# $p$-adic gauge theory in number theory

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# Dirichlet's theorem

$\boldsymbol{F}$: totally real field, $\boldsymbol{O_F}$: the integer ring, $[\boldsymbol{F} : \mathbb{Q}] = \boldsymbol{d}$.

$\boldsymbol{p}$: a prime number.

Dirichlet's unit theorem:

$$\boldsymbol{O_F^\times \simeq \{\pm 1\} \times \mathbb{Z}^{d-1}}$$

Moreover, for a basis $\langle \boldsymbol{\epsilon_1 \ldots, \epsilon_{d-1}} \rangle$ of the free part, he showed

$$\mathbf{rank}(\mathbf{log}|\boldsymbol{\iota}(\boldsymbol{\epsilon_j})|)_{\boldsymbol{1 \leq j \leq d-1,\ \iota : F \hookrightarrow} \mathbb{R}} = \boldsymbol{d - 1}$$

$\Leftrightarrow$ non-vanishing of the regulator

$(= \mathbf{det}\ (\boldsymbol{d-1}) \times (\boldsymbol{d-1})$-minor)

## Leopoldt's conjecture

$$\mathbf{rank}(\log_p \iota(\epsilon_j))_{1\le j\le d-1,\ \iota: F\hookrightarrow \hat{\bar{\mathbb{Q}}}_p} = d-1$$

$\Leftrightarrow$ non-vanishing of the $p$-adic regulator $R_p$

$$\Leftrightarrow \delta = 0$$

Here

$$\delta = \delta_{F,p} = \mathrm{rank}_{\mathbb{Z}_p} \ker(O_F^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \prod_{v|p} (o_{F_v}^\times)_p)$$

is the Leopoldt defect.

By the classical class field theory, $\mathbf{rank}_{\mathbb{Z}_p}\, \boldsymbol{\mathcal{X}_p = 1 + \delta_{F,p}}$.
Here $\boldsymbol{\mathcal{X}_p = (G^{\mathbf{ab}}_{\{v|p\}})_p}$.

$$\text{Leopoldt's conjecture}$$

$$\Leftrightarrow \text{ Any } \mathbb{Z}_{\boldsymbol{p}}\text{-extension is cyclotomic}$$

P. Colmez:

$$\text{Leopoldt's conjecture}$$

$$\Leftrightarrow \boldsymbol{p}\text{-adic zeta function of } \boldsymbol{F} \text{ has a pole at } \boldsymbol{s = 1}$$

$$\text{The conjecture } \in \text{ Iwasawa theory for } \mathbf{GL(1)}$$

## Known results

- $\delta_{F,p} = 0$ if $F$ is abelian over $\mathbb{Q}$ (Brumer, 1967).
- $\delta_{F,p} \leq [\frac{d}{2}]$ (Waldschmidt, 1984).

These results $\Leftarrow$ Transcendental number theory:

In Brumer's case, $p$-adic regulator $R_p$ is a product of $\bar{\mathbb{Q}}$-linear forms of $p$-adic log of units. Then apply a $p$-adic version of Baker's theorem.

## Question

Based on works of Serre, Ribet,

Mazur-Wiles: **GL(2)**-class field theory

$\Rightarrow$ Iwasawa's main conjecture for **GL(1)**

Question: Is **GL(2)**-class field theory effective to understand Leopoldt's conjecture?

## Digression: meaning of the title

**Theorem 0.1.** *(Donaldson)*
$M$*: closed, oriented, simply connected smooth 4-manifold*

*If the intersection form* $\langle\ ,\ \rangle$ *on* $H^2(M,\mathbb{Z})$ *is positive definite* $\Rightarrow$ $\langle\ ,\ \rangle$ *is standard.*

Use the moduli of ASD-$SU(2)$ connections to understand the topology of $M$.

**Remark 0.2.** $C = \mathrm{Spec}\, O_F \setminus \Sigma$ *should be considered as an analogue of hyperbolic 3-manifold* $N$*, so Thurston's theory of the moduli of flat bundles on* $N$ *is the right geometric analogy.*

*Leopoldt's conjecture* $\Leftrightarrow \dim_{\mathbb{Q}} H^1(N,\mathbb{Q}) = 1$.

## Partial answer

**Theorem 0.3.** $F$*: totally real field.*

$$p >> 0 \Rightarrow \delta_{F,p} = 0,$$

*i.e., Leopoldt's conjecture is true for* $(F, p)$.

**Remark 0.4.** *Theorem is proved under* $p \geq 3$, *and Assumption* $A_{(F,\ p)}$ *explained later (instead of* $p >> 0$*).*

## Strategy

The proof of the theorem is divided into:

- Part I : Existence of nice characters of $\boldsymbol{F}$
- Part II: Modularity of some 2-dimensional reducible residual representations
- Part III: Main argument

Based on recent (= last thirteen years after Wiles) development in **GL(2)**-class field theory. Also, Hida's theory of nearly ordinary Hecke algebras for **GL(2)** plays an essential role.

## Proof Plan

- Find a nice $\mathbf{2}$-dimensional *reducible* $\bar{\boldsymbol{\rho}}$.
- Define a deformation ring $\boldsymbol{R}$ from $\bar{\boldsymbol{\rho}}$. Let $\boldsymbol{I}$ be the defining ideal of the reducible locus (Eisenstein ideal). Determine the Krull dimension of $\boldsymbol{R}/\boldsymbol{I}$. Larger than the expected value if $\boldsymbol{\delta_{F,p} > 0}$.
- Determine the number of the generators of $\boldsymbol{I}$. Smaller than the expected value if $\boldsymbol{\delta_{F,p} > 0}$.
- Look at the tangent space of $\boldsymbol{R}$, and the local distributions of tangent vectors coming from $\boldsymbol{I}$. Conclude $\boldsymbol{\delta_{F,p} = 0}$ by showing the surjectivity of global to local restriction map.

## Part I

$E_\lambda$: a $p$-adic field, $o_\lambda$: the ring of the integers.

$\chi : G_F \longrightarrow o_\lambda^\times$: a character of finite order is *nice if*

- $\chi$ *is totally odd, of order prime to* $p$.
- $\chi$ *is unramified at* $\forall v|p$, $\chi(\mathrm{Fr}_v) \neq 1$.
- $H^1_f(F,\ \bar{\chi}^{\pm 1}) = 0$ *(*$\Leftrightarrow$ *the relative class number of* $F_\chi/F$ *is prime to* $p$ *if* $\chi$ *is quadratic).*

*Here* $\bar{\chi} = \chi \mod \lambda$.

## Existence of nice quadratic characters

Assumption $\boldsymbol{A_{(F,\ p)}}$: there is *at least one* nice character $\boldsymbol{\chi}$.

$\boldsymbol{A_{(F,\ p)}}$ is satisfied with some quadratic $\boldsymbol{\chi}$ in the following cases:

- $\boldsymbol{p = 3}$ (Davenport-Heilbronn, $\boldsymbol{\ldots}$).
- $\boldsymbol{p} \nmid$ the numerator of $\boldsymbol{\zeta_F(-1)}$ (P. Hartung, H. Naito).
- (Hopefully) $\boldsymbol{\zeta_p \notin F_v}$ for any $\boldsymbol{v|p}$, and $\boldsymbol{[F(\zeta_p) : F] > 2}$ (F.)

Some trace formula argument (Selberg or Lefschetz) is used.

## Part II

Assume $\boldsymbol{A_{(F,\ p)}}$, fix a nice character $\boldsymbol{\chi}$. Construct an indecomposable reducible $\bar{\boldsymbol{\rho}} : \boldsymbol{G_F} \longrightarrow \mathbf{GL_2}(\boldsymbol{k_\lambda})$ by the following conditions:

- $\bar{\boldsymbol{\rho}}$ takes a form

$$\mathbf{0} \longrightarrow \mathbf{1} \longrightarrow \bar{\boldsymbol{\rho}} \longrightarrow \bar{\chi} \longrightarrow \mathbf{0}.$$

- $\bar{\boldsymbol{\rho}}|_{\boldsymbol{I_v}}$ is split except one finite place $\boldsymbol{y}$ s.t. $\chi(\mathbf{Fr}_{\boldsymbol{y}})^{-1} \equiv \boldsymbol{q_y} \not\equiv \mathbf{1} \mod \boldsymbol{p}$.

$\boldsymbol{\Sigma} = \{\boldsymbol{v}|\boldsymbol{p}\} \cup \{$ramification set of $\bar{\boldsymbol{\rho}}\}$.

**Theorem 0.5.** *Assume* $\boldsymbol{d > 1}$, *and* $\boldsymbol{q_y}$*: sufficiently large. Then* $\bar{\boldsymbol{\rho}}$ *is (minimally) modular in the following sense:*

- $\exists \boldsymbol{\pi}$*: cuspidal rep. of* $\mathbf{GL_2}(\mathbb{A}_{\boldsymbol{F}})$, *unramified outside* $\boldsymbol{\Sigma} \cup \{\boldsymbol{v}|\infty\}$ *and of parallel weight 2,*
- $\boldsymbol{\pi}$ *is nearly ordinary at* $\forall \boldsymbol{v}|\boldsymbol{p}$,
- $\bar{\boldsymbol{\rho}} \simeq \bar{\boldsymbol{\rho}}_{\boldsymbol{\pi},\boldsymbol{\lambda}}$.

Proof: Rather complicated (congruences between Eisenstein series, level lowering argument, ...).

Related talk: Arithmetic Geometry Kyoto 09/2006.

E. Urban has a simpler construction.

## Part III: Preliminaries

$S$: an auxiliary set of finite places

- $\Sigma \cap S = \emptyset$, $\sharp S \leq \delta$.
- $\chi(\mathrm{Fr}_v) \equiv q_v \equiv -1 \mod p \quad \forall v \in S$.

$\Sigma_{\mathcal{D}} = \Sigma \cup S$.

$\mathcal{D} = \mathcal{D}_S$: minimal deformation data for $\bar{\rho}$

## Deformation conditions

$\boldsymbol{\rho : G_{\Sigma_{\mathcal{D}}} \longrightarrow \mathbf{GL_2}(A)}$: a deformation of $\boldsymbol{\bar{\rho}}$

Local conditions:

- At $\boldsymbol{v|p}$, $\boldsymbol{\rho}$ is nearly ordinary:

$$\boldsymbol{\rho|_{G_{F_v}} \sim \begin{pmatrix} \tilde{\chi}_{1,v} & * \\ 0 & \tilde{\chi}_2 \end{pmatrix}},$$

  $\boldsymbol{\tilde{\chi}_{1,v}}$ lifts $\boldsymbol{\bar{\chi}|_{G_{F_v}}}$ (converse to $\boldsymbol{\bar{\rho} \sim \begin{pmatrix} 1 & * \\ 0 & \bar{\chi} \end{pmatrix}}$).

- At $\boldsymbol{v} \in \boldsymbol{S}$, $\boldsymbol{\rho}$ is special:

$$\boldsymbol{\rho}|_{G_{F_v}} \sim \begin{pmatrix} \tilde{\boldsymbol{\chi}}_{\mathbf{1},\boldsymbol{v}} & * \\ \mathbf{0} & \tilde{\boldsymbol{\chi}}_{\mathbf{1},\boldsymbol{v}}(-\mathbf{1}) \end{pmatrix},$$

  $\tilde{\boldsymbol{\chi}}_{\mathbf{1},\boldsymbol{v}}$ lifts $\bar{\boldsymbol{\chi}}|_{G_{F_v}}$ (converse to $\bar{\boldsymbol{\rho}}$).

- At other places in $\boldsymbol{\Sigma_{\mathcal{D}}}$: finite deformation (special at $\boldsymbol{y}$, in other cases the restriction to the inertia is the "same as $\bar{\boldsymbol{\rho}}$".

Global condition: $\mathbf{det}\boldsymbol{\rho}$ is fixed (to $\boldsymbol{\chi} \cdot \boldsymbol{\chi}_{\mathbf{cycle},\boldsymbol{p}}^{-1}$).

## Nearly ordinary Hecke algebra

$\boldsymbol{R_{\mathcal{D}}}$ = universal deformation ring of $\bar{\boldsymbol{\rho}}$

$\boldsymbol{T_{\mathcal{D}}}$ = nearly ordinary cuspidal Hecke algebra (exists by Part II and a lifting argument of Diamond-Taylor (need to find a lift which is special at $\boldsymbol{S}$).

Since $\boldsymbol{\chi}$ is nice, $\exists$ universal modular representation

$$\boldsymbol{\rho_{\mathcal{D}}^{\textbf{modular}} : G_{\Sigma_{\mathcal{D}}} \longrightarrow \mathbf{GL}_2(T_{\mathcal{D}})}$$

(existence of free lattice), and we have a surjective map

$$\boldsymbol{R_{\mathcal{D}} \twoheadrightarrow T_{\mathcal{D}}.}$$

## $R = T$ and the consequence

**Proposition 0.6.** *Under the conditions,* $R_{\mathcal{D}} \simeq T_{\mathcal{D}}$*, and* $R_{\mathcal{D}}$ *is a complete intersection of Krull dimension* $d + 1$*.*

Proof: Standard application of Taylor-Wiles system.

Using the proposition, one can calculate the reducible (=Eisenstein) locus:

**Proposition 0.7.** $I_{\mathcal{D}} :=$ *Eisenstein ideal of* $T_{\mathcal{D}}$

$T_{\mathcal{D}}/I_{\mathcal{D}} = R_{\mathcal{D}}/I_{\mathcal{D}}$ *is* $o_{\lambda}$*-flat, complete intersection of Krull dimension* $= 1 + \delta - \sharp S$ *under a mild assumption on* $S$*.*

## Eisenstein locus

The Eisenstein locus is isomorphic to

$$\boldsymbol{o_\lambda[[\mathcal{X}_p]]/(L_y, \chi^{\mathrm{univ}}(\mathrm{Fr}_v) - 1 \quad (v \in S))},$$

where $\boldsymbol{L_y}$ is (Euler factor at $\boldsymbol{y}$ of) a $\boldsymbol{p}$-adic $\boldsymbol{L}$-function.

(Recall $\boldsymbol{\mathcal{X}_p = (G^{\mathrm{ab}}_{\{v|p\}})_p}$, $\mathbf{rank}_{\mathbb{Z}_p} \boldsymbol{\mathcal{X}_p = 1 + \delta}$.)

If $\mathbf{Fr}_{\boldsymbol{y}}$ and $\mathbf{Fr}_{\boldsymbol{v}}$ for $\boldsymbol{v \in S}$ are linearly independent in $\boldsymbol{\mathcal{X}_p}$
$\Rightarrow$ the Krull dimension is as in the proposition.

## TW-system

A TW system is a family $(\boldsymbol{R_Q}, \boldsymbol{M_Q})_{\boldsymbol{Q} \in \boldsymbol{X}}$ which satisfies

- $\boldsymbol{Q}$: a finite set of finite places of $\boldsymbol{F}$.
- $\boldsymbol{Q} = \emptyset \Rightarrow (\boldsymbol{R_\emptyset}, \boldsymbol{M_\emptyset}) = (\boldsymbol{R}, \boldsymbol{M})$.
- $\boldsymbol{R_Q}$: complete noeth. local $\boldsymbol{o_\lambda}[\boldsymbol{\Delta_Q}]$-algebra, ($\boldsymbol{\Delta_Q}$: finite abelian $\boldsymbol{p}$-group).
- $\boldsymbol{M_Q}$: non-zero $\boldsymbol{R_Q}$-module, *finite free* over $\boldsymbol{o_\lambda}[\boldsymbol{\Delta_Q}]$.

## Complete intersection-freeness theorem

Under certain assumptions, $\boldsymbol{R}$ is a complete intersection, and $\boldsymbol{M}$ is $\boldsymbol{R}$-free ($\Rightarrow$ $\boldsymbol{R}$ is equal to $\boldsymbol{T} = \mathbf{image}(\boldsymbol{R} \longrightarrow \mathbf{End}_{\boldsymbol{o_\lambda}} \boldsymbol{M})$). In particular, $\boldsymbol{M}$ is a faithful $\boldsymbol{R}$-module, and $\boldsymbol{R}$ can not be quite large.

## TW-system with local variables

Our choice to prove $\boldsymbol{R} = \boldsymbol{T}$:

- $\boldsymbol{Q} \in \boldsymbol{X} \Leftrightarrow \boldsymbol{Q} \cap \Sigma_{\boldsymbol{\mathcal{D}}} = \emptyset$,
  $\boldsymbol{Q} \subset \{\boldsymbol{v} : \boldsymbol{q_v} \equiv \mathbf{1} \mod \boldsymbol{p},\ \chi(\mathbf{Fr}_{\boldsymbol{v}}) \neq \mathbf{1}\}$
- $\boldsymbol{R}_{\boldsymbol{\mathcal{D}}_{\boldsymbol{Q}}}$: the universal deformation ring with unrestricted conditions at $\boldsymbol{Q}$
- $\boldsymbol{\Delta}_{\boldsymbol{Q}} = \prod_{\boldsymbol{v} \in \boldsymbol{Q}} \boldsymbol{k}(\boldsymbol{v})_{\boldsymbol{p}}^{\times}$ ($\Leftrightarrow$ split maximal torus of $\mathbf{SL(2)}$)
- $\boldsymbol{M}_{\boldsymbol{Q}}$: obtained from the middle dimensional cohomology of compact modular varieties of complex dimension $\leq \mathbf{1}$

## Part III: Control of Eisenstein ideals

Speculation: $\boldsymbol{I_{\mathcal{D}}}$ may be generated by

$$\dim \boldsymbol{R_{\mathcal{D}}} - \dim \boldsymbol{R_{\mathcal{D}}/I_{\mathcal{D}}}$$

$$= \boldsymbol{d} + 1 - (1 + \boldsymbol{\delta} - \sharp \boldsymbol{S}) = \boldsymbol{d} + \sharp \boldsymbol{S} - \boldsymbol{\delta}$$

elements (the argument is correct if $\boldsymbol{R_{\mathcal{D}}}$ were smooth).

**Theorem 0.8.** *For a suitably chosen $\boldsymbol{S}$, $\boldsymbol{I_{\mathcal{D}}}$ is generated by $\boldsymbol{d} + \sharp \boldsymbol{S} - \boldsymbol{\delta}$-elements.*

Proof: Use virtual smoothing argument using "TW-system with global variables".

## TW-system with global variables

Estimate of $\boldsymbol{I_{\mathcal{D}}}$ is done by using a modified system:

$\boldsymbol{\mathcal{X}_Q}$ = the maximal quotient of $(\boldsymbol{G^{\mathbf{ab}}_{\{v|p\}\cup Q}})_{\boldsymbol{p}}$ which is split at $\boldsymbol{S} \cup \{\boldsymbol{y}\}$.
$\tilde{\boldsymbol{\Delta}}_{\boldsymbol{Q}}$ = the image of $\boldsymbol{\Delta_Q}$ in $\boldsymbol{\mathcal{X}_Q}$.

$\tilde{\boldsymbol{R}}_{\boldsymbol{\mathcal{D}_Q}} = \boldsymbol{R_{\mathcal{D}_Q}} \otimes_{\boldsymbol{o_\lambda[\Delta_Q]}} \boldsymbol{o_\lambda}[\tilde{\boldsymbol{\Delta}}_{\boldsymbol{Q}}]$, etc. $\Rightarrow$ Modified TW system $(\tilde{\boldsymbol{R}}_{\boldsymbol{\mathcal{D}_Q}}, \tilde{\boldsymbol{M}}_{\boldsymbol{Q}})$.

Looks more like Euler systems.

## Conditions on $S$

There is a subspace in the dual cohomology group

$$\mathcal{L}_{\bar\rho} \subset H^1(F, \bar\rho^\vee(1)) \subset H^1(F, \mathrm{ad}^0 \bar\rho(1))$$

(the Leopoldt part). $\dim \mathcal{L}_{\bar\rho} \leq \delta$, and $S$ should satisfy

- $\mathrm{Fr}_y$ and $\mathrm{Fr}_v$ for $v \in S$ are linearly independent in $\mathcal{X}_p$.
- All classes in $\mathcal{L}_{\bar\rho}$ are unramified at $S$,

$$\mathcal{L}_{\bar\rho} \simeq \prod_{s \in S} H^1_f(F_s, \bar\rho^\vee(1)).$$

Consequence: $H^1_{\mathcal{D}^*}(F, \mathrm{ad}^0 \bar\rho(1)) \cap \mathcal{L}_{\bar\rho} = \{0\}$.

## Proof plan of Theorem 0.8

Choose a nice $\boldsymbol{S}$ by the Chebotarev density theorem. Then the Leopoldt part in $\boldsymbol{H^1_{\mathcal{D}^*}(F, \mathrm{ad}^0\,\bar{\rho}(1))}$ is zero, and there are no obstructions to make $(\tilde{\boldsymbol{R}}_{\mathcal{D}_Q}, \tilde{\boldsymbol{R}}_{\mathcal{D}_Q}/\tilde{\boldsymbol{I}}_{\mathcal{D}_Q})$ larger $\Rightarrow$ Theorem 0.8.

**Proposition 0.9.** $\boldsymbol{Q}$: *general,* $\tilde{\boldsymbol{I}}_{\mathcal{D}_Q}$: *the Eisenstein ideal of* $\tilde{\boldsymbol{R}}_{\mathcal{D}_Q}$.

*Then* $\tilde{\boldsymbol{I}}_{\mathcal{D}_Q}$ *is generated by* $\boldsymbol{d} + \sharp\boldsymbol{S} - \boldsymbol{\delta}$*-elements.*

Proof: Reduction to the minimal case using the flatness of $\tilde{\boldsymbol{R}}_{\mathcal{D}_Q}$ and $\tilde{\boldsymbol{R}}_{\mathcal{D}_Q}/\tilde{\boldsymbol{I}}_{\mathcal{D}_Q}$ over $\boldsymbol{o}_{\boldsymbol{\lambda}}[\tilde{\boldsymbol{\Delta}}_Q]$.

## Part III: Control of Tangent spaces

The tangent space $V_Q$ of $\tilde{R}_{\mathcal{D}_Q}$ is

$$\ker(H^1(F, \mathrm{ad}^0 \bar{\rho}) \to \prod_{v \notin Q} H^1(F_v, \mathrm{ad}^0 \bar{\rho})/L_v)$$

for the local tangent spaces $\{L_v\}$.

**General Principle (Vague form)**:

Assume $(R_Q, M_Q)_{Q \in X}$ satisfies the assumption of the complete intersection-freeness criterion.

For a finite set $P$ of finite places, the union of $\mathrm{image}(V_Q \xrightarrow{\mathrm{res}_Q} \prod_{v \in P} L_v)$ for $Q \in X$ spans $\prod_{v \in P} L_v$.

## The principle ⇒ Leopoldt

Use TW-system $(\tilde{R}_{\mathcal{D}_Q}, \tilde{M}_Q)$ with global variables.

There is a commutative diagram

$$\begin{array}{ccc} V_Q & \xrightarrow{\mathbf{res}_Q} & \prod_{v\in P} L_v \\ \downarrow & & \prod \mathbf{pr}_v \downarrow \\ (\tilde{I}_{\mathcal{D}_Q} \otimes k_\lambda)^\vee & \xrightarrow{f_Q} & \prod_{v\in P} L'_v \end{array}$$

Here $P = \{v|p\} \cup S$, $L'_v = H^1(F_v, \bar{\chi}) \overset{\mathbf{pr}_v}{\twoheadleftarrow} L_v$.

$\mathbf{dim}_{k_\lambda}(\tilde{I}_{\mathcal{D}_Q} \otimes k_\lambda)^\vee = d + \sharp S - \delta$,

$\mathbf{dim}_{k_\lambda} \prod_{v\in P} L'_v = d + \sharp S$.

$\exists Q$, $\mathbf{res}_Q$: surjective $\Rightarrow \exists Q$, $f_Q$: surjective $\Rightarrow \delta = 0$.

# Example of the Principle

Assume $\bar{\boldsymbol{\rho}}$ and $\mathbf{ad}^{\mathbf{0}}\,\bar{\boldsymbol{\rho}}$: abs. irred (with large monodromy).

$\boldsymbol{P}$: a finite set of finite places, $\boldsymbol{\mathcal{D}}$: minimal

$\Rightarrow$ there is some $\boldsymbol{Q}$ s.t. $\boldsymbol{H}^{\mathbf{1}}_{\boldsymbol{\mathcal{D}_Q}}(\boldsymbol{F}, \mathbf{ad}^{\mathbf{0}}\,\bar{\boldsymbol{\rho}}) \xrightarrow{\mathbf{res}_{\boldsymbol{Q}}} \prod_{\boldsymbol{v}\in\boldsymbol{P}} \boldsymbol{L}_{\boldsymbol{v}}$ is surjective.

Sketch: Take a subspace $\boldsymbol{W} \subset \prod_{\boldsymbol{v}\in\boldsymbol{P}} \boldsymbol{L}_{\boldsymbol{v}}$ s.t. the image of $\mathbf{res}_{\boldsymbol{Q}}$ is in $\boldsymbol{W}$ for any $\boldsymbol{Q}$.

$\boldsymbol{H}^{\mathbf{1}}_{\tilde{\boldsymbol{\mathcal{D}}}^*}(\boldsymbol{F}, \mathbf{ad}^{\mathbf{0}}\,\bar{\boldsymbol{\rho}}(\mathbf{1})))$: the dual cohomology group, the condition at $\boldsymbol{P}$ is the annihilator of $\boldsymbol{W}$ (modified from $\prod_{\boldsymbol{v}\in\boldsymbol{P}} \boldsymbol{L}_{\boldsymbol{v}}$ to $\boldsymbol{W}$).

Using the Chebotarev density theorem, a Taylor-Wiles system $(R_Q, M_Q)_{Q\in X}$ is constructed with
$r = \dim H^1_{\mathcal{D}}(F, \mathrm{ad}^0\,\bar{\rho})$,
$r' = \sharp Q = \dim H^1_{\tilde{\mathcal{D}}^*}(F, \mathrm{ad}^0\,\bar{\rho}(1)))$.

$$r' - r = \dim(\prod_{v\in P} L_v/W)$$

holds by Euler characteristic formula. $r = r'$ by a ring theoretic consideration $\Rightarrow W = \prod_{v\in P} L_v$.

**Remark 0.10.** *This method depends heavily on the Chebotarev density argument.*

## Proof Plan revisited

- Find a nice $\mathbf{2}$-dimensional *reducible* $\bar{\boldsymbol{\rho}}$.
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