Galois representation in arithmetic geometry.

1. Local-global in arithmetic.

"An $\ell$-adic representation is described by its $L$-function".

An analogy between algebraic curve and Spec $\mathbb{Z}$.

- function field $K : \mathbb{Q}$
- closed points : prime numbers and infinite places
- $K \subset$ local field $K_x : \mathbb{Q} \subset p$-adic field $\mathbb{Q}_p$ and $\mathbb{R} = \mathbb{Q}_\infty$

Two features:

Each point $p$ is a "circle" since the fundamental group $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ of Spec $\mathbb{F}_p$ is a pro-cyclic group generated by the Frobenius.

A global representation is uniquely determined by the local data by the Cebotarev density theorem: The Frobenius conjugacy classes form a dense open subset of the global Galois group.

$\ell$-adic representations.

A continuous representation $G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C})$ has open kernel and finite image. Not useful to study arithmetic geometry. An $\ell$-adic representation is a continuous representation $G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_\ell)$ unramified outside finite set of prime numbers, where $\ell$ denotes a prime number. Except finitely many primes, eigenpolynomials

$$\det(1 - \varphi_p t : V) \in \mathbb{Q}_\ell[t]$$

is defined. Consequence of the Cebotarev density theorem: upto semi-simplification, an $\ell$-adic representation $V$ is determined by the local $L$-factors $L_p(V, t) = \det(1 - \varphi_p t : V) \in \mathbb{Q}_\ell[t]$ at primes $p$ where $V$ is unramified. In most cases, $\det(1 - \varphi_p t : V)$ is in $\mathbb{Q}[t]$ and is independent of $\ell$, i.e. $\ell$-adic representation is a member of a compatible system.

$L$-function of $V$:

$$L(V, s) = \prod_p L_p(V, p^{-s})^{-1}.$$ 

Example 1. $E$ elliptic curve over $\mathbb{Q}$ e.g. $E = X_0(11)$ defined by the equation $y^2 = 4x^3 - 4x^2 - 40x - 79$. $T_\ell E = \lim_n E[\ell^n](\overline{\mathbb{Q}})$. $T_\ell E$ is an $\ell$-adic representation of $G_\mathbb{Q}$. If one forget the $G_\mathbb{Q}$-action, it is isomorphic to $\mathbb{Z}_\ell^2$ as a module. For a prime number $p$ prime to the discriminant of $E$,

$$\det(1 - \varphi_p t : T_\ell E) = 1 - a_p(E)t + pt^2$$

where $a_p(E)$ is an integer defined by $\sharp E(\mathbb{F}_p) = 1 - a_p(E) + p$.

$$L(E, s) = \prod_p (1 - a_p(E)p^{-s} + p^{1-2s})^{-1}.$$ 

Example 2. $f(\tau) = \sum_{n=1}^{\infty} a_n(f)q^n$ ($q = \exp 2\pi \sqrt{-1}\tau$) normalized eigen cusp form of weight 2 with trivial character that is an eigenvector for every Hecke operator e.g. $f_{11}(\tau) = q\prod_{n=1}^{\infty}(1 - q^n)(1 - q^{11n})^2$. $V_f$ $\ell$-adic representation associated to $f$. For $p$ prime to the level of $f$,

$$\det(1 - \varphi_p t : V_f) = 1 - a_p(f)t + pt^2.$$
$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - a_p(f)p^{-s} + p^{1-2s})^{-1}.$$  

Taniyama-Shimura Conjecture. (proved by Wiles-Taylor-Diamond-Conrad-Breuil)An $\ell$-adic representation of type in Example 1 is necessarily of type in Example 2. Or, equivalently, for an elliptic curve $E$, there exists a cusp form $f$ such that

$$L(E, s) = L(f, s).$$

The other implication was established by Eichler-Shimura. E.g. For $E$ in Example 1, $f_{11}$ in Example 2 works.  

2. Etale cohomology as an $\ell$-adic representation. "The Weil conjecture implies that the $L$-function of the etale cohomology is the Hasse-Weil $L$-function." 

$X$ projective smooth algebraic variety over $\mathbb{Q}$. Etale cohomology $H^m(X\bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ is defined. As a vector space, simply $H^m(X\bar{\mathbb{Q}}, \mathbb{Q}_\ell) = H^m(X^{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell$. The $\ell$-adic representation $H^m(X\bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ is unramified at a prime $p$ where $X$ has good reduction. 

The $L$-function of the $\ell$-adic representation $H^m(X\bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ is the Hasse-Weil $L$-function $L(H^m(X), s)$.

Example 1. If $E$ is an elliptic curve over $\mathbb{Q}$, we have $H^1(E\bar{\mathbb{Q}}, \mathbb{Q}_\ell) = \text{Hom}(T_\ell E, \mathbb{Q}_\ell)$. In other words,

$$L(H^1(E), s) = L(E, s).$$

Example 2. For an integer $N \geq 1$, let $X_0(N)$ be the modular curve of level $N$. $(X_0(N)^{an}$ is a compactification of $\Gamma_0(N) \backslash H$ where $\Gamma_0(N) = \{ (a \ b) \in \text{SL}_2(\mathbb{Z}) | c \equiv 0 \mod N \}$. Then,

$$H^1(X_0(N)\bar{\mathbb{Q}}, \mathbb{Q}_\ell) = \bigoplus_{f:Nf|N} \text{Hom}(V_f, \mathbb{Q}_\ell)^{\oplus \sharp \{ d|N/Nf \}}.$$  

Decomposition is given by Hecke operators. In other words,

$$L(H^1(X_0(N), s) = \prod_{f:Nf|N} L(f, s)^{\sharp \{ d|N/Nf \}}.$$  

Hasse-Weil $L$-function. Let $X \mod p$ be the reduction modulo a good prime $p$, that is a projective smooth variety over $\mathbb{F}_p$. Let

$$Z(X \mod p, t) = \exp \left( \sum_{n=1}^{\infty} \frac{\chi(X(\mathbb{F}_p^n))}{n} t^n \right)$$  

denote the congruence $\zeta$-function. By the Weil conjecture proved by Deligne, we have

$$Z(X \mod p, t) = \frac{P_1(X \mod p, t) \cdots P_{2d-1}(X \mod p, t)}{P_0(X \mod p, t) \cdot P_2(X \mod p, t) \cdots P_{2d}(X \mod p, t)} 2$$
where \( d = \dim X, P_m(X \bmod p, t) \in \mathbb{Z}[t] \). The decomposition is characterized by the property that, if we put \( P_m(X \bmod p, t) = \prod_i (1 - \alpha_{i,p,m} t) \), the complex eigenvalue of \( \alpha_{i,p,m} \) is \( p^{\frac{m}{2}} \). This is an analogue of the Riemann hypothesis.

\[
L(H^m(X), s) = \prod_p P_m(X \bmod p, p^{-s})^{-1}.
\]

Note: Bad factors are missing.

Further, we have

\[
\det(1 - Fr_p t : H^m(X_{\overline{Q}}, \mathbb{Q}_l)) = P_m(X \bmod p, t)
\]

and consequently,

\[
L(H^m(X_{\overline{Q}}, \mathbb{Q}_l), s) = L(H^m(X), s).
\]

The local factor at a prime of good reduction is determined by the Weil conjecture, up to semi-simplicification.

Semi-simplicity conjecture: (Tate) The action of \( Fr_p \) on \( H^m(X_{\overline{Q}}, \mathbb{Q}_l) \) is semi-simple.

The semi-simplicity conjecture implies that, the \( \ell \)-adic representation \( H^m(X_{\overline{Q}}, \mathbb{Q}_l) \) of \( G_{\mathbb{Q}_p} \) is determined by \( \det(1 - Fr_p t : H^m(X_{\overline{Q}}, \mathbb{Q}_l)) \).

The Hasse-Weil functions are conjectured to have analytic continuation and to satisfy a functional equation. To formulate a function equation, we need to include the bad primes and to introduce the \( \Gamma \)-factor that is a contribution of the infinite place.

\( \Gamma \)-factor: (Serre) \( V = H^m(X^{an}, \mathbb{Q}) \) is a pure Hodge structure of weight \( m \) with \( \text{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle \)-action. Put \( \Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2) \) and \( \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s) \). Define

\[
\Gamma_{\mathbb{R}}(H^m(X), s) = \prod_{p < m/2} \Gamma_{\mathbb{C}}(s - p)^{h_p^+} \Gamma_{\mathbb{C}}(s - m/2)^{h_p^-} \Gamma_{\mathbb{R}}(s - m/2 + 1)^{h_-}.
\]

If \( m \) is odd, we have only the first term. If \( m \) is even \( h^\pm \) is the dimension of the subspace of \( V^{\frac{m}{2} \cdot \frac{m}{2}} \) where \( \sigma \) acts as \((-1)^{n/2} \).


Bad factors of the Hasse-Weil \( L \)-function. (Serre)

\( P_p(H^m(X), t) = \det(1 - Fr_p t : H^m(X_{\overline{Q}}, \mathbb{Q}_l)_p^t) \) where \( P^p \) indicates the inertia fixed part.

Functional equation. (Serre)

Put \( \Lambda(H^m(X), s) = L(H^m(X), s) \cdot \Gamma_{\mathbb{R}}(H^m(X), s) \). Define \( N = \prod_{\text{bad } p} p^{f_p} \) where \( f_p \) is the Artin conductor of \( H^m(X_{\overline{Q}}, \mathbb{Q}_l) \) at \( p \). Then we expect to have a function equation

\[
\Lambda(H^m(X), s) = \pm N^{\frac{m+1}{2} - s} \Lambda(H^m(X), m + 1 - s).
\]

Question. (Serre) Are \( P_p(H^m(X), t) \) and \( f_p \) well-defined?

This question fits in more general problems.

(i) Description of local Galois representation.

"The monodromy-weight conjecture together with a part of the Tate conjecture implies an affirmative answer to Question."
(ii) Invariants of ramification.
  
  "We have a geometric formula computing the conductor."

(i) Absolute Galois group of a local field. To

\[ \mathbb{Q}_p \subset \mathbb{Q}_p^{ur} = \mathbb{Q}_p(\zeta_n (p \nmid n)) \subset \mathbb{Q}_p^{tr} = \mathbb{Q}_p^{ur}(\mathbb{F}_p^{\times} (p \nmid n)) \subset \overline{\mathbb{Q}}, \]

corresponds

\[ G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_p) \supset I = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_p^{ur}) \supset P = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_p^{tr}) \supset 1. \]

\( I \) is called the inertia and \( P \) is called the wild inertia. The quotients \( G_{\mathbb{Q}_p}/I = G_{\mathbb{F}_p} \) and \( I/P = \varprojlim_{n \mu_n \mu_n}(\overline{\mathbb{F}}_p) \) are pro-cyclic and \( P \) is a huge pro-p group. Take an isomorphism \( \varprojlim_{n \mu_n} \rightarrow \mathbb{Z}_\ell \) and let \( t_\ell : I \rightarrow \mathbb{Z}_\ell \) denote the composition. Also take a lifting \( F \in G_{\mathbb{Q}_p} \) of \( F_{\mathbb{F}_p} \). The inverse image \( W_{\mathbb{Q}_p} = \langle F, I \rangle \) of \( \langle F_{\mathbb{F}_p} \rangle \subset G_{\mathbb{F}_p} \) is called the Weil group.

We assume \( \ell \neq p \). The \( p \)-adic Hodge theory deals with the case \( \ell = p \) (Faltings’s Kuwait lecture on 28 October 2003, Fontaine's Kuwait lecture on 26 February 2003)


good semi-simplicity conjecture.(Tate) The action of \( F \) on \( H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \) is semi-simple.

Monodromy theorem.(Grothendieck) Let \( \ell \) be a prime number different from \( p \) and \( \rho : G_{\mathbb{Q}_p} \rightarrow GL_n(\mathbb{Q}_\ell) \) be a continuous representation. Then, there exists a pair of representation \( \rho' : W_{\mathbb{Q}_p} \rightarrow GL_n(\mathbb{Q}_\ell) \) and a nilpotent endomorphism \( N \in M_n(\mathbb{Q}_\ell) \) such that \( \rho(F^n\sigma) = \rho'(F^n\sigma)\exp(t_\ell(\sigma)N) \).

\( \rho \) is uniquely determined by \( (\rho', N) \). \( \rho' \) is determined by \( \text{Tr} \rho' \) up to semi-simplification.

Monodromy filtration: For \( N \) an nilpotent endomorphism of \( V \) \( (N^{n+1} = 0) \), the filtration \( W_r V = \sum_{p - q = r} \text{Ker} N^{p+1} \cap \text{Im} N^q \) is the unique increasing filtration satisfying the following property:

\( N(W_r V) \subset W_{r-2} V \) for all \( r \in \mathbb{Z} \), \( W_0 V = V, W_{-n-1} V = 0 \), and the induced map \( N^r : Gr^r W V \rightarrow Gr^r W V \) is an isomorphism for \( r \geq 0 \).

Monodromy-weight conjecture: (Deligne) The eigenvalues of \( F \) on \( Gr^r W H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \) are of weight \( m + r \). Namely are an algebraic integer and their complex absolute values are \( p^{-\frac{m+r}{2}} \).

MWC is an analogue of the Weil conjecture for a variety over a local field. MWC is know if \( m \leq 2 \). MWC implies that \( N \) is determined by \( \rho' \). Further Semi-simplicity conjecture implies that \( \rho' \) on \( H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \) is determined by \( \text{Tr} \rho = \text{Tr} \rho' \) on \( H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \).

Theorem 1. Assume MWC and further assume that the projectors to the Künneth components are algebraic. Then, \( P_p(H^m(X), t) \) and \( f_p \) are well-defined. More precisely, the function \( \text{Tr}(\sigma, (\text{Ker} N : H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))) \) on \( \sigma \in W_{\mathbb{Q}_p} \) is \( \mathbb{Q} \)-valued and is independent of \( \ell \).

Proof. Alteration and the weight spectral sequence (Steenbrink-Rapoport-Zink).

(ii) Conductor.

\[ f_p = \dim H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) - \dim H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} + \sum_{m=0}^{2d} (-1)^m Sw_{p} H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell). \]

Take a regular proper model \( X_{\mathbb{Z}} \) and put

\[ \text{Art}_p(X) = \chi(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) - \chi(X_{\overline{\mathbb{F}}_{p^m}}, \mathbb{Q}_\ell) + \sum_{m=0}^{2d} (-1)^m Sw_{p} H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell). \]
SwV = \sum_v v \times \dim V^{G_v} / V^{G_v} \; G_v \text{ filtration by ramification groups. SwV = 0 if } \text{ and only if } P \text{ acts trivially on } V.

Theorem 2 (Kato-T). If the closed fiber \( X_{\mathbb{F}_p} \) has normal crossings as a divisor of \( X \), we have

\[
\text{Art}_P(X) = \deg(-1)^d c_{d+1}^{X}_{X_{\mathbb{F}_p}}(\Omega_{X/\mathbb{Z}}).
\]

The right hand side is the degree of a 0-cycle class supported on the closed fiber. Theorem 2 is conjectured by S. Bloch without the extra assumption. A generalization of the conductor-discriminant formula in algebraic number theory. The Tate-Ogg formula for an elliptic curve is a special case.

Tomorrow: A related formula in a more geometric setting.