On parameter F-jumping numbers

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This talk is based on the following papers:

- C. Huneke, M. Mustaţă, S. Takagi and K. Watanabe, F-thresholds, tight closure, integral closure, and multiplicity bounds.
- C. Huneke, S. Takagi and K. Watanabe, Multiplicity bounds in graded rings.
- M. Mustaţă, S. Takagi and K. Watanabe, F-thresholds and Bernstein-Sato polynomials.

Notation:

R: Noetherian domain of char. $p > 0$, $\dim R = d$

$\mathfrak{a}, J \subseteq R$: ideals s.t. $\mathfrak{a} \subseteq \sqrt{J}$

$\overline{J}$ denotes the integral closure of J

J^* denotes the tight closure of J

$$(x \in J^* \iff {}^{\exists}c \in R \setminus \{0\}, {}^{\forall}q = p^e \gg 0, cx^q \in J^{[q]})$$

<u>F-thresholds:</u>

For $q = p^e$, $J^{[q]} := (x^q \mid x \in J)$

$$\nu_{\mathfrak{a}}^{J}(q) := \max\{r \in \mathbb{N} | \mathfrak{a}^r \not\subseteq J^{[q]}\}$$

(if $\mathfrak{a} \subseteq J^{[q]}$, then we put $\nu_{\mathfrak{a}}^{J}(q) = 0$)

$$\mathrm{c}_{+}^{J}(\mathfrak{a}) := \limsup_{q \to \infty} \frac{\nu_{\mathfrak{a}}^{J}(q)}{q}, \quad \mathrm{c}_{-}^{J}(\mathfrak{a}) := \liminf_{q \to \infty} \frac{\nu_{\mathfrak{a}}^{J}(q)}{q}$$

If $\mathrm{c}_{+}^{J}(\mathfrak{a}) = \mathrm{c}_{-}^{J}(\mathfrak{a})$, we denote this value by $\mathrm{c}^{J}(\mathfrak{a})$.

$\boxed{\text{Ex.}}$ $R = \mathbb{F}_p[X_1, \ldots, X_d]$ and $\mathfrak{m} = (X_1, \ldots, X_d)$

$$(X_1, \ldots, X_d)^{d(q-1)+1} \subseteq (X_1^q, \ldots, X_d^q)$$

$$(X_1 \cdots X_d)^{q-1} \notin (X_1^q, \ldots, X_d^q)$$

$$\Longrightarrow \nu_{\mathfrak{m}}^{\mathfrak{m}}(q) = d(q-1) \text{ and } c^{\mathfrak{m}}(\mathfrak{m}) = d$$

Basic Properties

(1) $c_{\pm}^{J}(\overline{\mathfrak{a}}) = c_{\pm}^{J}(\mathfrak{a})$

(2) $c_{\pm}^{\overline{J}}(\mathfrak{a}) \lneqq c_{\pm}^{J}(\mathfrak{a})$ in general

(3) If R is local and J is generated by a full s.o.p.

$\Longrightarrow c^{J}(J) = d$

$\boxed{\text{Q.}}$ When does $c^{J}(\mathfrak{a})$ exist?

A. $\mathrm{c}^J(\mathfrak{a})$ exists

- if $\mathfrak{a}$ is a principal ideal, or
- if R_P is F-pure for all primes P not containing $\mathfrak{a}$

Prop. Suppose R is local and $J \subseteq I \subseteq R$ are ideals s.t. J is generated by a full s.o.p.

(1) If R is excellent and analytically irreducible,

$$I \subseteq J^* \Longleftrightarrow \mathrm{c}_+^I(J) = d$$

(2) If R is formally equidimensional,

$$I \subseteq \overline{J} \Longleftrightarrow \mathrm{c}_+^J(I) = d$$

Suppose $(R, \mathfrak{m})$ is an F-finite local domain or a graded domain $R = \bigoplus_{n \geq 0} R_n$ with R_0 a field and $\mathfrak{m} = \bigoplus_{n \geq 1} R_n$

Parameter F-jumping numbers:

$F^e : H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R)$ e-times iterated Frobenius

For $t \geq 0$,

$0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} \subseteq H^d_{\mathfrak{m}}(R)$ R-submodule defined as follows:

$$z \in 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} \iff {}^{\exists}c \in R \setminus \{0\}, {}^{\forall}q = p^e \gg 0, c\mathfrak{a}^{\lceil tq \rceil} F^e(z) = 0$$

$H^d_{\mathfrak{m}}(R) \times \omega_R \longrightarrow H^d_{\mathfrak{m}}(R)$ duality pairring

$\tau(\omega_R, \mathfrak{a}^t) := \mathrm{Ann}_{\omega_R} 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} \subseteq \omega_R$

$$\mathrm{fjn}^J(\omega_R, \mathfrak{a}) := \min\{t \geq 0 \mid \tau(\omega_R, \mathfrak{a}^t) \subseteq J\omega_R\}$$

Basic Properties

(1) $\tau(\omega_R, \overline{\mathfrak{a}}^t) = \tau(\omega_R, \mathfrak{a}^t)$

In particular, $\mathrm{fjn}^J(\omega_R, \overline{\mathfrak{a}}) = \mathrm{fjn}^J(\omega_R, \mathfrak{a})$

(2) $\mathrm{fjn}^{\overline{J}}(\omega_R, \mathfrak{a}) \lneqq \mathrm{fjn}^J(\omega_R, \mathfrak{a})$ in general

(3) If J is generated by a full s.o.p.

$\Longrightarrow \mathrm{fjn}^J(\omega_R, J) = d$

(4) R is F-rational (i.e., $I^* = I$ for all par. ideals I)

iff R is CM and $\tau(\omega_R, R) = \omega_R$

(5) Fix $0 \neq {}^\forall c \in \tau_{\mathrm{par}}(R) := \bigcap_{J \subseteq R:\text{par. ideal}} J : J^*$

$(F^e)^\vee : F^e_*\omega_R \longrightarrow \omega_R$ dual of $F^e : R \longrightarrow F^e_* R$

$$\tau(\omega_R, \mathfrak{a}^t) = \sum_{e \geq 1} (F^e)^\vee (F^e_*(c\mathfrak{a}^{\lceil tp^e \rceil}\omega_R))$$

(6) (cf. Hyry–Villamayor)

Suppose $(R, \mathfrak{m})$ is a CM normal local ring and F-rational in the punctured spectrum

$r(R) := \min\{r \geq 0 \mid \mathfrak{m}^r \omega_R \subseteq \tau(\omega_R, R)\}$

$\overline{I^{n+d+r(R)-1}} \subseteq I^n$ for ${}^\forall I \subseteq R$ and ${}^\forall n \geq 0$

Multiplier submodules:

S: normal domain ess. finite type over a field of char. 0

$\mathfrak{a}, J \subseteq S$: ideals s.t. $\mathfrak{a} \subseteq \sqrt{J}$, $\quad t \geq 0$

$\pi : Y \to X := \operatorname{Spec} S$ log resolution of $\mathfrak{a}$

s.t. $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$

$$\mathcal{J}(\omega_S, \mathfrak{a}^t) := \pi_*(\omega_Y \otimes \mathcal{O}_Y(-\lfloor tF \rfloor)) \subseteq \omega_S$$

$$\lambda^J(\omega_S, \mathfrak{a}) := \min\{t \geq 0 \mid \mathcal{J}(\omega_S, \mathfrak{a}^t) \subseteq J\omega_S\}$$

Prop. (Hara–Yoshida) Suppose $(S_p, \mathfrak{a}_p, J_p)$ is a reduction of $(S, \mathfrak{a}, J)$ to char. $p \gg 0$

(1) $\tau(\omega_{S_p}, \mathfrak{a}_p^t) \subseteq \mathcal{J}(\omega_S, \mathfrak{a}^t)_p$ for $^\forall t \geq 0$. In particular,

$$\mathrm{fjn}^{J_p}(\omega_{S_p}, \mathfrak{a}_p) \leq \lambda^J(\omega_S, \mathfrak{a})$$

(2) If $p \gg 0$ (how large p has to be depends on t),

$$\tau(\omega_{S_p}, \mathfrak{a}_p^t) \subseteq \mathcal{J}(\omega_S, \mathfrak{a}^t)_p$$

A comparison of $\mathrm{c}^J(\mathfrak{a})$ and $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$:

Prop. (1) If R is regular, then $\mathrm{c}^J(\mathfrak{a}) = \mathrm{fjn}^J(\omega_R, \mathfrak{a})$
(2) In general, $\mathrm{c}_+^J(\mathfrak{a}) \gneqq \mathrm{fjn}^J(\omega_R, \mathfrak{a})$

Ex. $R = k[X, Y, Z]/(XY - Z^2)$, $\mathfrak{m} = (x, y, z)$
$\Longrightarrow \mathrm{c}^{\mathfrak{m}}(\mathfrak{m}) = 3/2$ and $\mathrm{fjn}^{\mathfrak{m}}(\omega_R, \mathfrak{m}) = 1$

Thm. If R_P is F-rational for all primes P not containing $\mathfrak{a}$ and if J is generated by a full s.o.p.,

$$\mathrm{c}_+^J(\mathfrak{a}) = \mathrm{fjn}^J(\omega_R, \mathfrak{a})$$

Q. What if R is not F-rational away from $V(\mathfrak{a})$?

Multiplicity bounds:

Thm. (de Fernex–Ein–Mustaţă)
$(S, \mathfrak{n})$: d-dim. regular local ring ess. finite type over a field of char. 0

$$e(\mathfrak{a}) \geq \left(\frac{d}{\lambda^{\mathfrak{n}}(\omega_S, \mathfrak{n})} \right)^d \text{ for all } \mathfrak{n}\text{-primary ideals } \mathfrak{a}$$

Q. What if S is singular?

Ex. $S = \mathbb{C}[X, Y, Z]/XY - Z^2$ and $\mathfrak{n} = (x, y, z)$.

$$e(\mathfrak{n}) = 2 \lneqq 4 = \left(\frac{2}{1} \right)^2 = \left(\frac{d}{\mathrm{fjn}^J(\omega_S, \mathfrak{n})} \right)^d$$

$\boxed{\text{Conj.}}$ $(R, \mathfrak{m})$: d-dim. F-finite local domain
$\mathfrak{a}, J \subseteq R$: $\mathfrak{m}$-primary ideals
If J is generated by a full s.o.p.,

$$e(\mathfrak{a}) \geq \left(\frac{d}{\mathrm{fjn}^J(\omega_R, \mathfrak{a})} \right)^d e(J)$$

$\boxed{\text{Rem.}}$

(1) Conj. is true if $\dim R = 1$ or if R is regular and $J = \mathfrak{m}$

(2) We may assume $\mathfrak{a}$ is generated by a full s.o.p.

(3) If J is not generated by a full s.o.p, Conj. can fail

Ex. $R = k[X_1, \ldots, X_d]$, $\mathfrak{a} = \mathfrak{m}$ and $J = \mathfrak{m}^l$ $(l \geq 2)$

$$e(\mathfrak{a}) = 1 \lneqq \left(\frac{dl}{d+l-1}\right)^d = \left(\frac{d}{\mathrm{fjn}^J(\omega_R, \mathfrak{a})}\right)^d e(J)$$

Main Thm. $R = \bigoplus_{n \geq 0} R_n$: graded domain with R_0 a field of char. $p > 0$
$\mathfrak{a}, J \subseteq R$: ideals generated by full homog. s.o.p.

$$e(\mathfrak{a}) \geq \left(\frac{d}{\mathrm{fjn}^J(\omega_R, \mathfrak{a})}\right)^d e(J)$$

$\boxed{\text{Cor.}}$ $R = \bigoplus_{n \geq 0} R_n$: normal graded domain with R_0 a field of char. 0

$\mathfrak{a}, J \subseteq R$: ideals generated by full homog. s.o.p.

$$e(\mathfrak{a}) \geq \left(\frac{d}{\mathrm{fjn}^J(\omega_R, \mathfrak{a})} \right)^d e(J)$$

Outline of the proof of Main thm.

(a) $\mathfrak{a} = (x_1, \ldots, x_d)$ and $J = (f_1, \ldots, f_d)$

$$\frac{e(\mathfrak{a})}{e(J)} = \frac{\deg x_1 \cdots \deg x_d}{\deg f_1 \cdots \deg f_d}$$

(b) $\mathrm{fjn}^{J}(\omega_R, \mathfrak{a}) \geq t$

$\Longrightarrow {}^{\exists}c \in R \setminus \{0\}$ homog. element

s.t. $c\mathfrak{a}^{\lceil tq \rceil} \subseteq J^{[q]}$ for all $q = p^e \gg 0$

(c) $R^{+\mathrm{gr}}$ is a big CM R-algebra (Hochster–Huneke)

We may assume R is CM

(d) Compare the Koszul complex of $f_1^q, \ldots, f_d^q$ and the Taylor resolution of a monomial ideal in x_i's

$\Longrightarrow$ Can compare the degrees of x_i's and f_i's.