Local Fourier transform and epsilon factors

joint work with Ahmed Abbes

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Abstract

The local epsilon factors appear in the constant term of the functional equation of the L-functions. Laumon proved a formula expressing the local epsilon factor using a local variant of \( \ell \)-adic Fourier transform by a global and arithmetic method and deduced a product formula for the constant term. We will discuss a local and geometric method to prove Laumon’s formula, under a certain assumption.

Plan

1. Laumon’s formula.
2. local Fourier transform.
3. Witt vectors and Ramification.
4. blow-up.

1 Laumon’s formula.

\( \ell \)-adic Fourier transform: \( k \) a field of characteristic \( p \), later we will assume \( p > 2 \):

\[
(\ell\text{-adic sheaf on } \mathbb{A}^1_k) \longrightarrow (\ell\text{-adic sheaf on } \mathbb{A}^1_k).
\]

Analogue of \( \int_k f(x)\psi(xy)dx \). Function-sheaf dictionary.

local version: \( K, K^\vee \) local fields of characteristic \( p > 0 \), \( G_K = \text{Gal}(\overline{K}/K), G_{K^\vee} = \text{Gal}(\overline{K^\vee}/K^\vee) \):

\[
(\ell\text{-adic representations of } G_K) \longrightarrow (\ell\text{-adic representations of } G_{K^\vee}).
\]

L-function: \( k = \mathbb{F}_q \) finite field. \( C \) curve over \( k \). \( \mathcal{F} \) \( \ell \)-adic sheaf on \( C \).

\[
L(\mathcal{F}, t) = \prod_{x \in C} \det(1 - \text{Fr}_x t : \mathcal{F}_x)^{-1} \in \mathcal{O}_\ell(t).
\]

Functional equation and the product formula:

\[
L(\mathcal{F}, t) = \varepsilon(\mathcal{F})t^{-\chi(C_k, \mathcal{F})}L(\mathcal{F}^*, (qt)^{-1}).
\]
The local epsilon factors $\varepsilon_x(F_x)$ play an important role e.g. in the local Langlands correspondence.

Laumon’s formula.

$$\det(- FR : F\psi(F)) = \varepsilon_x(F_x).$$

(1.1)

**Goal:** Reprove Laumon’s formula, under a certain assumption, by a *local* and *geometric* method, using a new construction from ramification theory.

## 2 local Fourier transform

$k = \mathbb{F}_q$.

$\mathbb{P}_k$ projective line, $x$ inhomogeneous coordinate. $T$ completion at 0. $K$ the fraction field of $O_T$.

$\mathbb{P}_k^\vee$ the dual projective line, $x^\vee$ inhomogeneous coordinate. $T^\vee$ completion at $\infty$.

$K^\vee$ the fraction field of $O_{T^\vee}$.

$\mathcal{F}_i$ on $T^\text{et}$: zero-extension of the $\ell$-adic sheaf corresponding to $V$.

$\mathcal{L}_{\psi_0}(xy)$ on $\mathbb{P}^1 \times \mathbb{P}^{1\vee}$: zero extension of the Artin-Schreier sheaf on $A^1 \times A^{1\vee} \subset \mathbb{P}^1 \times \mathbb{P}^{1\vee}$ defined by the equation $X^p - X = xy$ and an additive character $\psi_0 : \mathbb{F}_p \to \overline{\mathbb{Q}_\ell}^\times$.

**Local Fourier transform:**

$$F_{\psi_0}(\mathcal{F}_i) = \psi^1(\text{pr}_1^* \mathcal{F}_i \otimes \mathcal{L}_{\psi_0}(xy), \text{pr}_2).$$

The space of nearby cycles with respect to the second projection: $\ell$-adic representation of $G_{K^\vee}$.

We consider the case where $V = \text{Ind}_{G_L}^{G_K} L_\chi$ is monomial.

$L$: a finite separable extension of $K$. $\chi : G_L \to \overline{\mathbb{Q}_\ell}^\times$ a character.

**More precise goal:** We assume $\chi$ is wildly ramified. We put $S = \text{Spec} \ O_L$ and let $f : S \to T$ be the map defined by the inclusion $K \subset L$. First, we define a map $g : S \to T^\vee$ and construct a diagram

$$\begin{array}{ccc}
S \times S & \xrightarrow{f \times g} & S \\
\downarrow & & \downarrow f \\
T \times T^\vee & \xrightarrow{f} & T \\
\downarrow & & \downarrow f \\
S & \xrightarrow{g} & T^\vee.
\end{array}$$

$$F_{\psi_0}(\mathcal{F}_i)$$
Then, we compute the pull-back
\[ g^* F_{\psi_0}(\mathcal{F}_1) = \psi^1(\text{pr}_1(G_\chi)_! \otimes \mathcal{L}_{\psi_0}(xy), \text{pr}_2) \]
and prove the following.

**Theorem 1** There exists an isomorphism
\[
F_{\psi_0}(\mathcal{F}) \rightarrow g_*(G_\chi \otimes L_{\psi_0}(f^*xg^*y) \otimes K(\sqrt{\frac{f^*dx}{2g^*dy}}) \otimes Q)
\]
under a certain assumption formulated in the next section.

Here the Kummer sheaf \( K(\sqrt{\frac{f^*dx}{2g^*dy}}) \) is defined by the square root of \( \frac{f^*dx}{2g^*dy} \in L^x \) and \( Q = H^1_c(A^1, K(x) \otimes L_{\psi_0}(x)) \) is the rank 1 representation on which the Frobenius acts as the quadratic Gauss sum \( -\sum_{x \in F_q} \psi_0(\text{Tr} x/y, x^2) \).

Some partial results have been studied by Fu Lei.

We deduce Laumon’s formula (1.1) from the isomorphism (2.1) in Theorem 1. The key ingredients are:
- Explicit and standard computation of the local epsilon factors.
- Explicit reciprocity law for Artin-Schreier-Witt theory.
- Induction formula for the local epsilon factors.

### 3 Witt vectors and Ramification

Decompose the character \( \chi = \chi_t \cdot \chi_w \) where \( \chi_t \) is tamely ramified and \( \chi_w \) is defined by a Witt vector \( a = (a_0, \ldots, a_m) \in W_{m+1}(L) \) via the Artin-Schreier-Witt theory \( W_{m+1}(L) \rightarrow H^1(L, \mathbb{Z}/p^{n+1}\mathbb{Z}) \) for a fixed embedding \( \mathbb{Z}/p^{n+1}\mathbb{Z} \rightarrow \overline{\mathbb{Q}}_\ell^x \). We assume that the Swan conductor \( n = \text{Sw} \chi = \text{Sw} \chi_w > 0 \) and that \( a \in \text{Fil}^n W_{m+1}(L) \) namely
\[
p^{m-i} \text{ord} a_i \leq -n \quad \text{for } i = 0, \ldots, m.
\]

We put
\[
F^m da = \sum_{i=0}^m \alpha a^{p^{m-i}}d \log a_i = \alpha d \log t
\]
using a notation for the de Rham-Witt complexes. The assumption implies \( \text{ord} \alpha = -n \).

We define \( y \in L^x \) by
\[
F^m da + c \cdot db = 0
\]
and put
\[
d = \text{ord} \frac{d \log b}{d \log t}, \quad d' = \text{ord} \frac{d \log c}{d \log t}.
\]

We assume
\[
2d + pd' \leq (p - 2)n.
\]
The assumption means that the wild ramification of \( \chi \) is deeper than those of \( f \) and \( g \).
4 Blow-up

Define a map $g : S \to T^\vee$ by $c \in L^x$ and consider the diagram (2.1). For simplicity, we assume $L$ is a tamely ramified and totally ramified extension of $K$ with respect to $g$. Let $(S \times S)'$ denote the blow up of $S \times S$ at the closed point $s$ of the diagonal and $(S \times S)^\sim \subset (S \times S)'$ denote the complement of the proper transforms of $S \times s$ and $s \times S$. Let $\gamma : S \to S \times T^\vee$ be the graph of $g$. We consider the diagram

\[
\begin{array}{ccc}
(S \times S)^\sim & \to & S \\
\downarrow \gamma & & \downarrow f \\
S & \to & T \\
g & \downarrow & \\
& & T^\vee
\end{array}
\]

Let $\Gamma \subset S \times T^\vee$ be the graph of $g : S \to T^\vee$ and $\Gamma^\sim \subset (S \times S)^\sim$ denote the proper transform of the inverse image $\Gamma \times_{T^\vee} S$. Then, by the assumption $d' = 0$, $\Gamma^\sim$ is finite etale over $S$. By the dimension formula for the local Fourier transform, we have $\dim F_{\psi_0}(\mathcal{F}_l) = \deg g$. The cohomology sheaf $\psi^1$ is supported on the closed fiber $\Gamma^\sim \times_S s$.

Let $\delta^\sim : S \to (S \times S)^\sim$ be the log diagonal map. We will define an isomorphism

\[
(4.1) \quad \psi^1(\text{pr}_1^* \mathcal{G}_X \otimes \mathcal{L}_{\psi_0}(b_1c_2))_{\delta^\sim(s)} \to \mathcal{G}_X \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{K}(\frac{1}{2} db) \otimes \mathcal{Q}.
\]

This will imply the isomorphism (2.1). We put $\mathcal{H} = \mathcal{H}(\text{pr}_2^* \mathcal{G}_X, \text{pr}_1^* \mathcal{G}_X)$ on $\eta \times \eta \subset S \times S$. Then the left hand side is isomorphic to

\[
\mathcal{G}_X \otimes \mathcal{L}_{\psi_0}(b_2c_2) \otimes \psi^1(\mathcal{H} \otimes \mathcal{L}_{\psi_0}((b_1 - b_2)c_2))_{\delta^\sim(s)}
\]

and the isomorphism (4.1) is reduced to

\[
(4.2) \quad \psi^1(\mathcal{H} \otimes \mathcal{L}_{\psi_0}((b_1 - b_2)c_2))_{\delta^\sim(s)} \to \mathcal{K}(\frac{1}{2} db) \otimes \mathcal{Q}.
\]

According to the decomposition $\chi = \chi_t \cdot \chi_w$, we have $\mathcal{G}_X = \mathcal{G}_t \otimes \mathcal{G}_w$ and $\mathcal{H}_X = \mathcal{H}_t \otimes \mathcal{H}_w$. Since $\mathcal{H}_t$ is extended to a smooth sheaf on $(S \times S)^\sim$ and since the stalk $\mathcal{H}_{t,\delta^\sim(s)}$ is trivial, we may assume $\chi = \chi_w$ is defined by a Witt vector $a$.

We define an isomorphism (4.2) assuming $n = 2r$ is even. We blow-up $r$-times the closed point $s \in S \subset (S \times S)^\sim$ in the log diagonal to define $(S \times S)^{(r)} \subset (S \times S)^{(r)}[r] \to (S \times S)^\sim$. Then some elementary computation on Witt vectors shows that $\mathcal{H} \otimes \mathcal{L}_{\psi_0}((b_1 - b_2)c_2)$ is extended to a smooth sheaf on $(S \times S)^{(r)}$. Further, the restriction to the exceptional divisor $\Theta^{(r)} \subset (S \times S)^{(r)}$ is isomorphic to the Artin-Schreier sheaf $\mathcal{L}_{\psi_0}(\frac{1}{2} \beta w^2)$ where $\beta = t^w b'$ and $d'$. Thus the assertion follows from the isomorphism

\[
H^1_c(A^1, \mathcal{L}_{\psi_0}(\frac{1}{2} \beta w^2)) \to \mathcal{K}(\frac{1}{2} db) \otimes \mathcal{Q}.
\]