithmic diagonal

Let \( U \) be a separated smooth scheme of dimension \( d \) of finite type over \( K \) and \( X \) be a separated regular flat scheme of finite type over \( O_K \) containing \( U \) as the complement of a divisor \( D \) with simple normal crossing. Let \( (X \times_{O_K} X)^{\sim} \) be the log self-product and the log diagonal closed immersion \( \Delta_X : X \rightarrow (X \times_{O_K} X)^{\sim} \). Its generic fiber \( X_K \rightarrow (X_K \times_K X_K)^{\sim} \) is a regular immersion of codimension \( d \).

We give a local description. Assume \( X = \text{Spec} A \) and \( D \) is defined by \( \prod_{i \in I} t_i \).

Then, we have

\[
(X \times_{O_K} X)^{\sim} = A \otimes_{O_K} A[U_i^{\pm 1} (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes t_i) (i \in I))
\]

and the log diagonal map is defined by the map

\[
A \otimes_{O_K} A[U_i^{\pm 1} (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes t_i) (i \in I)) \rightarrow A
\]

sending \( a \otimes 1 \) and \( 1 \otimes a \) to \( a \in A \) and \( U_i \) to 1 for \( i \in I \).
2.2 Logarithmic localized intersection product

**Theorem 1** Let the notation be as above. Then,

1. There exists a unique map

\[(\mathcal{F}, \Delta_X)^{\log} : G((X \times_{O_K} X)^\sim) \longrightarrow G(X_F)\]  

such that for a coherent module $\mathcal{F}$ and an integer $q > d$, we have

\[(([\mathcal{F}], \Delta_X))^{\log} = (-1)^q([\text{Tor}_q^{O(X \times_{O_K} X)^\sim}(\mathcal{F}, O_X)] - [\text{Tor}_{q+1}^{O(X \times_{O_K} X)^\sim}(\mathcal{F}, O_X)]).\]

2. The map (2.1) induces a map

\[((\mathcal{F}, \Delta_X))^{\log} : G((X_K \times_K X_K)^\sim) \longrightarrow G(X_F).\]

Further, it maps $F_{d+i}G((X_K \times_K X_K)^\sim)$ into $F_iG(X_F)$ for $i \in \mathbb{Z}$.

Now we define the map (0.1).

**Corollary 2** Let $f : U \rightarrow V$ be a finite étale morphism. Then, there exists a unique map

\[((\mathcal{F}, \Delta_Z))^{\log} : Z_d(V \times_U V) \longrightarrow F_0G(\nabla_F) \otimes_{\mathbb{Q}}\]

that makes the diagram

\[(2.3)\]

\[
\begin{array}{ccc}
Z_d(V \times_U V) & \longrightarrow & F_0G(\nabla_F) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\otimes_{O_{V \times_K V}O_{W \times_K W}} & & \downarrow \text{projection} \\
F_dG(W \times_U W) & \longrightarrow & F_0G(Y_F) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\uparrow \text{restriction} & & \uparrow \text{projection} \\
F_dG((Z_K \times_K Z_K)^\sim) & \longrightarrow & F_0G(Z_F) \\
\end{array}
\]

commutative for an arbitrary diagram

\[(2.4)\]

\[
\begin{array}{ccc}
W & \subset & Z \\
g \downarrow & & \downarrow \\
V & \subset & Y \\
f \downarrow & & \downarrow \\
U & \subset & X \\
\end{array}
\]

of schemes over $O_K$ satisfying the following conditions:
(2.4.1) $X$ is a proper flat scheme over $O_K$ containing $U$ as the complement of a Cartier divisor $B$. The generic fiber $X_K$ is smooth and the divisor $B_K$ has simple normal crossings.

(2.4.2) $Y$ is a proper flat scheme over $O_K$ containing $V$ as a dense open subscheme. Namely, $Y$ is an object of $\mathcal{C}_V$.

(2.4.3) $Z$ is a proper regular scheme over $O_K$ containing $W$ as the complement of a divisor $D$ with simple normal crossings.

(2.4.4) The quadrangles are Cartesian.

(2.4.5) The proper map $g: W \to V$ is generically finite of constant degree $[W : V]$.

3 Ramification theory

Let $K$ and $F$ be as above. Let $f: V \to U$ be a finite étale morphism of separated smooth schemes purely of dimension $d$ of finite type over $K$. We define generalizations of classical invariants recalled in Section 1 to higher dimension by using the map

\[(0.1) \quad ((, \Delta_{\nabla}))^{\log}: Z_d(V \times_U V) \longrightarrow F_0G(\nabla_F) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

3.1 different

We define the wild different by

\[(3.1) \quad D^{\log}_{V/U} = (([V \times_U V] - [\Delta_V], \Delta_{\nabla}))^{\log} = f^*(([[\Delta_U], \Delta_{\nabla}))^{\log} - (([\Delta_V], \Delta_{\nabla}))^{\log}
\]

in $F_0G(\nabla_F) \otimes_{\mathbb{Z}} \mathbb{Q}$. For a finite étale morphism $g: V' \to V$, we have obviously a chain rule

\[(3.2) \quad D_{V'/V} = D_{V'/V} + g^*D_{V/U}.
\]

3.2 Swan character class

If $V$ is a Galois covering of Galois group $G$, we define the Swan character class $s_{V/U}(\sigma) \in F_0G(\nabla_F) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

\[(3.3) \quad s_{V/U}(\sigma) = s_G(\sigma) = \begin{cases} D^{\log}_{V/U} & \text{if } \sigma = 1, \\ -((\Gamma_\sigma, \Delta_{\nabla}))^{\log} & \text{if } \sigma \neq 1. \end{cases}
\]

We have $s_G(\sigma) = 0$ unless the order of $\sigma$ is a power of $p$. We expect but do not know $s_G(\sigma) = s_G(\sigma')$ if $\langle \sigma \rangle = \langle \sigma' \rangle$. We have equalities analogous to (1.7) and (1.8).
3.3 Swan class

Let $F$ be a smooth $\ell$-adic sheaf on $U$. We take a finite étale Galois covering $f : V \rightarrow U$ trivializing the reduction $\overline{F}$. Let $G$ be the Galois group and $M$ be the representation of $G$ corresponding to $\overline{F}$. Then, we define the Swan class $SwF \in F_0G(\overline{U}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the naive Swan class $Sw^{\text{naive}}F \in F_0G(\overline{U}) \otimes_{\mathbb{Z}} \mathbb{Q}(\zeta_p^{\infty})$ by

$$SwF = \frac{1}{|G|} \sum_{\sigma \in G} f_* s_G(\sigma) \left( \dim M^\sigma - \frac{\dim M^\sigma/M^{\sigma p}}{p-1} \right),$$

$$Sw^{\text{naive}}F = \frac{1}{|G|} \sum_{\sigma \in G_{(p)}} f_* s_G(\sigma) \tr^{Br}(\sigma : \dim M)$$

where $G_{(p)} = \{ \sigma \in G | \text{the order of } \sigma \text{ is a power of } p \}$ and $\tr^{Br}$ denotes the Brauer trace. They are independent of $V$ by an analogue of (1.8). The Swan class $SwF$ is the image of the naive Swan class by the projection $\mathbb{Q}(\zeta_p^{\infty}) \rightarrow \mathbb{Q}$.

We expect the following generalization of the Hasse-Arf theorem.

**Conjecture 3** The Swan class $SwF \in F_0G(\overline{U}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is in the image of $F_0G(\overline{U})$

**Theorem 4** Conjecture 3 is true if $\dim U = 1$.

*Proof.* By the induction formula below, it is reduced to the rank 1 case. In the rank 1 case, one can compute the Swan class explicitly in terms of the ramification divisor in the sense of Kato.

Conjecture 3 implies a conjecture of Serre:

The Artin central function for an isolated fixed point is a character.

We have a conductor formula.

**Theorem 5** If $V = \text{Spec } K$, we have

$$SwR\Gamma_c(U_\overline{\mathbb{F}_\ell}, \mathcal{F}) = \deg Sw\mathcal{F} - \rank \mathcal{F} \cdot \deg((\Delta_U, \Delta_{\overline{V}}))^\log.$$

Further if $\mathcal{F} = \mathbb{Q}_\ell$, we obtain

$$SwR\Gamma_c(U_\overline{\mathbb{F}_\ell}, \mathbb{Q}_\ell) = -\deg((\Delta_U, \Delta_{\overline{V}}))^\log.$$

*Proof.* A logarithmic Lefschetz trace formula for open variety and the associativity for the localized intersection product.

We expect have the following relative version.

**Conjecture 6** Let $f : U \rightarrow V$ be a smooth morphism of relative dimension $d$ of separated smooth schemes of finite type over $K$. We assume that there exist a proper smooth scheme $X$ over $V$ containing $U$ as the complement of a divisor $D$ with relative simple normal crossings.

Then, for a smooth $\overline{\mathbb{Q}_\ell}$-sheaf or a smooth $\overline{\mathbb{F}_\ell}$-sheaf $\mathcal{F}$, we have

$$\rank Rf_*\mathcal{F} \cdot ((\Delta_V, \Delta_{\overline{V}}))^\log - SwRf_*\mathcal{F} = f_*(\rank \mathcal{F} \cdot ((\Delta_U, \Delta_{\overline{V}}))^\log - Sw\mathcal{F}).$$
We can prove Conjecture 6 if $\dim V = 0$.
Note that we have
\[
\text{rank} Rf_! \mathcal{F} = \text{rank} \mathcal{F} \cdot \text{rank} Rf_! \mathbb{Q}_\ell = \text{rank} \mathcal{F} \cdot (-1)^d \deg c_d(\Omega^1_{X/V}(\log D)).
\]
The equality (3.8) is equivalent to the combination of
\[
\begin{align*}
(3.9) \quad & Sw Rf_! \mathcal{F} = f_* Sw \mathcal{F} + \text{rank} \mathcal{F} \cdot Sw Rf_! \mathbb{Q}_\ell, \\
(3.10) & Sw Rf_! \mathbb{Q}_\ell = (-1)^d \deg c_d(\Omega^1_{X/V}(\log D)) \cdot ((\Delta_V, \Delta_V))^{\log} - f_* ((\Delta_U, \Delta_U))^{\log}.
\end{align*}
\]
If $d = 0$, in other words, if $f : V \to U$ is finite étale, the right hand side of (3.10) is $f_* D_{V/U}^{\log}$. 

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