Notes on the characteristic cycle of a constructible sheaf

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Abstract

We study some properties of the characteristic cycle of a constructible complex on a smooth variety over a perfect field; push-forward and product.

The characteristic cycle of a constructible complex on a smooth scheme over a perfect field is defined as a cycle on the cotangent bundle \([12]\) supported on the singular support \([2]\). It is characterized by the Milnor formula \((2.4)\) for the vanishing cycles defined for morphisms to curves. We briefly recall the definitions of singular support and characteristic cycle in Definitions 1.5 and 2.1 respectively.

We study some properties of characteristic cycles. First, we formulate Conjecture 2.2 on the compatibility with proper direct image. We prove it in some cases, for example, morphisms from surfaces to curves under a mild assumption in Theorem 2.6. We also prove a formula \((3.14)\) for the external product in Theorem 3.6.

We briefly sketch the idea of proofs, which use the global index formula \((2.11)\) computing the Euler-Poincaré characteristic in all cases. For the compatibility Theorem 2.6, it amounts to prove a conductor formula \((2.14)\) at each point of the curve. By choosing a point and killing ramification at the other points using Epp’s theorem \([5]\), we deduce the conductor formula \((2.14)\) from the index formula \((2.11)\).

For the external product, first we show that the external product is micro-supported on the external product of the singular supports of the factors. We deduce this from projection formulas for nearby cycles over general base schemes in \([13]\) recalled in Section 3.1. We prove the formula \((3.14)\) for characteristic cycle first for product of curves. Here we use the fact that the only unknown coefficients are those for the fibers at closed points of the product and conclude using the index formula. From this case, we deduce a conductor formula \((3.18)\) for the additive convolution \([10]\), \([8]\). The general case is reduced to \((3.18)\) by the Thom-Sebastiani formula \([8]\).

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1 Complements on singular support

We briefly recall properties of the singular support of a constructible complex on a smooth scheme over a perfect field. For more detail, we refer to \([2]\) and \([12]\).
Recall that a closed subset of a vector bundle said to be conical if it is stable under the action of the multiplicative group $G_m$. For a morphism $f: X \to Y$ of noetherian schemes, we say that $f$ is proper (resp. finite) on a closed subset $C \subset X$ if its restriction on $C$ is proper (resp. finite) with respect to one or equivalently any closed subscheme structure. For morphisms $h: W \to X$ and $f: W \to Y$ of noetherian schemes and a closed subset $C \subset X$ such that $f$ is proper on $h^{-1}(C)$, the image of $C$ by the algebraic correspondence $X \leftarrow W \to Y$ is defined to be the closed subset $f(h^{-1}(C)) \subset Y$.

**Definition 1.1 ([2, 1.1]).** Let $X$ be a smooth scheme over a field $k$. Let $C \subset T^*X$ be a closed conical subset of the cotangent bundle.

1. Let $f: X \to Y$ be a morphism of smooth schemes over $k$ and let $C' \subset T^*Y$ be a closed conical subset.

   For $x \in X$, we say that $f: X \to Y$ is $(C,C')$-transversal if for $\omega \in T^*Y \times Y$ at $y = f(x) \in Y$, the conditions $\omega \in C'$, $f^*\omega \in C$ imply $\omega = 0$. We say $f: X \to Y$ is $(C,C')$-transversal if $f: X \to Y$ is $(C,C')$-transversal at every $x \in X$. Or equivalently, if the intersection $df^{-1}(C) \cap f^*C'$ of the inverse image of $C$ by $df: X \times_Y T^*Y \to T^*X$ and $f^*C' = X \times_Y C'$ in $X \times_Y T^*Y$ is a subset of the 0-section.

2. We say that a morphism $h: W \to X$ of smooth schemes over $k$ is $C$-transversal if $h$ is $(T^*_W W, C)$-transversal.

   We set $h^*C = W \times_X C \subset W \times_X T^*X$ and, if $h: W \to X$ is $C$-transversal, we define a closed conical subset

\[
(1.1) \quad h^*C \subset T^*W
\]

to be the image of $C \subset T^*X$ by the algebraic correspondence $T^*X \leftarrow W \times_X T^*X \to T^*W$.

Assume that every irreducible component of $C$ is of the same dimension as $X$. Then, we say that a $C$-transversal morphism $h: W \to X$ of smooth schemes over $k$ is properly $C$-transversal [12, Definition 4.1] if every irreducible component of $h^*C = W \times_X C$ is of the same dimension as $W$.

3. We say that a morphism $f: X \to Y$ of smooth schemes over $k$ is $C$-transversal if $f$ is $(T^*Y, C)$-transversal.

   For a proper morphism $r: X \to Y$ of smooth schemes over $k$, we define a closed conical subset

\[
(1.2) \quad r_oC \subset T^*Y
\]

to be the image of $C \subset T^*X$ by the algebraic correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$.

If $h: W \to X$ is $C$-transversal, then $dh: W \times_X T^*X \to T^*W$ is finite on $h^*C$ and $h^*C \subset T^*W$ is defined.

**Lemma 1.2.** Let $C \subset T^*X$ be a closed conical subset. For morphisms $h: W \to X$ and $f: W \to Y$ of smooth schemes over $k$, the following conditions are equivalent:

1. The morphism $h: W \to X$ is $C$-transversal and $f: W \to Y$ is $h^*C$-transversal.
2. The morphism $(h,f): W \to X \times Y$ is $C \times T^*Y$-transversal.

If the equivalent conditions in Lemma 1.2 is satisfied, we say that the pair $(h,f)$ is $C$-transversal.
Proof. The condition (1) is equivalent to the following the condition:

(1') For \( \alpha \in h^*C \), if its image in \( T^*W \) is 0, then we have \( \alpha = 0 \). For \( \beta \in W \times_Y T^*Y \), if its image in \( T^*W \) is contained in the image of \( h^*C \), then we have \( \beta = 0 \).

We consider the restriction of the morphism \((W \times_X T^*X) \times_W (W \times_Y T^*Y) = W \times_{X \times Y} T^*(X \times Y) \rightarrow T^*W \) to the subset \( h^*C \times_W (W \times_Y T^*Y) = W \times_{X \times Y} (C \times T^*Y) \). Then, the condition (2) is equivalent to the following the condition:

(2') For \((\alpha, \beta) \in h^*C \times_W (W \times_Y T^*Y)\), if its image in \( T^*W \) is 0, then we have \((\alpha, \beta) = 0\). Hence the assertion follows.

Lemma 1.3. Let \( f: X \rightarrow Y \) be a morphism of smooth schemes over \( k \) and let \( \gamma: X \rightarrow X \times Y \) be the graph of \( f \). For closed conical subsets \( C \subset T^*X \) and \( C' \subset T^*Y \) the following conditions are equivalent:

(1) \( f \) is \((C, C')\)-transversal.
(2) \( \gamma \) is \( C \times C'\)-transversal.

Further, if the condition (2) is satisfied, the closed subset \( \gamma^\circ(C \times C') \subset T^*X \) equals the subset \( C + f^*C' \subset T^*X \) consisting of the sum \( \alpha + \beta \) of \( \alpha \in C \) and \( \beta \in f^*C' \).

Proof. The condition (1) is equivalent to the following the condition:

(1') For \( \beta \in f^*C' \), if \( df(\beta) \in C \), then we have \( \beta = 0 \).

The condition (2) is equivalent to the following the condition:

(2') For \( \alpha \in C \) and \( \beta \in f^*C' \), if \( \alpha + df(\beta) = 0 \), then we have \( \alpha = 0 \) and \( \beta = 0 \).

Hence the conditions (1) and (2) are equivalent. Since \( \gamma^\circ(C \times C') = C \times_X f^*C' \), we obtain \( \gamma^\circ = C + f^*C' \).

Lemma 1.4. Let \( r: W \rightarrow X \) and \( f: X \rightarrow Y \) be morphisms of smooth schemes over \( k \) and assume \( r \) is proper. For a closed conical subset \( C \subset T^*W \), the following conditions are equivalent:

(1) The composition \( f \circ r \) is \( C\)-transversal.
(2) \( f \) is \( r_0C\)-transversal.

Proof. We consider the commutative diagram

\[
\begin{array}{ccc}
T^*W & \supset & C \\
\downarrow \text{dr} & & \\
W \times_Y T^*Y & \longrightarrow & W \times_X T^*X \supset dr^{-1}(C) \\
\downarrow & & \downarrow \\
X \times_Y T^*Y & \xrightarrow{df} & T^*X \supset r_0C.
\end{array}
\]

Since the square is cartesian, the inverse image of \( dr^{-1}(C) \) by the upper horizontal arrow is a subset of the 0-section if and only if the inverse image of \( r_0C \) by the lower one is a subset of the 0-section.

Let \( \Lambda \) be a finite field of characteristic \( \ell \) invertible in \( k \). We say that a complex \( F \) of \( \Lambda \)-modules on the étale site of \( X \) is constructible if the cohomology sheaf \( H^qF \) is constructible for every \( q \) and vanishes except for finitely many \( q \). More generally, if \( \Lambda \) is a finite local ring of residue field \( \Lambda_0 \) of characteristic \( \ell \) invertible in \( k \), the singular support and the characteristic cycle of a complex \( F \) of \( \Lambda \)-modules of finite tor-dimension equals to those of \( F \otimes^L_\Lambda \Lambda_0 \).
Definition 1.5 ([2, 1.2]). Let $X$ be a smooth scheme over a field $k$ and let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$.

1. We say that $\mathcal{F}$ is micro-supported on a closed conical subset $C$ of the cotangent bundle if for every $C$-transversal pair $(h, f)$ of morphisms $h: W \to X$ and $f: W \to Y$ of smooth schemes over $k$, the morphism $f: W \to Y$ is locally acyclic relatively to $h^*\mathcal{F}$.

2. The singular support $SS\mathcal{F}$ is defined to be the smallest closed conical subset $C \subset T^*X$ of the cotangent bundle such that $\mathcal{F}$ is micro-supported on $C$.

Let $X$ be a normal noetherian scheme and $U \subset X$ be a dense open subscheme. Let $G$ be a finite group and $V \to U$ be a $G$-torsor. The normalization $Y \to X$ in $V$ carries a natural action of $G$. For a geometric point $\bar{x}$ of $X$, the stabilizer $I \subset G$ of a geometric point $\bar{y}$ of $Y$ above $\bar{x}$ is called an inertia subgroup at $\bar{x}$.

Lemma 1.6. Let $G$ be a finite group and $(1.3)$

\[
\begin{array}{ccc}
W & \leftarrow & V \\
\downarrow & & \downarrow \\
X & \leftarrow & U
\end{array}
\]

be a cartesian diagram of smooth schemes over a field $k$ where the horizontal arrows are dense open immersions, the right vertical arrow $V \to U$ is a $G$-torsor and the left vertical arrow $r: W \to X$ is proper. Assume that for every geometric point $x$ of $X$, the order of the inertia group $I_x \subset G$ is prime to $\ell$.

Let $\mathcal{F}$ be a locally constant sheaf on $U$ such that the pull-back $r^*\mathcal{F}$ is a constant sheaf. Then, for the intermediate extension $j_!\mathcal{F} = j_!(\mathcal{F}[\dim U])[-\dim U]$ on $X$, we have an inclusion

$$SSj_!\mathcal{F} \subset r_0(T^*_WW).$$

Proof. Since the assertion is étale local on $X$, we may assume that $G = I_x$ is of order prime to $\ell$. Then, the canonical morphism $\mathcal{F} = (r^*_jr^*\mathcal{F})^G \to r^*_jr^*\mathcal{F}$ is a splitting injection and induces a splitting injection $j_!\mathcal{F} \to j_!r^*_jr^*\mathcal{F}$. Hence, we have

$$SS(j_!\mathcal{F}) \subset SS(j_!r^*_jr^*\mathcal{F}).$$

Since every irreducible subquotient of the shifted perverse sheaf $j_!r^*_jr^*\mathcal{F}$ is isomorphic to an irreducible subquotient of a shifted perverse sheaf $\mathcal{H}^0(Rr_0j_!r^*\mathcal{F}[\dim U])[-\dim U]$ extending $r^*_jr^*\mathcal{F}$, we have

$$SS(j_!r^*_jr^*\mathcal{F}) \subset SS(\mathcal{H}^0(Rr_0j_!r^*\mathcal{F})) \subset SS(Rr_0j_!r^*\mathcal{F})$$

by [2, Theorem 1.3 (ii)]. Since $j_!r^*\mathcal{F}$ is a constant sheaf on $W$ and $r$ is proper, we have

$$SS(Rr_0j_!r^*\mathcal{F}) \subset r_0SS(j_!r^*\mathcal{F}) \subset r_0(T^*_WW)$$

by [2, Lemma 2.2 (ii), Lemma 2.1 (iii)]. Thus the assertion follows. □
2 Compatibility with direct image

Let \( h: W \to X \) and \( f: W \to Y \) be morphisms of smooth schemes over a field \( k \). Let \( C \subseteq X \) be a closed subset such that \( f \) is proper on \( h^{-1}(C) \) and let \( C' = f(h^{-1}(C)) \subseteq Y \) be the image of \( C \) by the algebraic correspondence \( X \leftarrow W \to Y \). If \( \dim W = \dim X - c \), the intersection theory defines the pull-back and push-forward morphisms

\[
CH_\bullet(C) \xrightarrow{h^!} CH_\bullet_c(h^{-1}(C)) \xrightarrow{f_*} CH_\bullet_c(C').
\]

We call the composition the morphism defined by the algebraic correspondence \( X \leftarrow W \to Y \). If every irreducible component of \( C \) is of dimension \( n \) and if every irreducible component of \( C' \) is of dimension \( m = n - c \), the morphism (2.1) defines a morphism \( Z_n(C) \to Z_m(C') \) of free abelian groups of cycles.

Let \( f: X \to Y \) be a proper morphism of smooth schemes over a field \( k \). Assume that every irreducible component of \( X \) is of dimension \( n \) and that every irreducible component of \( Y \) is of dimension \( m \). By applying the construction of (2.1) to the algebraic correspondence \( T^*X \leftarrow X \times_Y T^*Y \to T^*Y \) and \( C \subseteq T^*X \), we obtain a morphism

\[
f_*: CH_n(C) \to CH_m(f_*C).
\]

Further if every irreducible component of \( C \subseteq T^*X \) is of dimension \( n \) and if every irreducible component of \( f_*C \subseteq T^*Y \) is of dimension \( m \), we obtain a morphism

\[
f_*: Z_n(C) \to Z_m(f_*C).
\]

Let \( \mathcal{F} \) be a constructible complex on a smooth scheme \( X \) over \( k \). By [2, Theorem 1.2 (ii)], every irreducible component of the singular support \( C = SS\mathcal{F} \) is of dimension \( n \).

In the following, we assume that \( k \) is perfect.

**Definition 2.1.** Let \( X \) be a smooth scheme over a perfect field \( k \) and let \( C \subseteq T^*X \) be a closed conical subset.

1. Let \( j: U \to X \) be an étale morphism, \( u \in U \) be a closed point and \( f: U \to Y \) be a morphism to a smooth curve over \( k \). Then, we say that \( u \) is an isolated characteristic point of \( f \) with respect to \( C \) if the pair \( X \leftarrow U - \{u\} \to Y \) is \( C \)-transversal and if the pair \( X \leftarrow U \to Y \) is not \( C \)-transversal.

2. ([12, Theorem 3.1, Theorem 6.1]) Let \( \mathcal{F} \) be a constructible complex on \( X \). Assume that \( C \) contains the singular support \( SS\mathcal{F} \) as a subset and that every irreducible component \( C_a \) of \( C = \bigcup_a C_a \) is of dimension \( n \). Then, the characteristic cycle \( CC\mathcal{F} = \sum_a m_a C_a \) is a unique \( \mathbb{Z} \)-linear combination satisfying the Milnor formula

\[
- \dim \text{tot} \phi_u(j^*\mathcal{F}, f) = (j^*CC\mathcal{F}, df)_{T^*U,u}
\]

for every pair \((j, f)\) of morphisms as in 1. with at most isolated characteristic point at \( u \in U \).

In the left hand side of (2.4), \( \phi_u(j^*\mathcal{F}, f) \) denotes the stalk of the complex of vanishing cycles and \((j^*CC\mathcal{F}, df)_{T^*U,u}\) in the right hand side denotes the intersection number with the section \( df \) of \( T^*U \) defined by the pull-back of a basis of the line bundle \( T^*Y \), supported on the closed fiber at \( u \).
We say that a constructible complex $\mathcal{F}$ is \textit{locally constant} if every cohomology sheaf $\mathcal{H}^q\mathcal{F}$ is locally constant. In this case, we have

\begin{equation}
CC\mathcal{F} = (-1)^n \text{rank } \mathcal{F} \cdot [T_X^*X]
\end{equation}

where $T_X^*X$ denotes the 0-section and $n = \dim X$. Assume $\dim X = 1$ and let $U \subset X$ be a dense open subset where $\mathcal{F}$ is locally constant. For a closed point $x \in X$, the Artin conductor $a_x\mathcal{F}$ is defined by

\begin{equation}
a_x\mathcal{F} = \text{rank } \mathcal{F}|_U - \text{rank } \mathcal{F}_x + \text{Sw}_x\mathcal{F}
\end{equation}

where $\text{Sw}_x\mathcal{F}$ denotes the alternating sum of the Swan conductor at $x$. Then, we have

\begin{equation}
CC\mathcal{F} = -\left( \text{rank } \mathcal{F} \cdot [T_X^*X] + \sum_{x \in X - U} a_x\mathcal{F} \cdot [T_x^*X] \right)
\end{equation}

where $T_x^*X$ denotes the fiber.

Let $f: X \to Y$ be a proper morphism of smooth schemes over $k$ and assume that every irreducible component of $Y$ is of dimension $m$. Then, the direct image

\begin{equation}
f_*CC\mathcal{F} \in CH_m(f_*C)
\end{equation}

of the characteristic cycle $CC\mathcal{F}$ is defined by the algebraic correspondence $T^*Y \leftarrow X \times_Y T^*X$. Further if every irreducible component of $f_*C \subset T^*Y$ is of dimension $m$, the direct image

\begin{equation}
f_*CC\mathcal{F} \in Z_m(f_*C)
\end{equation}

is defined as a linear combination of cycles.

\textbf{Conjecture 2.2.} Let $f: X \to Y$ be a proper morphism of smooth schemes over a perfect field $k$. Assume that every irreducible component of $X$ is of dimension $n$ and that every irreducible component of $Y$ is of dimension $m$. Let $\mathcal{F}$ be a constructible complex on $X$ and $C = SS\mathcal{F}$ be the singular support.

1. We have

\begin{equation}
CCRf_*\mathcal{F} = f_*CC\mathcal{F}
\end{equation}

in $CH_m(f_*C)$.

2. In particular, if every irreducible component of $f_*C \subset T^*Y$ is of dimension $m$, we have an equality (2.10) of cycles.

If $Y = \text{Spec } k$ is a point, the equality (2.10) is nothing but the index formula

\begin{equation}
\chi(X_k, \mathcal{F}) = (CC\mathcal{F}, T_X^*X)_{T^*X}.
\end{equation}

This is proved in [12, Theorem 4.21] under the assumption that $X$ is projective. For a closed immersion $i: X \to P$ of smooth schemes over $k$, Conjecture 2.2 holds [12, Lemma 3.18.2]. Hence, for a proper morphism $g: P \to Y$ of smooth schemes over $k$, Conjecture 2.2 for $\mathcal{F}$ and $f = g \circ i: X \to Y$ is equivalent to that for $i_*\mathcal{F}$ and $g: P \to Y$. For the singular support, an inclusion $SSRf_*\mathcal{F} \subset f_*SS\mathcal{F}$ is proved in [2, Theorem 1.3 (ii)].
Lemma 2.3. Assume that $f$ is finite on the support of $\mathcal{F}$. Then Conjecture 2.2.2 holds.

In particular, if $f : X \to Y$ is a finite flat and generically étale morphism of smooth curves, Conjecture 2.2.2 holds.

Proof. We may assume that $k$ is algebraically closed. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $f_* CC\mathcal{F}$ satisfies the Milnor formula (2.4) for $Rf_* \mathcal{F}$.

Let $Z \subset X$ denote the support of $\mathcal{F}$. Let $V \to Y$ be an étale morphism and $g : V \to T$ be a morphism to a smooth curve $T$ with isolated characteristic point $v \in V$ with respect to $f_* C$. By replacing $Y$ by $V$, we may assume $V = Y$.

By Lemma 1.4 and by the assumption that $Z$ is finite over $Y$, the composition $g \circ f : X \to T$ has isolated characteristic points at the inverse image $Z \times_Y v$. Hence, the composition $g \circ f : X \to T$ is locally acyclic relatively to $\mathcal{F}$ on a neighborhood of the fiber $X \times_Y v$ except at $Z \times_Y v$ and we have a canonical isomorphism

$$\phi_v(Rf_* \mathcal{F}, g) \to \bigoplus_{u \in Z \times_Y v} \phi_u(\mathcal{F}, g \circ f).$$

Thus by the Milnor formula (2.4), we have

$$- \dim \text{tot} \phi_v(Rf_* \mathcal{F}, g) = \sum_{u \in Z \times_Y v} - \dim \text{tot} \phi_u(\mathcal{F}, g \circ f)$$

$$= \sum_{u \in Z \times_Y v} (CC\mathcal{F}, d(g \circ f))_{T^* X, u} = (f_* CC\mathcal{F}, dg)_{T^* Y, v}$$

and the assertion follows.

If $Y$ is a curve, Conjecture 2.2.2 may be rephrased as follows. Let $C = SS\mathcal{F}$ be the singular support and assume that on a dense open subscheme $V \subset Y$, the restriction $f_V : X_V = X \times_Y V \to V$ of $f$ is $C$-transversal. Then, $f_* C \times_Y V$ is a subset of the $0$-section. Thus the condition that every irreducible component of $f_* C$ is of dimension 1 is satisfied. Further $f_V$ is locally acyclic relatively to $\mathcal{F}$. Since $f$ is proper, $Rf_* \mathcal{F}$ is locally constant on $V$ by [1, Théorème 2.1] and we have

$$CCRf_* \mathcal{F} = - \left( \text{rank } Rf_* \mathcal{F} \cdot [T^*_Y Y] + \sum_{y \in Y \setminus V} a_y Rf_* \mathcal{F} \cdot [T^*_y Y] \right).$$

For a closed point $y \in Y$, the Artin conductor $a_y Rf_* \mathcal{F}$ is defined by

$$a_y Rf_* \mathcal{F} = \chi(X_{\tilde{y}}, \mathcal{F}) - \chi(X_{\bar{y}}, \mathcal{F}) + Sw_y H^*(X_{\tilde{y}}, \mathcal{F}).$$

In the right hand side, the first two terms denote the Euler-Poincaré characteristics of the geometric generic fiber and the geometric closed fiber respectively and the last term denotes the Swan conductor at $y$.

Let $df$ denote the section of $T^* X$ on a neighborhood of the inverse image $X_y$ defined by the pull-back of a basis $dt$ of the line bundle $T^* Y$ for a local coordinate $t$ on a neighborhood of $y \in Y$. Then, the intersection product $(CC\mathcal{F}, df)_{T^* X, y}$ supported on the inverse image of $X_y$ is well-defined since $SS\mathcal{F}$ is a closed conical subset.
Lemma 2.4. Let \( C = SSF \) be the singular support and assume that \( f_Y : X_V \to V \) is projective, smooth and \( C \)-transversal.

1. The equality (2.10) is equivalent to the equality

\[
-\alpha_y Rf_* F = (CC F, df)_{T^* X, y}
\]

at each point \( y \in Y - V \), where the right hand side denotes the intersection number supported on the inverse image of \( y \).

2. Let \( \delta_y \) denote the difference of (2.14). If \( X \) and \( Y \) are projective, we have \( \sum_{y \in Y - V} \delta_y \), \( \deg y = 0 \).

Proof. 1. Let \( V' \subset V \) be the complement of the images of irreducible components \( C_a \) of the singular support \( C = \bigcup_a C_a \) such that the image of \( C_a \) is a closed point of \( Y \). Then, for every closed point \( w \in V' \), the immersion \( i_w : X_w \to X \) is properly \( C \)-transversal and we have \( CC i_w^* F = i_w^* CC F \) by [12, Theorem 4.4]. Further, we have

\[
f_* CC F = -\left((i_w^* CC F, T_{X_0}X_w) \cdot [T_Y Y] - \sum_{y \in Y - V} (CC F, df)_{T^* X, y} \cdot [T_Y Y]\right).
\]

If we assume that \( f_Y : X \times_Y V \to V \) is projective, the index formula (2.11) implies

\[
\text{rank } Rf_* F = \chi(X, i_w^* F) = (CC i_w^* F, T_{X_0}X_w)_{T^* X_0}.
\]

Thus, it suffices to compare (2.15) and (2.12).

2. We have

\[
(CCRf_* F, T_Y Y)_{T^* Y} = \chi(Y, Rf_* F) = \chi(X, F) = (CC F, T_X X)_{T^* X} = (f_* CC F, T_Y Y)_{T^* Y}
\]

by the index formula (2.11) and the projection formula. Thus it follows from 1. \( \square \)

If \( f \) has at most isolated characteristic points, (2.14) is an immediate consequence of the Milnor formula (2.4).

We prove some cases of Conjecture 2.2.2 assuming that \( X \) is a surface.

Proposition 2.5. Let \( X \) be a normal scheme of finite type over a perfect field \( k \), \( Y \) be a smooth curve over \( k \) and \( f : X \to Y \) be a flat morphism over \( k \). Let \( V \subset Y \) be a dense open subscheme such that \( f_Y : X_V \to V \) is smooth.

1. There exist a finite flat surjective morphism \( g : Y' \to Y \) of smooth curves over \( k \) and a dense open subscheme \( X'' \subset X' \) of the normalization \( X' \) of \( X \times_Y Y' \) satisfying the following condition:

   (1) We have inclusions \( X'_Y = X' \times_Y V \subset X'' \subset X' \) and \( X'' \subset X' \) is dense in every fiber of \( X' \to Y' \). The morphism \( X'' \to Y' \) is smooth.

   2. Let \( F \) be a perverse sheaf on \( X_V \) and let \( C \subset T^* X_V \) be a closed conical subset on which \( F \) is micro-supported. Assume that \( f_Y : X_V \to V \) is \( C \)-transversal. Then, there exist \( g : Y' \to Y \) and \( X'' \subset X' \times_Y Y' \) as in 1. satisfying the condition (1) above and the following condition:

   (2) Let \( j'' : X_V \to X'' \) denote the open immersion and \( C' = SS j''_! F' \) be the singular support of the intermediate extension \( j''_! F' \) of the pull-back \( F' \) of \( F \) to \( X'_Y \). Then, the morphism \( X'' \to Y' \) is \( C' \)-transversal.
Proof. By devissage and approximation, we may assume that the complement $Y - V$ consists of a single closed point $y$ and that the closed fiber $X_y$ is irreducible. The assertion is local on a neighborhood in $X$ of the generic point $\xi$ of $X_y$.

1. It follows from [5].

2. Since $f_V: X_V \to V$ is smooth, the $C$-transversality of $f_V$ and the condition that $\mathcal{F}$ is micro-supported on $C$ are preserved after base change by [12, Lemma 2.7.2, Lemma 2.11.4]. After replacing $Y$ by $Y'$ and $X$ by $X''$ as in 1., we may assume that $X \to Y$ is smooth. Shrinking $X$ and $Y$ further if necessary, we may assume that $\mathcal{F}$ is locally constant.

Let $W_V \to X_V$ be a $G$-torsor for a finite group $G$ such that the pull-back of $\mathcal{F}$ on $W_V$ is a constant sheaf. Let $r: W \to X$ be the normalization of $X$ in $W_V$. Applying 1 to $W \to Y$ and shrinking $X$ if necessary, we may assume that there exists a finite flat surjective morphism of smooth curves $Y' \to Y$ such that the normalization $W'$ of $W \times_Y Y'$ is smooth over $Y'$.

Let $r': W' \to X'$ be the canonical morphism. Since the ramification index at the generic point $\xi'$ of an irreducible component of the fiber $X'_0$ is 1, the inertia group at $\xi'$ is of order a power of $p$. Hence, after shrinking $X$ if necessary, we may assume that for every geometric point $w'$ of $W'$, the order of the inertia group is a power of $p$. Hence, by Lemma 1.6, we have $C' = SS(j'_0, \mathcal{F}) \subset r'_* (T_{W'} W')$. Since $W' \to Y'$ is smooth, the morphism $f': X' \to Y'$ is $C'$-transversal by Lemma 1.4(1)⇒(2).

**Theorem 2.6.** Let the notation be as in Conjecture 2.2 and let $C = SS \mathcal{F}$ be the singular support. Assume that $\dim X = 2$, $\dim Y = 1$ and that there exists a dense open subscheme $V \subset Y$ such that $f_V: X_V \to V$ is smooth and $C$-transversal. Then, Conjecture 2.2.2 holds.

**Proof.** We may assume $\mathcal{F}$ is a perverse sheaf by [2, Theorem 1.3 (ii)]. Since the resolution of singularity is known for curves and surfaces, we may assume $Y$ is projective. Since a proper smooth surface over a field is projective, the surface $X$ is projective. Let $y \in Y - V$ be a point. It suffices to show the equality (2.14).

By Proposition 2.5 and approximation, there exists a finite flat surjective morphism of proper smooth curves étale at $y$ and satisfying the conditions in Proposition 2.5 on the complement $Y - \{y\}$. Since the normalization $X'$ of $X \times_Y Y'$ is projective, we may take a projective smooth scheme $P$ and decompose $f': X' \to Y$ as a composition $X' \to P \to Y$ of a closed immersion $i: X' \to P$ and $g: P \to Y$.

Let $U = V \cup \{y\}$. Let $\mathcal{F}'$ be the pull-back of $\mathcal{F}$ to $X'_U = X' \times_Y U$ and let $j'_0, \mathcal{F}'$ be the intermediate extension with respect to the open immersion $j': X'_U \to X'$. It suffices to show Conjecture 2.2.2 holds for $P \to Y$ and $G = i_* j'_0, \mathcal{F}'$. Outside the inverse image of $y$, the morphism $g: P \to Y$ has at most isolated characteristic points with respect to the singular support $SS \mathcal{G}$ by the condition (2) in Proposition 2.5 and Lemma 1.4(1)⇒(2). Thus, we have $\delta_y = 0$ for any closed point $y' \in Y'$ not on $y$. This implies $[Y': Y] \cdot \delta_y = 0$ by Lemma 2.4.2. Thus the assertion follows.

**Theorem 2.7.** Let the notation be as in Conjecture 2.2 and let $C = SS \mathcal{F}$ be the singular support. Assume that $\dim X = \dim Y = 2$, that $f: X \to Y$ is proper surjective and that every irreducible component of $f_* C$ is of dimension 2. Then, Conjecture 2.2.2 holds.

**Proof.** By Lemma 2.3, the assertion holds except possibly for the coefficients of the fibers $T^*_v Y$ of finitely many closed points $y \in Y$ where $X \to Y$ is not finite. Let $v \in Y$ be a closed point and we show that the coefficients of the fibers $T^*_v Y$ are equal.
Since the resolution of singularity is known for surfaces and since a proper smooth surface over a field is projective, we may assume that $Y$ and hence $X$ are projective. By replacing $X$ by the Stein factorization of $X \to Y$ except on a neighborhood of $v$, we define $X \to X' \to Y$ such that $f' : X' \to Y$ is finite on the complement of $v$ and $r: X \to X'$ is an isomorphism on the inverse image of a neighborhood of $v$.

Since $X'$ is projective, we may take a projective smooth scheme $P$ and decompose $f' : X' \to Y$ as a composition $X' \to P \to Y$ of a closed immersion $i: X' \to P$ and $g: P \to Y$. Conjecture 2.2.2 holds for $G = i_* Rr_* F$ and $g: P \to Y$ except possibly for the coefficients of the fiber $T^n_Y$ by Lemma 2.3. Namely, we have $gCCG = CCRg_* G = CCRf_* F$ except possibly for the coefficients of the fiber $T^n_Y$. By the index formula (2.11), we have

$$((CCRf_* F, T^n_Y)_{T^n_Y} = \chi(Y, Rf_* F),$$

$$= \chi(P, G) = (CCG, T^n_P)_{T^n_P} = (gCCG, T^n_Y)_{T^n_Y}.$$ 

Thus, we have an equality also for the coefficients of the fiber $T^n_Y$.

We give a characterization of characteristic cycle using functoriality.

**Proposition 2.8.** Let $k$ be a perfect field and $\Lambda$ be a finite field of characteristic $\ell$ invertible in $k$. Suppose that for every smooth scheme $X$ over $k$ and constructible complex $F$ of $\Lambda$-modules on $X$, a linear combination $A(F) = \sum_a m_a C_a$ of irreducible components of the singular support $SS F = C = \bigcup_a C_a \subset T^* X$ is attached. If the following conditions (1)-(4) are satisfied, then we have

$$(2.16) \quad A(F) = CC F.$$ 

1. For every étale morphism $j: U \to X$, we have $A(j^* F) = j^* A(F)$.
2. For every projective morphism $f: X \to Y$ to smooth scheme $Y$ of dimension $m$ such that $\dim f_* C = m$, we have $A(Rf_* F) = f_* A(F)$.
3. For every properly $C$-transversal closed immersion $i: V \to X$ of smooth subscheme of codimension 2 and the blow-up $\pi: W \to X$ at $V$, we have $\pi^* A(F) = \pi^! A(F)$.
4. For $X = \text{Spec} k$, we have $A(F) = \text{rank} F \cdot T^* X$.

The characteristic cycles satisfy the conditions (1), (3) and (4). The condition (2) is Conjecture 2.2.2. As the proof below shows, it suffices to assume the condition (2) in the case where $f$ is a closed immersion or $\dim Y \leq 1$.

**Proof.** The conditions (2) and (4) imply the index formula $\chi(X_k, F) = (A(F), T^*_X X)_{T^*_X}$.

First, we show (2.16) for locally constant $F$. Since $SS F = T^*_X X$, we have $A(F) = m_0 C_0$ for $C_0 = T^*_X X$. Since $F$ is étale locally isomorphic to $\bigoplus L^{\otimes q}[-q]$, to show $m_0 = (-1)^n \text{rank} F$, we may assume that $X = \mathbb{P}^n$ and that $F = \bigoplus L^{\otimes q}[-q]$ by (1). Then, the index formula implies $m_0 = (-1)^n \text{rank} F$ since $\chi(\mathbb{P}^n) \neq 0$.

Next, we show (2.16) assuming $\dim X = 1$. Let $U \subset X$ be a dense open subscheme where $F$ is locally constant. Then, we have

$A(F) = \chi(F, T^*_X X + \sum_{x \in X - U} m_x \cdot T^*_x X).$

We show $m_x$ equals the Artin conductor $a_x F$. If $F|_U$ is unramified at $x$ and if $F_x = 0$, the same argument as above shows $m_x = \text{rank} F|_U = a_x F$. We show the general case. By
Lemma 2.4.1 implies that the pencil meets \( X \) satisfying the following properties as in the proof of [12, Theorem 4.21]: The axis \( A \) induced by the adjunction \( \Psi \) by [11, Proposition 3.1].

As the Milnor formula (2.4) and we have (2.16).

Then, by the condition (3) for \( C \) and (2) for \( C \), we refer to [7], [8], [11].

Thus, the index formula implies \( m_a = a_x F \).

We show the general case. By (1), we may assume \( X \) is affine. We consider an immersion \( X \rightarrow A^n \subset P^n \). Then, by (1) and (2), we may assume \( X \) is projective. Set \( SS F = C = \bigcup a C_a \) and take a projective embedding \( X \rightarrow P \) and a pencil \( L \subset P^r \) satisfying the following properties as in the proof of [12, Theorem 4.21]: The axis \( A_L \) of the pencil meets \( X \) transversely, that the closed immersion \( i: V = X \cap A_L \rightarrow X \) is \( C \)-transversal, that the morphism \( p_L: X_L \rightarrow L \) defined by the pencil has at most isolated characteristic points, that the isolated characteristic points are not contained in the inverse image of \( V \) and are unique in the fibers of \( p_L \), and that for each irreducible component \( C_a \) there exists an isolated characteristic point \( u \) where a section \( dp_L \) of \( T^* X \) meets \( C_a \). Then, by the condition (3) for \( \pi : X_L \rightarrow X \) and (2) for \( p_L : X_L \rightarrow L \), the same proof as Lemma 2.4.1 implies

\[-\dim \text{tot}\phi_u(F, p_L) = -a_v R(p_L)_x \pi^* F = (A(F), dp_L)_{T^* X, u} = m_a \cdot (C_a, dp_L)_{T^* X, u} \neq 0\]

for \( v = p_L(u) \). This means that the coefficient \( m_a \) is characterized by the same condition as the Milnor formula (2.4) and we have (2.16).

3 External products

3.1 Nearby cycles and projection formulas

Let \( f : X \rightarrow S \) be a morphism of schemes. For the definition and properties of the vanishing topos \( X \times_S S \) and the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\psi_f} & f \\
\downarrow & & \downarrow \\
X \times_S S & \xrightarrow{p_2} & S \\
\leftarrow & id & \rightarrow \\
S & \xrightarrow{id} & S \\
\end{array}
\]

we refer to [7], [8], [11].

Assume that \( f : X \rightarrow S \) is a morphism of finite type of noetherian schemes. Let \( \Lambda \) be a finite field of characteristic \( \ell \) invertible on \( S \) and let \( F \) and \( G \) be complexes bounded above of \( \Lambda \)-modules on \( X \) and on \( S \) respectively. A canonical morphism

\[
R\psi_f \mathcal{F} \otimes^\Lambda_{\Lambda} p_2^* G \rightarrow R\psi_f (\mathcal{F} \otimes^\Lambda_{\Lambda} f^* G)
\]

on \( X \times_S S \) is defined as the adjoint of \( \psi_f^*(R\psi_f \mathcal{F} \otimes^\Lambda_{\Lambda} p_2^* G) = \psi_f^* R\psi_f \mathcal{F} \otimes^\Lambda_{\Lambda} f^* G \rightarrow \mathcal{F} \otimes^\Lambda_{\Lambda} f^* G \)

induced by the adjunction \( \psi_f^* R\psi_f \mathcal{F} \rightarrow \mathcal{F} \), since \( \psi_f \) is of finite cohomological dimension by [11, Proposition 3.1].

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Lemma 3.1 ([13, Proposition 4]). Let \( f : X \to S \) be a morphism of finite type of noetherian schemes and let \( \mathcal{F} \) and \( \mathcal{G} \) be complexes bounded above of \( \Lambda \)-modules on \( X \) and on \( S \) respectively. We assume that the formation of \( R\Psi_f \mathcal{F} \) commutes with finite base change. Then, the canonical morphism

\[
R\Psi_f \mathcal{F} \otimes^L_A p_2^* \mathcal{G} \to R\Psi_f (\mathcal{F} \otimes^L_A f^* \mathcal{G})
\]
on \( X \times_S S \) is an isomorphism.

Further, let \( h : W \to X \) be a morphism of schemes and consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Psi_f} & X \\
\downarrow{p_2} & & \downarrow{\Psi_f} \\
W \times_X X & \xrightarrow{h} & X \times_X X \\
\downarrow{f} & & \downarrow{f} \\
W \times_S S & \xrightarrow{h} & X \times_S S
\end{array}
\]

of vanishing toposes. By [11, Proposition 3.1], \( f_* \) is of finite cohomological dimension. Let \( \mathcal{F} \) and \( \mathcal{G} \) be complexes of \( \Lambda \)-modules on \( X \) and on \( W \) respectively. We define a base change morphism

\[
R\Psi_f \mathcal{F} \otimes^L_A p_2^* \mathcal{G} \to Rf_*(p_2^* \mathcal{F} \otimes^L_A p_2^* \mathcal{G})
\]
on \( W \times_S S \) as the adjoint of the morphism \( f_*(R\Psi_f \mathcal{F} \otimes^L_A p_2^* \mathcal{G}) = \text{id} \) defined as follows. We identify \( f_*(R\Psi_f \mathcal{F} \otimes^L_A p_2^* \mathcal{G}) \) by the isomorphism \( p_2^* \to R\Psi_f \text{id} \) [7, Proposition 4.7] defined as the adjoint of \( \text{id} \). Then, the morphism in question is induced by the adjunction \( f_*(Rf_* \to \text{id}) \).

Lemma 3.2 ([13, Proposition 5]). Let \( f : X \to S \) be a morphism of finite type of noetherian schemes and \( h : W \to X \) be a morphism of schemes. Let \( \mathcal{F} \) and \( \mathcal{G} \) be complexes bounded above of \( \Lambda \)-modules on \( X \) and on \( W \) respectively. We assume that the formation of \( R\Psi_f \mathcal{F} \) commutes with finite base change. Then, the canonical morphism

\[
R\Psi_f \mathcal{F} \otimes^L_A p_2^* \mathcal{G} \to Rf_*(p_2^* \mathcal{F} \otimes^L_A p_2^* \mathcal{G})
\]
on \( W \times_S S \) is an isomorphism.

We recall an interpretation of local acyclicity in terms of vanishing topos.

Proposition 3.3 ([12, Proposition 1.7]). Let \( f : X \to S \) be a morphism of schemes. Then, for a complex \( \mathcal{F} \in D^+(X) \) bounded below, the following conditions (1) and (2) are equivalent:

1. The morphism \( f : X \to S \) is locally acyclic relatively to \( \mathcal{F} \).
2. The formation of \( R\Psi_f \mathcal{F} \) commutes with every finite base change \( T \to S \) and the canonical morphism \( p_2^* \mathcal{F} \to R\Psi_f \mathcal{F} \) is an isomorphism.
The canonical morphism \( p_1^*F_T \to R\Psi_{f^*}F_T \) is an isomorphism for every finite morphism \( T \to S \), the cartesian diagram
\[
\begin{array}{ccc}
X & \xleftarrow{f} & X_T \\
S & \xleftarrow{f_T} & T
\end{array}
\]
and the pull-back \( F_T \) of \( F \) on \( X_T \).

**Corollary 3.4.** Let \( f: X \to S \) be a morphism of finite type of noetherian schemes and let \( F \) be a bounded complex of \( \Lambda \)-modules on \( X \). Assume that \( f: X \to S \) is locally acyclic relatively to \( F \).

1. Let \( G \) be a complex bounded above of \( \Lambda \)-modules on \( S \). Then, the canonical morphism (3.2) induces an isomorphism
\[
p_1^*F \otimes \Lambda^L p_2^*G \to R\Psi_f(F \otimes \Lambda^L f^*G)
\]
on \( X \times_S S \).

2. Let \( h: W \to X \) be a morphism of schemes and let \( G \) be a complex bounded above of \( \Lambda \)-modules on \( W \). Then, the canonical morphism (3.4) defines an isomorphism
\[
p_1^*h^*F \otimes \Lambda^L p_1^*G \to Rf_*(p_2^*F \otimes \Lambda^L p_1^*G)
\]
on \( W \times_S S \).

**Proof.** By the assumption of local acyclicity and Proposition 3.3 (1)⇒(2), the formation of \( R\Psi_fF \) commutes with finite base change and the canonical morphism \( p_1^*F \to R\Psi_fF \) is an isomorphism.

1. By Lemma 3.1, (3.2) induces an isomorphism (3.5).

2. By Lemma 3.2, and by the canonical isomorphism \( h^* p_1^* \to p_1^* h^* \), the right hand side of (3.4) is identified with that of (3.6). Thus, the assertion follows. \( \square \)

We briefly recall the definition of additive convolution from [8, 4.1]. Let \( k \) be a field and let \( A_1 = A^{(0)}_1 \) and \( A_2 = A^{(0)}_2 \) denote the henselizations of the affine line and of the affine plane at the origins. Let \( f: X \to A_1 \) and \( g: Y \to A_1 \) be morphisms of finite type. We regard the fiber product \( (X \times Y)_2 = (X \times Y) \times_{A_1 \times A_1} A_2 \) as a scheme over \( A_1 \) by the composition of the second projection and the morphism \( a: A_2 \to A_1 \) induced by the addition \( +: A^2 \to A^1 \). Morphisms of vanishing toposes
\[
\begin{array}{c}
X \times_{A_1} A_1 \\
\xleftarrow{pr_1} (X \times Y)_2 \times_{A_2} A_2 \xrightarrow{pr_2} Y \times_{A_1} A_1 \\
\xrightarrow{\bar{a}} (X \times Y)_2 \times_{A_1} A_1
\end{array}
\]
are defined by projections and by \( a: A_2 \to A_1 \).

Let \( \Lambda \) be a finite field of characteristic invertible in \( k \). For bounded complexes \( F \) and \( G \) of \( \Lambda \)-modules on \( X \times_{A_1} A_1 \) and on \( Y \times_{A_1} A_1 \), let \( F \boxtimes G \) denote \( pr_1^*F \otimes pr_2^*G \) on \( (X \times Y)_2 \times_{A_2} A_2 \) and define the additive convolution \( F \ast G \) on \( (X \times Y)_2 \times_{A_1} A_1 \) by
\[
F \ast G = R\bar{a}_*(F \boxtimes G).
\]
3.2 External products

Let \( k \) be a field and let \( \Lambda \) be a finite field of characteristic invertible in \( k \).

**Proposition 3.5.** Let \( X \) and \( Y \) be smooth schemes over \( k \) and \( \mathcal{F} \) and \( \mathcal{G} \) be constructible complexes of \( \Lambda \)-modules on \( X \) and on \( Y \) respectively. Assume that \( \mathcal{F} \) is micro-supported on a closed conical subset \( C \subset T^*X \). Then \( \mathcal{F} \boxtimes^L \mathcal{G} = \text{pr}_1^* \mathcal{F} \boxtimes^L \text{pr}_3^* \mathcal{G} \) is micro-supported on \( C \times T^*Y \subset T^*(X \times Y) \).

**Proof.** It suffices to show that, for morphisms \( a: W \to X, b: W \to Y, c: W \to Z \) of smooth schemes over \( k \) such that the pair \( h = (a, b): W \to X \times Y \) and \( c: W \to Z \) is \( C \times T^*Y \)-transversal, the morphism \( c: W \to Z \) is locally acyclic relatively to \( h^* \mathcal{F} \). By Lemma 1.2, the pair of morphisms \( a: W \to X \) and \( f = (b, c): W \to Y \times Z \) is \( C \)-transversal. Since \( \mathcal{F} \) is assumed micro-supported on \( C \), the morphism \( f = (b, c): W \to Y \times Z \) is locally acyclic relatively to \( a^* \mathcal{F} \).

Let \( Z' \to Z \) be any finite morphism and we consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{a'} & W' = W \times_Z Z' \\
\downarrow{h'} & & \downarrow{f'} \\
X \times Y & \xrightarrow{q} & Y \times Z'
\end{array}
\]

By Proposition 3.3 (3)\( \Rightarrow \) (1), it suffices to show that the canonical morphism

\[
(3.8) \quad p_1^! h'^* (\mathcal{F} \boxtimes \mathcal{G}) \to R\Psi_c h'^* (\mathcal{F} \boxtimes \mathcal{G})
\]

on \( W' \times_{Z'} Z' \) is an isomorphism. For the second term in (3.8), we have a canonical isomorphism

\[
(3.9) \quad R\overline{\tau}_* R\Psi_{f'} (a'^* \mathcal{F} \otimes f'^* q^* \mathcal{G}) \to R\Psi_c h'^* (\mathcal{F} \boxtimes \mathcal{G}).
\]

For the first term in (3.9), we apply Corollary 3.4.1 to \( f': W' \to Y \times Z' \) and to \( a'^* \mathcal{F} \) on \( W' \) and \( q^* \mathcal{G} \) on \( Y \times Z' \). Since \( f': W' \to Y \times Z' \) is locally acyclic relatively to \( a'^* \mathcal{F} \), the assumption of Corollary 3.4.1 is satisfied and we obtain a canonical isomorphism

\[
(3.10) \quad p_1^! a'^* \mathcal{F} \otimes p_0^* q^* \mathcal{G} \to R\Psi_{f'} (a'^* \mathcal{F} \otimes f'^* q^* \mathcal{G}).
\]

Further we apply Corollary 3.4.2 to \( f': W' \to Y \times Z' \) and \( r: Y \times Z' \to Z' \) and to \( q^* \mathcal{G} \) on \( Y \times Z' \) and \( a'^* \mathcal{F} \) on \( W' \). By the generic local acyclicity [4, Corollaire 2.16], the second projection \( r: Y \times Z' \to Z' \) is locally acyclic relatively to \( q^* \mathcal{G} \). Hence the assumption of Corollary 3.4.2 is satisfied and we obtain a canonical isomorphism

\[
(3.11) \quad p_1^! a'^* \mathcal{F} \otimes p_1^* f'^* q^* \mathcal{G} \to R\overline{\tau}_* (p_1^! a'^* \mathcal{F} \otimes p_0^* q^* \mathcal{G})
\]

on \( W' \times_{Z'} Z' \). Thus, (3.9)–(3.11) give an isomorphism

\[
(3.12) \quad p_1^! h'^* (\mathcal{F} \boxtimes \mathcal{G}) = p_1^! a'^* \mathcal{F} \otimes p_1^* b'^* \mathcal{G} \to R\Psi_c h'^* (\mathcal{F} \boxtimes \mathcal{G})
\]

and the assertion follows.
For linear combinations $A = \sum_{a} m_{a} \cdot C_{a}$ and $A' = \sum_{a'} m'_{a'} \cdot C'_{a'}$ of irreducible components of closed conical subsets $C = \bigcup_{a} C_{a} \subset T^{*}X$ and $C' = \bigcup_{a'} C'_{a'} \subset T^{*}Y$ of cotangent bundles, the external product $A \boxtimes A'$ is defined by

$$A \boxtimes A' = \sum_{a,a'} m_{a} m'_{a'} \cdot C_{a} \times C'_{a'}$$

as a linear combination supported on $C \times C' \subset T^{*}X \times T^{*}Y = T^{*}(X \times Y)$.

**Theorem 3.6.** Let $X$ and $Y$ be smooth schemes over a perfect field $k$ and $F$ and $G$ be constructible complexes of $\Lambda$-modules.

1. Assume that $F$ and $G$ are micro-supported on closed conical subsets $C \subset T^{*}X$ and on $C' \subset T^{*}Y$ respectively. Then $F \boxtimes_{\Lambda} G$ is micro-supported on $C \times C' \subset T^{*}X \times T^{*}Y = T^{*}(X \times Y)$.

2. We have

$$CC(F \boxtimes_{\Lambda} G) = CC(F) \boxtimes CC(G).$$

3. We have

$$SS(F \boxtimes_{\Lambda} G) = SS(F) \boxtimes SS(G).$$

The crucial case of Theorem 3.6.2 where $\dim X = \dim Y = 1$ is essentially [9, Exemples 2.3.8 (a)].

**Proof.** 1. By Proposition 3.5, the external product $F \boxtimes_{\Lambda} G$ is micro-supported on the intersection $(C \times T^{*}Y) \cap (T^{*}X \times C') = C \times C'$.

2. If one of $F$ and $G$ is locally constant, the equality (3.14) holds by [12, Theorem 4.4]. In particular, if $\dim X = 0$ or $\dim Y = 0$ the equality (3.14) holds. We may assume $k$ is algebraically closed.

First, we show the case where $\dim X = \dim Y = 1$. Since the question is local on $X$ and on $Y$, we may assume that $F$ and $G$ are locally constant on the complement $U = X - \{x\}$ and $V = Y - \{y\}$ of closed points $x \in X$ and of $y \in Y$ respectively. Then, $F$ is micro-supported on $C = T_{X}^{*}X \cup T_{x}^{*}X$ and $G$ is micro-supported on $C' = T_{Y}^{*}Y \cup T_{y}^{*}Y$ respectively. Since $F \boxtimes G$ is micro-supported on $C \times C'$ by 1., we have

$$CCF = m_{0}T_{X}^{*}X + m_{1}T_{x}^{*}X, \quad CCG = m'_{0}T_{Y}^{*}Y + m'_{1}T_{y}^{*}Y,$$

$$CC(F \boxtimes_{\Lambda} G) = m_{00}T_{X \times Y}^{*}(X \times Y) + m_{10}T_{x \times y}^{*}(X \times Y) + m_{01}T_{X \times y}^{*}(X \times Y) + m_{11}T_{x \times y}^{*}(X \times Y)$$

for some integers. It suffices to show $m_{ij} = m_{i} \cdot m'_{j}$.

Since $F$ is locally constant on $U$ and and $G$ is locally constant on $V$, the equality (3.14) holds on the union $(U \times Y) \cup (X \times V) \subset X \times Y$. Thus, we have $m_{ij} = m_{i} \cdot m'_{j}$ except possibly for $(i, j) = (1, 1)$.

We show $m_{11} = m_{1} \cdot m'_{1}$. We may assume that $F$ and $G$ are sheaves placed at degree 0 on $X$ and on $Y$ respectively. Let $\tilde{X} \supset X$ and $\tilde{Y} \supset Y$ be smooth compactifications. After replacing $X$ and $Y$ by neighborhoods of $x$ and of $y$ respectively, there exists a finite étale schemes $p: X' \rightarrow X$ and $q: Y' \rightarrow Y$ such that for the open immersions $j: X' \rightarrow \tilde{X}$ and $j': Y' \rightarrow \tilde{Y}$ to their smooth compactifications, the direct images $F' = j_{*}p^{*}F$ and $G' = j'_{*}q^{*}G$ are micro-supported on $\tilde{X} \times Y' = X' \times Y'$.
$j'_*q'_*\mathcal{G}$ are locally constant on the complements $\tilde{X}' - p^{-1}(x)$ and $\tilde{Y}' - q^{-1}(y)$ respectively. Then, we have

\[ CC(\mathcal{F}' \boxtimes_\Lambda \mathcal{G}') = m_{00}T_{X' \times \tilde{Y}'}(X' \times Y') + m_{10} \sum_{x' \rightarrow x} T_{x' \times \tilde{Y}'}(X' \times Y') \]

\[ + m_{01} \sum_{y' \rightarrow y} T_{x' \times y'}(X' \times Y') + m_{11} \sum_{x' \rightarrow x, y' \rightarrow y} T_{x' \times y'}(X' \times Y'). \]

By the index formula (2.11) and $\chi(\tilde{X} \times \tilde{Y}, \mathcal{F}' \boxtimes_\Lambda \mathcal{G}') = \chi(\tilde{X}, \mathcal{F}') \cdot \chi(\tilde{Y}, \mathcal{G}')$, we have $m_{11}[X': X][Y': Y] = m_1[X': X] \cdot m'_1[Y': Y]$. Thus the case $\dim X = \dim Y = 1$ is proved.

We show the general case. Write the singular supports $C = SS(\mathcal{F}) = \bigcup_a C_a$ and $C' = SS(\mathcal{G}) = \bigcup_a C'_a$ as the unions of irreducible components and set $CC(\mathcal{F}) = \sum_a m_a C_a$ and $CC(\mathcal{G}) = \sum_a m_a C'_a$. Then, by 1, we have $CC(\mathcal{F} \boxtimes \mathcal{G}) = \sum_{a, a'} m_{a, a'} C_a \times C'_a$ for some integers $m_{a, a'}$. It suffices to show $m_{b, b'} = m_b \cdot m_{b'}$ for each pair of irreducible components $C_b$ and $C_{b'}$.

After shrinking $X$, we may take a morphism $f : X \to \mathbb{A}^1$ such that $f$ has an isolated characteristic point $u$, that $f(u) = 0$ and that the section $df$ meets only $C_b$. Similarly, after shrinking $Y$, we may take a morphism $g : Y \to \mathbb{A}^1$ such that $g$ has an isolated characteristic point $v$, that $g(v) = 0$ and that the section $dg$ meets only $C_{b'}$. Let $h : X \times Y \to \mathbb{A}^1$ denote the morphism defined by the sum $f + g$. Since $dh = df + dg$, the morphism $h$ has an isolated characteristic point $(u, v)$ with respect to $C \times C' = \bigcup_{a, a'} C_a \times C'_a$ and that the section $dh$ meets only $C_b \times C'_{b'}$. Further, we have $(C_b \times C'_{b'}) \cdot (C_{b'}, dh) |_{T^*(X \times Y), (u, v)} = (C_b, df) |_{T^*X, u} \cdot (C_{b'}, dg) |_{T^*Y, v} \neq 0$. Thus, by the Milnor formula (2.4), it suffices to show

\[ (3.16) \quad - \dim \text{tot} \phi_{(u,v)}(\mathcal{F} \boxtimes \mathcal{G}, h) = (- \dim \text{tot} \phi_u(\mathcal{F}, f)) \cdot (- \dim \text{tot} \phi_v(\mathcal{G}, g)). \]

We canonically identify $u \times \mathbb{A}^1 \mathbb{A}^1$ with the strict localization $A^1_{(0)}$. Then the total dimension $\dim \text{tot} \phi_u(\mathcal{F}, f)$ equals the Artin conductor $a_0((R\Psi_f(\mathcal{F})) |_{u \times \mathbb{A}^1 \mathbb{A}^1})$ and similarly for the other terms. By [8, Theorem 4.5 (4.5.1)], we have an isomorphism

\[ (3.17) \quad R\Psi_f \mathcal{F} \ast R\Psi_{g} \mathcal{G} = R\Psi_h(\mathcal{F} \boxtimes \mathcal{G}). \]

The left hand side is the additive convolution (3.7). Thus, we obtain the equality (3.16) by applying Corollary below, which is an immediate consequence of Theorem 3.6 in the case $\dim X = \dim Y = 1$ already proved, to $K = (R\Psi_f(\mathcal{F})) |_{u \times \mathbb{A}^1 \mathbb{A}^1}$ and $L = (R\Psi_g(\mathcal{G})) |_{v \times \mathbb{A}^1 \mathbb{A}^1}$.

3. We may assume $\mathcal{F}$ and $\mathcal{G}$ are perverse sheaves. Then, since the singular support is the support of the characteristic cycle by [12, Proposition 3.19.2], the assertion follows from 2.

**Corollary 3.7** (cf. [10, Proposition (2.7.2.1)]). Let $K$ and $L$ be constructible complexes of $\Lambda$-modules on the strict localization $A^1_{(0)}$. Then, for the Artin conductor, we have

\[ (3.18) \quad -a_0(K \ast L) = (-a_0K) \cdot (-a_0L). \]

**Proof.** We may assume that $K$ and $L$ are the pull-backs of constructible complexes $F$ and $G$ on étale neighborhoods $X$ and $Y$ of 0 respectively. Then, it suffices to apply the Milnor formula (2.4) to $CC(\mathcal{F} \boxtimes \mathcal{G}) = CC\mathcal{F} \boxtimes CC\mathcal{G}$ and $a = x + y$. 

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For a separated morphism \( h : W \to X \) of finite type and for a constructible complex \( \mathcal{F} \) of \( \Lambda \)-modules, a canonical morphism \( h^* \mathcal{F} \otimes \text{R}h^! \Lambda \to \text{R}h_! \mathcal{F} \) is defined as the adjoint of the morphism \( \text{R}h_!(h^* \mathcal{F} \otimes \text{R}h^! \Lambda) \cong \mathcal{F} \otimes \text{R}h_! \text{R}h^! \Lambda \to \mathcal{F} \) induced by the adjunction \( \text{R}h_! \text{R}h^! \Lambda \to \Lambda \).

**Corollary 3.8.** Let \( f : X \to Y \) be a morphism of smooth schemes over \( k \) and let \( \gamma : X \to X \times Y \) be the graph of \( f \). Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on \( X \) and set \( C = SS(\mathcal{F}) \).

1. Let \( \mathcal{G} \) be a constructible complex of \( \Lambda \)-modules on \( Y \) and set \( C' = SS(\mathcal{G}) \). Assume \( f \) is \((C,C')\)-transversal. Then, the canonical morphism

\[
(3.19) \quad \gamma^*(\mathcal{F} \boxtimes \mathcal{G}) \otimes R\gamma^! \Lambda \to R\gamma^!(\mathcal{F} \boxtimes \mathcal{G})
\]

is an isomorphism and \( \mathcal{F} \otimes f^* \mathcal{G} = \gamma^*(\mathcal{F} \boxtimes \mathcal{G}) \) is micro-supported on \( C + f^* C' \subset T^* X \) consisting of the sum \( \alpha + \beta \) of \( \alpha \in C \) and \( \beta \in f^* C' \).

2. Assume that \( f \) is \( C \)-transversal. Then, for every constructible complex \( \mathcal{G} \) of \( \Lambda \)-modules on \( Y \), the conclusion of 1 is satisfied.

The morphism (3.19) is the same as [3, (5.3)]. The conclusion of 2. is shown to be equivalent to the local acyclicity of \( f \) in [3, Theorem B.2].

**Proof.** 1. The assumption that \( f \) is \((C,C')\)-transversal means that \( \gamma \) is \( C \times C' \)-transversal by Lemma 1.3. Since \( \mathcal{F} \boxtimes \mathcal{G} \) is micro-supported on \( C \times C' \) by Theorem 3.6.1, the morphism \( \gamma \) is \( \mathcal{F} \boxtimes \mathcal{G} \)-transversal by [12, Proposition 5.6 (1)\( \Rightarrow \) (2)]. Thus, the morphism (3.19) is an isomorphism. Further, \( \mathcal{F} \otimes f^* \mathcal{G} = \gamma^*(\mathcal{F} \boxtimes \mathcal{G}) \) is micro-supported on \( \gamma^!(C \times C') = C + f^* C' \) by [12, Lemma 2.11.4 (1)\( \Rightarrow \) (2)] and Lemma 1.3.

2. Since \( f \) is \((C,T^* Y)\)-transversal, it is \((C,C')\)-transversal for any closed conical subset \( C' \subset T^* Y \).

**Corollary 3.9.** Let \( \mathcal{F} \) and \( \mathcal{G} \) be constructible complexes of \( \Lambda \)-modules on a smooth scheme \( X \) over \( k \). Assume that the intersection \( SS(\mathcal{F}) \cap SS(\mathcal{G}) \subset T^* X \) of the singular supports is a subset of the 0-section \( T^*_X X \subset T^* X \). Then, the canonical morphism

\[
(3.20) \quad \mathcal{G} \otimes^L R\text{Hom}_X(\mathcal{F}, \Lambda) \to R\text{Hom}_X(\mathcal{F}, \mathcal{G})
\]

is an isomorphism.

**Proof.** Set \( C = SS(\mathcal{F}) \) and \( C' = SS(\mathcal{G}) \). The assumption \( C \cap C' \subset T^*_X X \) implies that the diagonal \( \delta : X \to X \times X \) is \( C' \times C \)-transversal. Since \( SSD_X \mathcal{F} = SS \mathcal{F} \) by [12, Corollary 2.27], the external product \( \mathcal{G} \boxtimes D_X \mathcal{F} \) is micro-supported on \( C' \times C \subset T^* X \times T^* X = T^*(X \times X) \) by Theorem 3.6.1. Since the canonical morphism \( \mathcal{G} \boxtimes D_X \mathcal{F} \to R\text{Hom}_{X \times X}(pr_2^* \mathcal{F}, pr_1^* \mathcal{G}) \) is an isomorphism by [6, (3.1.1)], the diagonal \( \delta : X \to X \times X \) is \( R\text{Hom}_{X \times X}(pr_2^* \mathcal{F}, pr_1^* \mathcal{G}) \)-transversal by [12, Proposition 5.6]. Thus, the assertion follows from [12, Proposition 5.3.2 (1)\( \Rightarrow \) (2)].

**References**


