k : 体　X/k　分離有限型スキーム

$\Lambda = \mathbb{Z}/\ell^n$　$\ell \neq \mathrm{char}\, k$

$D_{ctf}(X, \Lambda) = D(X)$　6 operations

$K_X = Rf^!\Lambda$　$f : X \to \mathrm{Spec}\, k$

Tate twist　shift

f smooth, rel dim $= d$ なら $K_X = \Lambda(d)[2d]$

$D_X = R\mathcal{H}om(-, K_X) : D(X) \to D(X)$

canonical class

$\mathcal{F}$: object of $D(X)$ に対し、$\underset{\text{canonical class}}{C(\mathcal{F})} \in H^0(X, K_X)$

X : smooth なら $H^{2d}(X, \Lambda(d))$

$\mathcal{H} = R\mathcal{H}om(pr_2^* \mathcal{F}, Rpr_1^! \mathcal{F})$　$D(X \times X)$ の obj.

$X \times X \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} X$　$f : X \to Y$　$\Gamma_f \subset X \times Y$

X : smooth　$\mathcal{F}$ smooth ($= \mathcal{F}$ l.c.c. sheaf of flat Λ-mod)
(有限生成)

のときは

$\mathcal{H} = \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^* \mathcal{F})(d)[2d]$

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$$\begin{array}{ccc} X\times X & \xrightarrow{pr_2} & X \\ \downarrow{\scriptstyle pr_1} & & \downarrow \\ X & \longrightarrow & \cdot \end{array}$$

標準同形

SGA5

$$\mathcal{H} = R\mathcal{H}om(pr_2^*\mathcal{F}, Rpr_1^!\mathcal{F}) \xleftarrow{\sim} \mathcal{F} \otimes^L D_X(\mathcal{F})$$

$$\text{"} \quad pr_1^*\mathcal{F} \otimes^L \underbrace{pr_2^* R\mathcal{H}om(\mathcal{F}, K_X)}$$

$$\text{"} \quad R\mathcal{H}om(pr_2^*\mathcal{F}, Rpr_1^!\Lambda)$$

$$\text{"} \quad pr_2^* K_X$$

$$pr_1^*\mathcal{F} \otimes Rpr_1^!\Lambda \to Rpr_1^!\mathcal{F}$$

標準射

$\delta : X \to X\times X$

$$R\delta^!\mathcal{H} = R\delta^! R\mathcal{H}om_{X\times X}(pr_2^*\mathcal{F}, Rpr_1^!\mathcal{F})$$

$$\xleftarrow[\text{SGA5}]{\sim} R\mathcal{H}om_X(\delta^* pr_2^*\mathcal{F}, \delta^! Rpr_1^!\mathcal{F})$$

$$\begin{array}{ccc} \delta_*\delta^* pr_2^*\mathcal{F} & \to & \delta_*\delta^! Rpr_1^!\mathcal{F} \\ \uparrow & \circlearrowleft & \downarrow \\ pr_2^*\mathcal{F} & \to & Rpr_1^!\mathcal{F} \end{array}$$

$1 \mapsto \delta_*\delta^* \quad \delta_*\delta^! \mapsto 1$

$$\uparrow$$

$R\mathcal{H}om(\mathcal{F}, \mathcal{F})$

$$H^0(X, R\mathcal{H}om(\mathcal{F},\mathcal{F})) \to H^0(X, R\delta^!\mathcal{H}) = H^0_{X\,\text{diagonal}}(X\times X, \mathcal{H}) \ni id_{\mathcal{F}}\text{の像}$$

$$\text{"}$$

$id_{\mathcal{F}} \in End(\mathcal{F})$

$R\delta^!$

$\searrow \delta^*$

$$\downarrow$$

$H^0(X\times X, \mathcal{H})$

$$\downarrow$$

$H^0(X, \delta^*\mathcal{H})$

$$\delta^*\mathcal{H} \to K_X \qquad \delta^*\mathcal{H} \xleftarrow{\sim} \delta^*(\mathcal{F} \otimes^L D_X \mathcal{F}) = \mathcal{F} \otimes^L D_X \mathcal{F}$$

$$\begin{array}{c} \mathcal{F}\otimes^L D_X\mathcal{F} \\ \| \\ R\mathcal{H}om(\mathcal{F}, K_X) \xrightarrow{\text{evaluation}} K_X \\ \end{array}$$

$\mathcal{F}$: smoothとすると $\simeq$

$$\mathcal{E}nd(\mathcal{F}) \otimes K_X \xrightarrow{\mathrm{Tr}\otimes 1} K_X$$

$$1 \in \mathrm{End}(\mathcal{F}) \simeq H^0_X(X \times X, \mathcal{H}) \to H^0(X, \delta^*(\mathcal{F} \otimes^L D_X(\mathcal{F}))) \xrightarrow{\mathrm{Tr}} H^0(X, K_X) \ni c(\mathcal{F})$$

$$Rf_! \colon Rf^! \to 1.$$

$$\mathrm{Tr} : H^0_c(X, K_X) \to \Lambda$$

X: properならば $H^0_c(X,K_X) = H^0(X, K_X)$

Lefschetz trace formula

X: properとすると, $\mathrm{Tr} : H^0(X, K_X) \to \Lambda$ による

$c(\mathcal{F})$ の像は $\chi(X_{\bar{k}}, \mathcal{F}) = \mathrm{rk}\, \underbrace{Rf_! \mathcal{F}}$

$f : X \to \mathrm{Spec}\, k$

$D_{ctf}(\mathrm{Spec}\, k, \Lambda)$

$$\begin{tikzcd} H^0(X, K_X) \arrow[r, "\mathrm{Tr}"] & \Lambda = \mathbb{Z}/\ell^n \\ CH_0(X) \arrow[u] \arrow[r, "\deg"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[u] \end{tikzcd}$$

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$j : U \to X$ open imm　U smooth $\dim = d$

(X proper)

$\mathcal{F}$: U 上の smooth Λ-sheaf (ℓ.c.c. + flat)

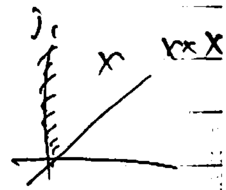
$j_!\mathcal{F}$ $D_{ctf}(X, \Lambda)$ の obj.

$$\begin{array}{ccccc}
\text{として},\ C_c(j_!\mathcal{F}) & \in & H^0_c(U, K_U) = H^{2d}_c(U, \Lambda(d)) & に定義する & \\
\downarrow & & \downarrow \qquad \leftrightarrow CH_0(X, D) & & D = X \setminus U \\
 & & \downarrow & & \\
c(j_!\mathcal{F}) & \in & H^0(X, K_X) \leftrightarrow CH_0(X) & &
\end{array}$$

$$\begin{array}{ccc}
U \times U & \xrightarrow{j_2} & U \times X \\
\downarrow & & \downarrow j_1 \\
X \times U & \to & X \times X
\end{array}$$

$$\mathcal{H} = R\mathcal{H}om_{X \times X}(pr_2^*(j_!\mathcal{F}), Rpr_1^!(j_!\mathcal{F}))$$

$$\simeq j_{1!} Rj_{2*} \underbrace{R\mathcal{H}om_{U \times U}(pr_2^*\mathcal{F}, Rpr_1^!\mathcal{F})}_{\mathcal{H}om(pr_2^*\mathcal{F}, pr_1^*\mathcal{F})(d)[2d]}$$



$$j_!\mathcal{F} \boxtimes^L D_X(j_!\mathcal{F}) = j_{1!} Rj_{2*}(\mathcal{F} \otimes D_U(\mathcal{F}))$$

$$\delta^*(\text{———}) = j_! \mathcal{E}nd(\mathcal{F})(d)[2d]$$

同じ構成により $H^{2d}_c(U, \mathcal{E}nd(\mathcal{F})(d))$ の元

$$\downarrow Tr$$

$H^{2d}_c(U, \Lambda(d))$　"

canonical class の計算

X smooth dim d $\mathcal{F}$ X上の smooth Λ-mod

$X \to X \times X$

$$H^{2d}_X(X\times X, \Lambda(d)) \overset{\text{purity}}{\simeq} H^0(X, \Lambda) \ni 1 \mapsto [X]$$

$$\underbrace{H^0(X, \underbrace{\delta^* \mathcal{H}om(pr_2^*\mathcal{F}, pr_1^*\mathcal{F})}_{=\mathcal{E}nd(\mathcal{F})})}_{=\mathrm{End}(\mathcal{F})} \times H^0_X(X\times X, \Lambda(d)[2d]) \to H^0_X(X\times X, \mathcal{P})$$

$$\mathcal{P} = \mathcal{H}om(pr_2^*\mathcal{F}, pr_1^*\mathcal{F})(d)[2d]$$

$\mathrm{id}_{\mathcal{F}} \in \mathrm{End}(\mathcal{F})$

$\in$ $\delta^* \otimes \delta^! \to \delta^!$

$(\mathrm{id}_{\mathcal{F}}, [X])$ の像 $= J$, $\mathcal{F}$ の class

$\downarrow \delta^*$

$\downarrow (\otimes \delta^*)$

$$\begin{tikzcd}
\mathrm{End}(\mathcal{F}) \times H^{2d}(X, \Lambda(d)) \arrow[r] \arrow[d, "\mathrm{Tr}\otimes 1"'] & H^{2d}(X, \mathcal{E}nd(\mathcal{F})(d)) \arrow[d, "\mathrm{Tr}"] \\
\Lambda \times H^{2d}(X, \Lambda(d)) \arrow[r] & H^{2d}(X, \Lambda(d))
\end{tikzcd}$$

$\mathrm{id} \in \mathrm{End}(\mathcal{F})$, (X, X) 交点積 $= (-1)^d c_d(\Omega^1_{X/k}) \in H^{2d}(X, \Lambda(d))$

$\mathrm{rk}\,\mathcal{F} \in \Lambda$, $C(\mathcal{F}) \in H^{2d}(X, \Lambda(d))$

$$C(\mathcal{F}) = \mathrm{rk}\,\mathcal{F} \cdot (-1)^d c_d(\Omega^1_{X/k})$$

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X proper Tr をとれば

$$\chi(X_{\bar k}, \mathcal{F}) = \mathrm{rk}\,\mathcal{F} \times \chi(X_{\bar k})$$

$$\chi(X_{\bar k}) = \deg (-1)^d c_d(\Omega^1_{X/k})$$

X (proper) smooth $D \subset X$ SNCD $U = X \setminus D$

$\mathcal{F}$: U 上の smooth sheaf D にそっての分岐は tame

$X \supset U$ 上のとおり $D = \bigcup_i D_i$ $K_i = \mathrm{Frac}\, \hat{\mathcal{O}}_{X, \xi_i}$

ξ_i : D_i の生成点

$V \to U$ finite étale $L_i = \Gamma(V \times_U \mathrm{Spec}\, K_i, \mathcal{O})$

有限 étale

K_i-algebra.

V の D にそっての分岐が tamely ramified とは

各 i に対し, L_i が K_i の tamely ramified ext. の

$\Updownarrow$

各 i に対し, L_i の K_i 上の log 分岐が $0+$ 以下

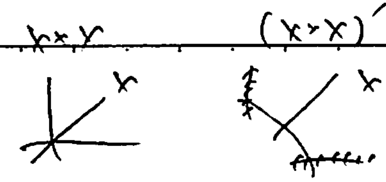
$\Updownarrow$ (⇑ 一般) ⇓ cポイント ヒント!

V の D にそっての log 分岐が $0+$ 以下

Abhyankhar's

+ Zariski-純

SGA1



$(X\times X)' \hookleftarrow X * X = (X\times X)^{\sim}$

open imm. $\quad \cup\, \tilde{j}$

π proper $\downarrow$ log blow-up 各 $D_i\times D_i$ を blow-up. $\quad X\times D, D\times X$ の proper transform を除いた. $\quad U\times U$

$X\times X$

$X\times X \quad \mathcal{H} = j_{1!} Rj_{2*} \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^* \mathcal{F})(d)[2d]$

$(X\times X)' \quad \mathcal{H}' = k_{1!} Rk_{2*} \tilde{j}_* \mathcal{H}om(-,-)(d)[2d]$

• coh をとると同じ
• $H^0_x(-,-)$ を使って構成ができる.

blow-up で分離された.

$(X\times X)' \xleftarrow{k_1} (X\times X)' \setminus (X\times D \text{ の p.t. だけ}) \xleftarrow{k_2} X * X$

Lem (Faltings-Pink)

標準射 $\mathcal{H} \to R\pi_* \mathcal{H}'$ は同形.

($\pi^* \mathcal{H} \to \mathcal{H}'$ や $R\pi^! \mathcal{H} \to \mathcal{H}'$ は同形でない.)

$\mathcal{H}$ と $R\pi_* \mathcal{H}'$ の差があるとすれば $D_i\times D_i$ 上だけ

$\mathcal{H}|_{D_i\times D_i} = 0$ で したがって adjunction で can. map が定義される.

$R\pi_* \mathcal{H}'|_{D_i\times D_i} = 0$ をいえばよい.

proper base change で 各点 の fiber 上 の coh $= 0$ をいえばよい.

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$X \times X$ の fiber $\subset$ $(X\times X)'$ の fiber

torus — $\mathbb{P}^1$ のいくつかの積

$$H^*(\mathbb{P}^1_{\bar k}, \overset{j_{0!}}{\overset{\shortparallel}{j_{\infty !} R j_{0*}}} \mathcal{L}_\chi) \qquad \mathcal{L}_\chi : \text{Kummer 層}$$

$t^n = T \qquad \mathbb{G}_m = \mathrm{Spec}\, k[T^{\pm 1}]$

$p \nmid n$

$n \neq 1 \quad H^*_c(\mathbb{G}_{m,\bar k}, \mathcal{L}_\chi) = 0$

cポート

$n = 0 \quad H^*(\mathbb{P}^1_{\bar k}, j_{\infty !} R j_{0*} \Lambda) = 0$

<u>Cor</u>

$$H^0_x(X \times X, \mathcal{H}) \xrightarrow{\sim} H^0_{\pi^{-1}(x)}((X\times X)', \mathcal{H}') \text{ は同形}$$

$H^0_x(X \times X, \mathcal{H}) = \mathrm{End}_X(j_! \mathcal{F})$

$H^0_x(X \times X, \mathcal{H}) = \mathrm{Hom}(\delta_x \Delta_x, \mathcal{H})$

$H^0_{\pi^{-1}(x)}((X\times X)', \mathcal{H}') = \mathrm{Hom}(\pi^* \delta_x \Delta_x, \mathcal{H}') = \mathrm{Hom}(\delta_x \Delta_x, R\pi_* \mathcal{H}')$

$$H^0_{\pi^{-1}(x)}((X\times X)', \mathcal{H}') \xrightarrow{\ \wr\!\!\text{?}\ } H^0_U(U \times U, \mathcal{H}|_{U\times U}) = \mathrm{End}_U(\mathcal{F})$$

$$\mathrm{Hom}(\delta_x \Delta_x, \mathcal{H}) \xrightarrow{\ \wr\ } H^0_U(U \times U, \mathcal{H}|_{U\times U})$$

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$$\begin{tikzcd} & & X\times X \\ & & (X\times X)^\wedge \arrow[u, phantom, "\wedge"] \arrow[d, "\pi"] \\ X \arrow[urr, bend left, "\hat\delta"] \arrow[urr, ""] \arrow[rr, "\delta"'] & & X\times X \end{tikzcd}$$

$C(\mathcal{F})$のもとで

$$\underline{H^0_x((X\times X)', \mathcal{H}')} \to H^0_{\pi^{-1}(x)}((X\times X)', \mathcal{H}')$$

$$\shortparallel \qquad\qquad\qquad \cong$$

$$H^0_x(X\times X, \underline{\mathcal{H}'|_{X\times X}}) \qquad H^0_x(X\times X, \mathcal{H})$$

$$\shortparallel$$

$$\boxed{\hat{j}_* \mathcal{H}om(-,-)}(d)[2d] \qquad\qquad H^0_U(U\times U, \mathcal{H}) \ni \mathrm{id}$$

$$R\hat\delta^! \hat{j}_* \mathcal{H}om(-,-) \xrightarrow{\sim} j_* \mathcal{E}nd_U$$

$U[x]$

来週

(示しまうけれたので)

$$\mathrm{id}_{\mathcal{F}} \in H^0(X, j_* \mathcal{E}nd_U(\mathcal{F})) = \mathcal{E}nd_U(\mathcal{F})$$