9/11 特性サイクルと特異台

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Beilinson : ... are holonomic Arxiv

T. Saito : Char cycle & sing supp ... in preparation

Notes : webpage Talks at Berlin

taken by Lars Kindler

・ 〃 講義

0.1 Introduction

k : 体 (標数 $p>0$, 完全 (代数閉))

X : smooth $/k$ local に $\mathbb{A}^n$ 上 étale

T^*X : cotangent b'dle $\mathrm{Spec}_X(\Omega^{1\vee}_{X/k})$

i.e. covariant vector b'dle $\mathbb{V}(\Omega^1_{X/k})$

$\Omega^1_{X/k}$: sheaf of diff. 1-form

local には $X \to \mathbb{A}^n = \mathrm{Spec}\, k[T_1, \cdots, T_n]$ étale

$T^*\mathbb{A}^n = \mathbb{A}^n$ 上基底 $dT_1, \cdots, dT_n$ のベクトル束

$\hookrightarrow X \times_{\mathbb{A}^n} T^*\mathbb{A}^n \Rightarrow T^*X$

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Λ：有限局所環　剰余体の標数 $\ell \neq \mathrm{char}\, k = p$
（体）

$\mathcal{F}$：X 上の Λ-加群の層の constructible complex
(tor-dimension finite)

constructible：

- X の locally closed subset による分割 $X = \coprod_i X_i$
  $\forall q$ $\underline{\mathcal{H}^q(\mathcal{F})}|_{X_i}$: $\underline{\text{locally constant}}$ constructible
  cohomology sheaf
  étale local に有限 Λ-加群にともなう定数層
- 有限個を除いて $\mathcal{H}^q(\mathcal{F}) = 0$

例　$X = \mathbb{A}^2 = \mathrm{Spec}\, k[x, y]$　T^*X　dx, dy
$\cup$
$U = \mathbb{G}_m \times \mathbb{A}^1 = \mathrm{Spec}\, k[x^{\pm 1}, y] \overset{j}{\hookrightarrow} X$ open imm.

$\zeta_p \in \Lambda$　1の原始 p 乗根

$V \to U$　p 次巡回被覆　$p = \mathrm{char}\, k > 0$
$t^p - t = \dfrac{y}{x^p}$

$\mathrm{Gal}(V/U) = \mathbb{F}_p \hookrightarrow \Lambda^\times$ に対応する rk 1 の
局所定数層 $\mathcal{G}$ on U

V 上の descent datum
降下材

$\mathcal{F} = j_! \mathcal{G}$ on X constructible.

$D = X - U$ $\quad \mathcal{F}|_U = \mathcal{G}$: l.c.c.

$\mathcal{F}|_D = 0$: 〃

0.2 Singular support

$C = SS(\mathcal{F}) \subset T^*X$

closed conical subset

$\mathbb{G}_m$ の作用で安定.

代数的には

$T^*X = \mathrm{Spec}\, S^\bullet \Omega^{1\vee}_{X/k}$ ← graded ring

$\cup$

$\exists\, I$: graded ideal

s.t. $C = \mathrm{Spec}\left(S^\bullet \Omega^{1\vee}_{X/k} / I\right)$

射影化

$\mathbb{P}(C) \subset \mathbb{P}(T^*X) = \mathrm{Proj}_X\, S^\bullet \Omega^{1\vee}_{X/k}$

$\mathrm{Proj}_X (S^\bullet \Omega^{1\vee}_{X/k} / I) = \mathbb{P}(C)$ $\qquad \mathbb{P}(T^*X) = (T^*X - T^*_X X)/\mathbb{G}_m$

0-section ($T^*_X X$)

$Y \subset X$ smooth subscheme $I \subset \mathcal{O}_X$

$T^*_Y X = \mathrm{Spec}_Y\, S^\bullet (I/I^2)^\vee$ conormal bd'le

$\cap$

$Y \times_X T^*X$

$T^*_X X$: $Y = X$

$C = \bigcup_a C_a$ irred components

$\dim C_a = \dim X \qquad \dim T^*X = 2\dim X$

Cは$\mathcal{F}$の性質を統制する.

$\left.\begin{array}{l} f: X \to Y \\ h: W \to X \end{array}\right\}$ 良い射について

Cに関する条件 ⇒ $\mathcal{F}$に関する条件

Y, W : 良い smooth

Cに関する条件 … 横断性(など)

non-characteristic

$\mathcal{F}$に関する条件

… f-local acyclicity $\overset{先}{\Rightarrow}$ $SS(\mathcal{F})$の定義

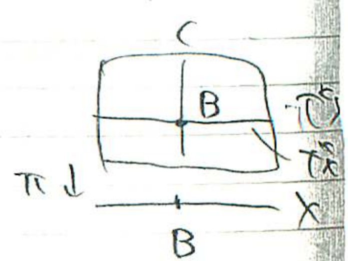
h-transversality 後

cf. Kashiwara-Schapira

Sheaves on Manifolds

$B = C \cap \underset{\overset{\parallel}{X}}{T^*_X X} \subset X$: C_a base

・$C \subset \pi^{-1}(B) \quad \pi : T^*X \to X$



・$B = \mathrm{Supp}(\mathcal{F})$

$\parallel$

$\mathcal{F}|_U = 0$となる最大の開集合の補集合

$\parallel$

$\overline{\{x \in X \mid \mathcal{F}|_x \neq 0\}}$

$SS(\mathcal{F})$ の構成　X 上 local　$X \underset{\text{immersion}}{\overset{i}{\hookrightarrow}} \mathbb{P} = \mathbb{P}^n$ proj. space

Radon 変換

$\mathbb{P}^\vee = \{H \mid H \subset \mathbb{P} \text{ hyperplane}\}$ dual proj. space

$= \mathbb{P}(E) = \mathrm{Proj}_k S^\cdot E^\vee$

$\mathbb{P} = \mathbb{P}(E^\vee)$

$\mathbb{P}\times\mathbb{P}^\vee \supset Q = \{(x, H) \in \mathbb{P}\times\mathbb{P}^\vee \mid x \in H\}$ universal family of hyper plane

$\downarrow p^\vee \quad \supset H$

$\mathbb{P}^\vee \supset \{H\}$

$$\begin{array}{ccccccc} X & \xleftarrow{p} & X\times\mathbb{P}^\vee & \supset & X\times_{\mathbb{P}} Q & \supset & X\cap H \\ \mathcal{F} & & \cap & & \cap & & \cap \\ & & \mathbb{P}\times\mathbb{P}^\vee & \supset & Q & \supset & H \\ & & & & \downarrow p^\vee & & \downarrow \\ & & & & \mathbb{P}^\vee & \supset & \{H\} \end{array}$$

univ. family of hyper plane sections

$R(\mathcal{F}) = Rp^\vee_* p^* \mathcal{F}$　Radon 変換

$E \subset X\times_{\mathbb{P}} Q$: $p^\vee : X\times_{\mathbb{P}} Q \to \mathbb{P}^\vee$ が $p^*\mathcal{F}$ に関して univ. loc. acyclic となる最大の開集合の補集合

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$$X \times_{\mathbb{P}} Q = \mathbb{P}(X \times_{\mathbb{P}} T^*\mathbb{P}) \leftarrow X\text{上の proj. bdle}$$

($X \times_{\mathbb{P}} T^*\mathbb{P}$: X上の vect bdle)

開集合 $\cup$ $\quad$ $\cup$

$E = \mathbb{P}(\hat{C}) \quad \exists \hat{C} \subset X \times_{\mathbb{P}} T^*\mathbb{P}$

$\parallel$

$E_{p^\vee}(p^*\mathcal{F})$

closed conical subset.
uniqueにはきまらない.

<u>例</u> $\mathbb{P}(\hat{C}) = \mathbb{P}(\hat{C} \cup (\text{0-section}))$

<u>定理</u> $C = SS(\mathcal{F})$ とする. $\hat{C}$ を

$$\begin{array}{ccc} X \times_{\mathbb{P}} T^*\mathbb{P} & \to & T^*X \\ \cup & & \cup \\ \hat{C} & \to & C \end{array}$$

Cartesian で定めると
$E = \mathbb{P}(\hat{C})$.

$\rightsquigarrow$ 定理 + $B = \mathrm{Supp}\,\mathcal{F} \Rightarrow C$ は uniqueに定まる

<u>例</u> 0. $\mathcal{F}$: loc. const. ($\Leftrightarrow$ $\forall q$ $\mathcal{H}^q(\mathcal{F})$: loc. const.)

X: conn. $\mathcal{F} \neq 0 \Rightarrow SS(\mathcal{F}) = T^*_X X$

$\mathcal{F} = 0 \Rightarrow SS(\mathcal{F}) = \phi$

1. $U \subset X \supset D$ $\quad$ D: div. with simple normal crossing

$\parallel$

$X - D$

$\mathcal{G}$: U 上の loc. const. sheaf

Dに沿って tamely ramified

$D = \bigcup_i D_i$ $\quad$ K_i: D_i の gen. pt ξ_i での局所体

$K_i = \mathrm{Frac}\, \hat{\mathcal{O}}_{X, \xi_i}$

$\bar{\eta}_i = \mathrm{Spec}(\bar{K}_i)$

$\mathcal{H}^q(\mathcal{F})_{\bar{\eta}_i} \curvearrowleft \mathrm{Gal}(\bar{K}_i/K_i)$ tamely ramified.

$\mathcal{F} = j_! \mathcal{G}$

X : conn. $\mathcal{G} \neq 0$

$\Rightarrow SS(\mathcal{F}) = \bigcup_{I \subset \{1, \dots, m\}} T^*_{D_I} X \quad D = \bigcup_{i=1}^{m} D_i$

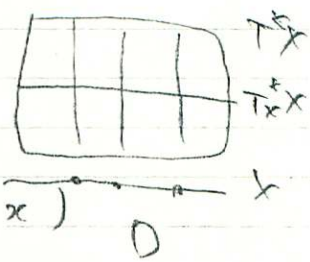
$D_I = \bigcap_{i \in I} D_i \quad T^*_{D_I} X \subset T^* X$ conormal bdle

2. $\dim X = 1$

$U \subset X$: $\mathcal{F}|_U$ loc. const. となる最大の開集合

$D = X - U$ quasi-finite

X : conn. $\mathcal{F}|_U \neq 0$ なら



$SS(\mathcal{F}) = T^*_X X \cup \bigcup_{x \in D} (T^* X \times_X x)$ fiber

3. I, II の $X = \mathbb{A}^2 \quad V \to U : t^p - t = \frac{y}{x^p}$

$p \neq 2 \quad SS(\mathcal{F}) = T^*_X X \cup \langle dy/D \rangle$ ← 標数p特有 non-Lagrangean

$D = (x = 0) \subset \mathbb{A}^2, \ T^*X : dx, dy$

$D \times_X T^* X = \underset{\underset{\langle dx/D \rangle}{\parallel}}{T^*_D X} \oplus \underset{\underset{T^*D}{\wr\wr}}{\langle dy/D \rangle}$

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0.3 Characteristic Cycle (k: perfect)

$SS(\mathcal{F}) = C = \bigcup_a C_a \subset T^*X$ 整係数

$\mathrm{Char}(\mathcal{F}) = \sum_a m_a [C_a]$ $m_a \in \begin{cases} \mathbb{Z}[\frac{1}{p}] & p = \mathrm{char}(k) \\ \mathbb{Z} & p = 0. \end{cases}$

1. 特徴づけ Milnor 公式

$f: X \to Y = \mathbb{A}^1$ isolated characteristic point

(上の étale scheme)

定数層の場合: Deligne SGA7 XVI

存在証明:

vanishing cycle の total dimension の stability

$\Uparrow$ ($\Leftarrow$ Milnor 公式)

Deligne-Laumon の Swan conductor の semi-continuity の弱い形の高次元化

2. char cycle の性質

・ひき戻しとの両立性「$\mathrm{char}\, h^*\mathcal{F} = h^!\mathrm{char}\,\mathcal{F}$」

・指数公式: X: projective なら

$\chi(X_{\bar{k}}, \mathcal{F}) \left(= \sum_q (-1)^q \dim H^q(X_{\bar{k}}, \mathcal{F})\right)$ (Λ: 体)

$= (\mathrm{char}(\mathcal{F}), T^*_X X)_{T^*X}$

$\sum m_a [C_a]$, $\dim X$