The characteristic class and micro local analysis on of an \(\ell\)-adic étale sheaf (with Ahmed Abbes)

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Plan:
1. Characteristic class.
2. Localization.
3. Rank 1 case.
4. Analogy with Microlocal analysis.

Notation: \(k\) field of characteristic \(p > 0\).
\(X\) separated of finite type over \(F\).
\(\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell\) etc. (\(\ell \neq p\))
\(\mathcal{F}\) \(\Lambda\)-sheaf on \(X\), or more generally, an object in a suitable derived category.

1 Characteristic class.

Let \(F\) be a perfect field of characteristic \(p > 0\). Let \(X\) be a separated scheme of finite type over \(F\) and \(\mathcal{F}\) be an \(\ell\)-adic sheaf on \(X\). Then the characteristic class \(C(\mathcal{F}) \in H^0(X, K_X)\) is defined as follows. Here and in the following \(K_X = a'\Lambda\) where \(a : X \to F\). Hence if \(X\) is smooth of dimension \(d\), the characteristic class \(C(\mathcal{F})\) is defined in \(H^{2d}(X, \Lambda(d))\).

We consider

\[
1 \in \text{Hom}(\mathcal{F}, \mathcal{F}) = H^0_X(X \times X, R\text{Hom}(p^*_2 \mathcal{F}, p^*_1 \mathcal{F})) = H^0_X(X \times X, R\text{Hom}(p^*_1 \mathcal{F}, p^*_2 \mathcal{F})).
\]

By the natural pairing, \(R\text{Hom}(p^*_2 \mathcal{F}, p^*_1 \mathcal{F}) \otimes R\text{Hom}(p^*_1 \mathcal{F}, p^*_2 \mathcal{F}) \to K_{X \times X}\), their pairing is defined and gives the characteristic class as

\[
C(\mathcal{F}) = \langle 1, 1 \rangle = H^0_X(X \times X, K_{X \times X}) = H^0(X, K_X).
\]

If \(X\) is smooth of dimension \(d\) and \(\mathcal{F}\) is smooth of rank \(r\), we have \(C(\mathcal{F}) = r \cdot (-1)^d c_d(\Omega^1_{X/F})\).
If $X$ is proper, the Lefschetz trace formula in SGA 5 gives
\[
\text{Tr}C(F) = \chi(X_F, F).
\]

The characteristic class $C(F) \in H^0(X, K_X)$ may be also regarded as the class of the composition
\[
\delta_* \Lambda_X \to \mathcal{H}om(pr_2^* F , pr_1^* F ) \to \delta_* K_X.
\]
The first map is the adjoint of $\Lambda_X \to \mathcal{H}om(pr_2^* F , pr_1^* F ) = \mathcal{H}om(F, F)$ and the second map is the adjoint of $\delta^* \mathcal{H}om(pr_2^* F, pr_1^* F ) = F \otimes \mathcal{H}om(F, K_X) \to K_X$.

### 2 Localization of the characteristic class.

We consider the following case. Assume $X$ is smooth and let $S \subset X$ be a closed subscheme. Let $U = X \setminus S$ be the complement and $j : U \to X$ be the open immersion. We assume $F = j_! F_U$ is the zero-extension of a smooth sheaf $F_U$ on $U$. Then, we expect that the difference $C(j_! F) - \text{rank} F \cdot C(j_! \Lambda)$ may be computed by the ramification of $F$ along $S$. Here, we construct a natural lifting of $C(j_! F) - \text{rank} F \cdot C(\Lambda)$ in $H^0_S(X, K_X)$.

Let $\mathfrak{X}$ be the formal completion of $X \times X$ with respect to the diagonal $\delta : X \to X \times X$ and $\mathfrak{X}^{\text{rig}}$ be the associated rigid space. Let $\psi : \mathfrak{X}^{\text{rig}} \to X \times X \setminus X$ be the canonical map and $\rho : \mathfrak{X}^{\text{rig}} \to X$ be the specialization. Then, we have a non-commutative diagram

\[
\begin{array}{ccc}
\mathfrak{X}^{\text{rig}} & \xrightarrow{\psi} & X \times X \setminus X \\
\rho \downarrow & & \downarrow g \\
X & \xrightarrow{\delta} & X \times X
\end{array}
\]

of associated étale topoi where $g$ denotes the open immersion. The nearby cycle functor $\Psi : D(\mathfrak{X}) \to D(\mathfrak{X}^{\text{rig}})$ is defined by $\Psi F = \psi^* g^* F$ is defined on $\mathfrak{X}^{\text{rig}}$. The vanishing cycle functor $\Phi$ fits in the distinguished triangle
\[
\rightarrow \rho^* \delta^* F \to \Psi F \to \Phi F \to
\]
on $\mathfrak{X}^{\text{rig}}$. If $F = \delta^* G$, we have $\Psi F = 0$ and an isomorphism $\Phi F \to \rho^* G[1]$. Fujiwara’s theorem gives us a canonical isomorphism $\delta^* g_* F \to \rho_* \psi^* F$.

Applying the functor $\Phi$ to the composition $\delta_* \Lambda_X \to \mathcal{H}om(pr_2^* j_! F , pr_1^* j_! F ) \to \delta_* K_X$, we obtain a map $\Phi(\delta_* \Lambda_X ) \to \Phi(\delta_* K_X)$. By the isomorphism $\Phi \delta_* G = \rho^* G[1]$, it is equivalent to the map $\rho^* \Lambda \to \rho^* K_X$ and by the adjunction further to $\Lambda \to \rho_* \rho^* K_X$. By the assumption that $F$ is smooth on $U$, the complex $\mathcal{H}om(pr_2^* j_! F , pr_1^* j_! F )$ is smooth on $U \times U$. Thus, the restriction of $\Phi \mathcal{H}om(pr_2^* j_! F , pr_1^* j_! F )$ on $U$ is 0. Hence, the map $\Lambda \to \rho_* \rho^* K_X$ factors through $\Lambda_S$ and gives an element in $H^0_S(X, \rho_* \rho^* K_X)$.

By the isomorphism $\delta^* g_* \to \rho_\ast \psi^*$, the target $\rho_* \rho^* K_X$ is identified with $\delta^* g_* \Lambda_U \otimes K_X$. Thus, if $X$ is smooth of dimension $d$, we have a distinguished triangle $\Lambda_X \to K_X \to \rho_* \rho^* K_X$, where the class of the first map $\Lambda_X \to K_X$ is the canonical class
\[ c_X = (-1)^d c_d(\Omega^1_X) \in H^0(X, K_X). \] From the distinguished triangle \( \Lambda_X \to K_X \to \rho_* \rho^* K_X \to \), we deduce an isomorphism \( H^0_S(X, K_X) \to H^0_S(X, \rho_* \rho^* K_X). \) Thus we have obtained a localized class
\[ C_S(j, F) \in H^0_S(X, K_X). \]
We call it the localized characteristic class.

**Theorem 1.** The image of \( C_S(j, F) \in H^0_S(X, K_X) \) in \( H^0(X, K_X) \) is equal to \( C(j, F) - \text{rank}(D \cdot C(\Lambda)). \)

### 3 Computation of the characteristic class in rank 1 case.

As a model, we compute the characteristic class of a smooth sheaf of rank 1. Let \( X \) be a smooth scheme over a perfect field \( F \) and \( U \subset X \) be the complement of a divisor \( D \) with simple normal crossings. We consider a smooth sheaf \( F \) of rank 1 on \( U \).

First, we recall the Swan divisor \( D_F \) and the refined Swan character defined by Kato. Let \( D = \bigcup_i D_i \) be the irreducible components and \( K_i = \text{Frac} \, \mathcal{O}_{X, \xi_i} \) be the local field at the generic point \( \xi_i \) of \( D_i \). Let \( F_i = \kappa(\xi_i) \) be the residue field of \( K_i \). The \( F_i \)-vector space \( \Omega_{F_i/F}(\log) = \Omega^1_{X/F}(\log D)_{\xi_i} \otimes F_i \) is of dimension \( d \) and fits in an exact sequence \( 0 \to \Omega_{F_i/F} \to \Omega_{F_i}(\log) \to F_i \to 0 \). For each \( D_i \), the stalk of \( F \) defines a continuous character \( \chi_i : G_{K_i}^{\text{ab}} \to \Lambda^\times \). If its \( p \)-part \( \chi_i' \) has order at most \( p^{m+1} \), it defines an element \( H^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z}) \).

By the Artin-Schreier-Witt theory, we have a natural surjection \( W_{m+1}(K_i) \to H^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z}) \). Brylinski defined an increasing filtration
\[ F_r W_{m+1}(K_i) = \{(x_0, \ldots, x_m)|p^{m-i}\text{ord}x_i \geq -r \text{ for } i = 0, \ldots, m\}. \]

**Theorem 2.** 1. On \( H^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z}) = \text{Hom}(G_{K_i}^{\text{ab}}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \) the following three filtrations are equal:
   a. The image of \( F_* \).
   b. The dual of the filtration defined by Kato.
   c. The dual of the logarithmic upper numbering filtration defined by Abbes-Saito.

More precisely, for an integer \( r \geq 1 \), we have \( G_{K, \log}^{\text{ab}, j} = G_{K, \log}^{\text{ab}, r} \) for \( j \in (r-1, r] \) and
\[ F_r H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) = \text{Hom}(G_{K, \log}^{\text{ab}, r}, \mathbb{Z}/p^{m+1}\mathbb{Z}). \]

2. Further the map
\[ R_r : Gr^F_r W_{m+1}(K) \to \text{Hom}(m_K^r/m_K^{r+1}, \Omega_F(\log)) \]
defined by
\[ R_r(x_0, \ldots, x_m) = x_0^{p^m}d\log x_0 + \cdots + x_m d\log x_m \]
is well defined. It induces an injection
\[ \text{rsw}_r : Gr^F_r H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \text{Hom}(m_K^r/m_K^{r+1}, \Omega_F(\log)). \]
The Swan divisor $D_{\mathcal{F}} = \sum_i r_i D_i$ is defined by Kato by putting $r_i$ to be the minimum integer $r \geq 0$ satisfying $x_i \in F, H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. Furthermore, he shows there exists a global map $rsw_{\mathcal{F}} : O(-D_{\mathcal{F}})|_{D_w} \to \Omega^1_X(\log D)|_{D_w}$ whose stalks are $rsw_{x_i}$ at each $i$.

**Theorem 3** If $rsw_{\mathcal{F}} : O(-D_{\mathcal{F}}) \to \Omega^1_X(\log D)$ is locally an isomorphism onto a direct summand, we have

\[
C(j!\mathcal{F}) - C(j!\Lambda) = (-1)^{d-1}c_{d-1}(\text{Coker}(rsw_{\mathcal{F}})) = (\text{Image } rsw_{\mathcal{F}}, 0\text{-section})_{T^*X(\log)}.
\]

Basic fact in the proof of Theorems 2 and 3 is the following. Let $(X \times X')' \to X \times X$ be the blow-up at every $D_i \times D_i$. Then the diagonal map $X \to X \times X$ is uniquely lifted to the log diagonal map $X \to (X \times X)'$. Let $(X \times X)'' \to (X \times X)'$ be the blow-up at $D_{\mathcal{F}} \subset X$ in the log diagonal. Then, the exceptional divisor of $(X \times X)'' \to (X \times X)'$ is a compactification of an $\mathbb{A}^d$-bundle $E = \text{Spec } S(\Omega^1_X(\log D)(D_{\mathcal{F}})|_{D_w})$ over $D_w$.

Further the smooth sheaf $\mathcal{H}om(pr^*_2\mathcal{F}, pr^*_1\mathcal{F})$ on $U \times U \subset (X \times X)''$ is unramified along $E$ and the restriction on $E$ of the smooth extension is the Artin-Schreier sheaf defined by the linear form $rsw_{\mathcal{F}}$ on $E$.

### 4 Analogy with Microlocal analysis.

Over $\mathbb{C}$, the Riemann-Hilbert correspondence gives an equivalence of categories:

\[
(\text{regular holonomic } \mathcal{D}_X\text{-modules}) \to (\text{perverse sheaves of } \mathbb{C}_X\text{-modules}).
\]

Let a $\mathcal{D}_X$-module $\mathcal{M}$ be corresponding to $\mathcal{F}$. Then, on the $\mathcal{D}_X$-module side, the characteristic cycle $\text{Char}(\mathcal{M})$ is defined as a cycle on the cotangent bundle $T^*X$ as the class of $gr^*(\mathcal{M})$ regarded as an $O_{T^*X} = gr^*(\mathcal{D}_X)$ The cohomology class $[\text{Char}(\mathcal{M})] \in H^{2d}(T^*X, \mathbb{Z}(d)) = H^{2d}(X, \mathbb{Z}(d))$ gives the characteristic class $C(\mathcal{F})$. Kashiwara-Schapira define the microsupport $\text{SS}(\mathcal{F})$ that is the same as $\text{Char}(\mathcal{M})$ directly without resorting the Riemann-Hilbert correspondence, in the following way. They start with $\mathcal{H} = R\mathcal{H}om(pr^*_2\mathcal{F}, pr^*_1\mathcal{F})$ on $X \times X$. Then by deforming $X \to X \times X$ to $X \to TX$ and applying the nearby cycle functor to $\mathcal{H}$, they define $\nu hom(\mathcal{F}, \mathcal{F})$. Further applying the Fourier-Sato transform, they obtain $\mu hom(\mathcal{F}, \mathcal{F})$ on $T^*X$.

Verdier has studied a similar construction in a $\ell$-adic setting. However, one can not capture wild ramification in this way.

In rank 1 case, we obtain $\overline{\mathcal{H}}|_E$ on the twisted tangent bundle. By applying the Fourier-Deligne transform, one gets a section $rsw_{\mathcal{F}}$ of a twisted cotangent bundle, that defines a cycle on the cotangent bundle.