The characteristic class and the Swan class of an \( \ell \)-adic sheaf (with Abbes and with Kato)

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1982: Galois theory. undergraduate, seminar.
1985-6: Intersection theory. First time to study it.

Report on application of intersection theory to etale cohomology.

Plan:
0. Outline.
1. Swan class and Grothendieck-Ogg-Shafarevich formula. (with Kato)
2. Characteristic class and its relation with the Swan class. (with Abbes)

Notation:
\( F \) field of characteritic \( p > 0 \).
\( \Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell \) etc. (\( \ell \neq p \))
\( X \) variety over \( F \).
\( \mathcal{F} \) \( \Lambda \)-sheaf on \( X \), or more generally, an object in a suitable derived category.

0.1. \( X \) variety, \( U \subset X \) dense open smooth over \( F \).
\( \mathcal{F} \) smooth on \( U \).

The Swan class \( \text{Sw}(\mathcal{F}) \) is defined in \( CH_0(X \setminus U)_\mathbb{Q} \). If \( X \) is proper,

\[
\chi_c(U_{\overline{F}}, \mathcal{F}) \left( = \sum_{q=0}^{2d} (-1)^q \dim H^q_c(U\overline{F}, \mathcal{F}) \right) = \text{rank} \mathcal{F} \cdot \chi_c(U\overline{F}) - \text{degSw}(\mathcal{F}).
\]

0.2. The characteristic class \( C(\mathcal{F}) \in H^0(X, K_X) \) is defined by Abbes. Implicitly in SGA5. In complex geometry, it is defined by Kashiwara-Schapira.

\( K_X = Ra\Lambda, a : X \to \text{Spec } F \). If \( X \) is smooth of dimension \( d \), \( C(\mathcal{F}) \) is defined in \( H^{2d}(X, \Lambda(d)) \) If \( X \) is proper,

\[
\text{Tr } C(\mathcal{F}) = \chi(X_{\overline{F}}, \mathcal{F}) \left( = \sum_{q=0}^{2d} (-1)^q \dim H^q(X_{\overline{F}}, \mathcal{F}) \right).
\]

Let \( j : U \to X \) be the open immersion. Then, the relation

\[
C(j, \mathcal{F}) = \text{rank} \mathcal{F} \cdot C(j, \Lambda) - c_l \text{Sw}(\mathcal{F})
\]

in \( H^0(X, K_X) \) is verified in many cases. \( c_l : CH_0(X) \to H^0(X, K_X) \) cycle class map.
1. $U \subset X$: smooth over $F$, $\mathcal{F}$ on $U$ smooth.
   For simplicity, assume $\mathcal{F}$ is trivialized by a finite Galois covering $V \to U$ of Galois group $G$. $M$: representation of $G$ corresponding to $\mathcal{F}$.
   Further assume there is a commutative diagram
   $$
   \begin{array}{ccc}
   Y & \xleftarrow{\sigma} & V \\
   f \downarrow & & \downarrow \\
   X & \xrightarrow{\gamma} & U
   \end{array}
   $$
   where $f: Y \to X$ is proper, $Y$ is smooth and $V$ is the complement of a divisor with simple normal crossings. (In general, we consider $\mathcal{F}$ mod $\ell$ and use the Brauer trace and also consider alteration.)
   $\sigma \in G = \text{Gal}(U/V), \sigma \neq 1$.
   Figure 1.
   $\Gamma_\sigma$: graph of $\sigma$.
   $(Y \times Y)' \to Y \times Y$: Blow up at $D_1 \times D_1, \ldots, D_m \times D_m$ where $D_1, \ldots, D_m$ are the irreducible components of $D$.
   $\Delta_Y: Y \to (Y \times Y)'$: the log diagonal map.
   Figure 2.
   $\overline{\Gamma_\sigma}$: closure of $\Gamma_\sigma \subset V \times_U V$ in $(Y \times Y)'$.
   tame ramification: no intersection.
   wild ramification: non-empty intersection.
   Define
   $$
   s_{V/U}(\sigma) = -\langle \overline{\Gamma_\sigma}, \Delta_Y \rangle_{Y \times Y}' \in CH_0(Y - V),
   $$
   $$
   s_{V/U}(1) = -\sum_{\sigma \neq 1} s_{V/U}(\sigma) \text{ and }
   $$
   (1)
   $$
   \text{Sw}(\mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in G} f_* s_{V/U}(\sigma) \text{Tr}(\sigma : M) \in CH_0(X - U) \otimes \mathbb{Q}.
   $$
   In fact, $\text{Sw}(\mathcal{F})$ is defined as an element of $CH_0(E)_\mathbb{Q}$ where $E \subset X - U$ is the wild ramification locus.
   Problem: Compute the Swan class in terms of Abbes-Saito filtration. (Partial answer in the rank 1 case.)
   We have a generalization of the Grothendieck-Ogg-Shafarevich formula.

   **Theorem 1** If $X$ is proper,
   $$
   \chi_c(U, \mathcal{F}) = \chi_c(U) \cdot \text{rank } \mathcal{F} - \text{deg } \text{Sw}(\mathcal{F}).
   $$
   Main ingredient of proof. Lefschetz trace formula for an open variety, proved using a method of Pink-Faltings.
2. More generally, the characteristic class is defined for a cohomological correspondence.

We put $\mathcal{H} = R\text{Hom}(pr_2^*\mathcal{F}, pr_1^!\mathcal{F})$. Then, $u$ defines a map $\Lambda_C \to c^!\mathcal{H}$ and hence $u \in H^0_C(X \times X, \mathcal{H})$.

On the other hand, the canonical isomorphism $\mathcal{F} \otimes D\mathcal{F} \to \mathcal{H}$ and the evaluation map $\mathcal{F} \otimes D\mathcal{F} \to K_X$ induce a map $e : \delta^*\mathcal{H} \to K_X$.

We define a class $C(F, C, u) \in H^0_C(X, K_X)$ as $e \circ \delta^*u$.

Proposition 2 If $X$ is proper over $F$, $\text{Tr}(u^* : H^*(X_F, \mathcal{F})) = \text{Tr}(C(F, C, u))$.

$C(F, C, u)$ is the pairing $(\text{id}, u)$ in the notation of SGA5. A reformulation of the Lefschetz trace formula in SGA5. A special case of the compatibility of the construction of the characteristic class with proper push-forward.

$$C(j_!\mathcal{F}) \in H^0(X, K_X) \xleftarrow{\text{cycle map}} \text{CH}_0(X - U)_\mathbb{Q} \supset \text{Sw}(\mathcal{F})$$

$$\text{deg} \downarrow$$

$$\chi_c(U_F, \mathcal{F}) \in \mathbb{Q}_\ell \supset \mathbb{Q}$$

Conjecture 3 $U \subset X$: smooth over $F$, $\mathcal{F}$ smooth $\mathbb{Q}_\ell$-sheaf on $U$. Then, we have

$$C(j_!\mathcal{F}) = \text{rank } \mathcal{F} \cdot C(j_!\Lambda) - \text{Sw}\mathcal{F}$$
in $H^0(X, K_X)$.

Theorem 4 Conjecture 3 is true if there exists a finite etale Galois covering $V \to U$ satisfying one of the following conditions.

(Res) There exist a proper smooth scheme $Y$ over $F$, a divisor $D \subset Y$ with simple normal crossings, an isomorphism $V \to Y \setminus D$ and an action of $G$ on $Y$ extending that on $V$. The pull-back $\mathcal{F}_V$ of $\mathcal{F}$ on $V$ is tamely ramified along $D$.

(Triv) The pull-back $\mathcal{F}_V$ is constant.

Proof is similar to that of Theorem 1.

Assume $X$ is smooth and $D = X - U$ has simple normal crossings. If $\mathcal{F}$ is tamely ramified, we have

$$C(j_!\mathcal{F}) = \text{rank } \mathcal{F} \cdot (-1)^d c_d(\Omega^1_{X/F}(\log D))$$
in $H^{2d}(X, \Lambda(d))$. In particular,

$$C(j_!\Lambda) = (-1)^d c_d(\Omega^1_{X/F}(\log D))$$.
If dim $U = 1$ and rank $F = 1$, we can prove Theorem 4 integrally.

**Theorem 5** Let $X$ be a smooth curve and $U \subset X$ be a dense open. Let $F$ be a smooth $\Lambda$-sheaf of rank 1. Then, we have

\[(4) \quad C(j_* F) = C(j_* \Lambda) - Sw F\]

in $H^2(X, \Lambda(1))$.

**Sketch of Proof.** Assume for simplicity $U = X - \{x\}$. Put $n = Sw_x F \geq 0$.

$(X \times X)^{(0)} \to X \times X$ the blow-up at the image of $x$ by the diagonal map $X \to X \times X$. The diagonal map $X \to X \times X$ is extended to the log diagonal map $X \to (X \times X)^{(0)}$.

We define blow-up $(X \times X)^{(i)} \to (X \times X)^{(i-1)}$ for $i = 1, 2, \ldots, n$ inductively. $\delta^{(i)} : X \to (X \times X)^{(i)}$: immersion induced by the diagonal $E_i$: exceptional divisor.

$(U \times U)^{(n)}$: complement in $(X \times X)^{(n)}$ of the union of the proper transforms of $X \times x, x \times X$, and the exceptional divisors $E_i$ for $i = 0, 1 \ldots, n - 1$.

In the commutative diagram

\[
\begin{align*}
(X \times X)^{(n)} & \xleftarrow{j^{(n)}} (U \times U)^{(n)} \\
\downarrow f^{(n)} & \uparrow k^{(n)} \\
X \times X & \xleftarrow{j} U \times U,
\end{align*}
\]

the left vertical arrow is the composition of blow-ups and the others are open immersions.

**Proposition 6** We put $\mathcal{H} = \text{Hom}(pr_2^* F, pr_1^* F)$. Then, we have the following.

1. The $\Lambda$-sheaf $\mathcal{H}^{(n)} = k_+^{(n)} \mathcal{H}$ is a smooth $\Lambda$-sheaf of rank 1 on $(U \times U)^{(n)}$.
2. The restriction $\mathcal{H}^{(n)}|_{E_n}$ is an Artin-Schreier sheaf.
3. If $F$ is ramified at $x$, the canonical map $j_1^{(n)} \mathcal{H}^{(n)} \to Rj_1^{(n)} \mathcal{H}^{(n)}$ is an isomorphism.

**Proof of Proposition.** Identify $H^1(K_x, \mathbb{Z}/p^n \mathbb{Z}) = W_m(K_x)/F - 1$ and consider the filtration of Brylinski inducing the filtration by ramification.

**Proof of Theorem.** The characteristic class $C(j_! F)$ is defined by the composition $\delta^{(n)}_! \Lambda_X \to \mathcal{H} \otimes pr_1^* K_X \to \delta^{(n)}_* K_X$ and hence equal to the intersection product $(X, X)_{(X \times X)^{(n)}}$.

**Application of Theorem.** Proof of the GOS formula without using the Weil formula. (Brauer induction).