Characteristic cycle and singular support of an etale sheaf

X smooth /& perfect char \( p > 0 \)

\( K \) constructible complex of \( \Lambda \)-modules, for \( \text{Xet } \Lambda \text{ for } k \neq p \)

Goal. Define \( \text{Ch}_{\text{et}} X \) as a cycle on the cotangent bundle \( T^*X \)

Prove. Index formula for \( \text{Enri-Poincare} \text{ char.} \)

Milnor formula for vanishing cycles.

Goal has been attained for

- Tameley ramified case
- Curves
- Surfaces
- Higher dimension?

First step. Define singular support

- Conic closed subset of \( T^*X \)
- Underlying set of \( \text{Ch}_{\text{et}} X \)

Satisfying conditions \((SS!) \) \& \((SS4)\)

Thm 1. Assume \( S \subset T^*X \) satisfies \((SS4)\).

Then there exists a unique \( \text{Ch}_{\text{et}} X \) supported on \( S \) satisfying the Milnor formula. If \( X \) is proper, it also satisfies the index formula.

2. If \( S \) satisfies \((SS4)\), it also satisfies \((SS!)\).

Conversely, if \( S \cap (T^*X) \) satisfies \((SS!)\), it satisfies \((SS4)\).

This means that the goal will be attained if one can construct \( S \) satisfying \((SS!)\).

Ramification implies construction of \( S \) for surfaces.
1. Classical cases

2. Milnor formula.

3. Conditions (SS!) & (SS4)

4. Points i. Proof:

1. $X \supset U = X - D$, $j : U \subset X$. $K = j^! \mathcal{F} F \hat{\otimes} c\mathcal{O}_U$
   
   - tamely ramified along $D = UD$; simple n.c.d.
   
   $\text{Char } K = (-1)^d \text{rk } \sum \frac{[T^k_x X]}{I} \text{ canonical bundle}$

   - curve
   
   $\text{Char } K = \left( \inf \sum \dim_{\text{tot}} (f_\ast \mathcal{F}) \cdot [T^k_x X] \right)_{X \in \mathcal{D}} \text{ O-section fiber}$

2. $u \in U \to X \ (f \ast f =) \text{ df section of } T^* X \text{ on } U$
   
   flat if isolated chart, $\text{df}(U) \cap S$ isolated.

   $C$ smooth curve

   $- \dim \text{tot } \phi_u (\mathcal{F}, f) = (\text{Char } K, \text{ df })_{T^* C}$

   space of var. cycles; intersection number

3. $S \subset T^* X$ conic closed div.

   non characteristic morphism (generalization of smooth)

   $f : W \to X$, $g : X \to Y$

   SS! / SS4

   non char = good property for $K$

   $g$ local acyclicity (smooth = loc. acyclic rel to n.c. shift)

   $f$ can isom $f^! \mathcal{K} \otimes Rf^! \mathcal{A} \to RF^! \mathcal{K}$ is isom.

   $\mathcal{J}$ is propagation for $K$.

   (Poincare duality for smooth morphism)
non characteristic.

\[ f : W \to X, \quad d = \dim X, \quad b = \dim W \]

(1) \( f^* S = W \times_X S \) is finite over \( T^* W \) w.r.t.

canonical morphism \( W \times X \times T^* X \to T^* W \)

(2) Any cohomology of \( f^* S \) are of \( \dim b \).

\[ g : X \to Y \quad \text{flat,} \quad d = \dim X, \quad c = \dim Y \]

(1) inverse image of \( S \) by \( X \times_T Y \to TX \)

is a subset of the \( \mathcal{O} \)-section

(2) \( \forall y \in Y \), any cohomology of \( S \times_Y y \) is of \( \dim d - c \).

4.2. \[ g : X \to Y \quad \text{(smooth,)} \]

\[ i : Y \hookrightarrow Y \]

\[ i^* Y \]

- \( i \) non-chaotic \( \iff \) \( g \) non-chaotic on a \( \text{Whb} \) of \( W \).

- \( i \) propagating \( \iff \) \( g \) loc. acyclic

\[ \Rightarrow SGA 4 \cdot \frac{1}{2} \cdot \text{App à Th. finitude (without smoothness)} \]

\[ \Rightarrow \text{Reduction to \( Y \)} \text{curve,} \quad \mathcal{P} = \bigotimes \zeta^* \mathcal{R} \]

1. \( X \in \mathcal{P} \).

local \( \mathcal{R} \)ech transform.

- Vanishing cycles over general base scheme

- \( (\text{semi}) \) continuity of Swan conductor

\[ f : W \to X \quad \text{non-chaotic} \Rightarrow \text{Ch} \cdot f^* \mathcal{K} = (-1)^{d-b} f^! \text{Ch} \cdot \mathcal{K} \]

- \( f \) immersion, induction on codim.

\[ \text{codim} = 1 \quad \text{reduction to } d = 2, \quad \text{global argument} \]

- Index formula: induction on \( \text{dim} \) using G-O-S