Abstracts

Lifshitz tails for Schrödinger operators with non-sign definite random potentials

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(joint work with Frédéric Klopp)

We discuss recent results on Lifshitz tails for alloy-type (or generalized alloy-type) Schrödinger operators with the local potential which does not have fixed sign ([1, 2]). Usual proof of Lifshitz tail relies on the monotonicity of the random perturbation with respect to random variables, and thus we cannot apply these method directly to show Lifshitz singularities for our model.

Here we consider

$$H_\omega = -\triangle + V_p + V_\omega \quad \text{on } L^2(\mathbb{R}^d)$$

where V_p is a $\mathbb{Z}^d$-periodic background potential, and

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma v(x-\gamma).$$

Here $\{\omega_\gamma\}$ are i.i.d. random variables with the common distribution μ, and $v \in C_c^0(\Lambda_1(0))$, where $\Lambda_1(0)$ is the unit cube with the center at the origin. We suppose $\operatorname{Supp}\mu \subset [a,b]$ with $\{a,b\} \subset \operatorname{Supp}\mu$, and V_p and v are symmetric with respect to reflections about $\{x \mid x_j = 0\}$, $j = 1, \dots, d$.

Let H_λ^N be the operator $-\triangle + V_p + \lambda v$ on $L^2(\Lambda_1(0))$ with Neumann boundary conditions. We denote $E(\lambda) = \inf \sigma(H_\lambda^N)$. We note $E(\lambda)$ is a concave function in λ, and hence $E_- := \min\{E(\lambda) \mid \lambda \in [a,b]\}$ is attained either at $\lambda = a$ or b.

Theorem 1 ([1]) If $E(a) \neq E(b)$ then the Lifshitz tail holds at the bottom of the spectrum, i.e., $\inf \sigma(H_\omega) = E_-$ almost surely, and

$$\limsup_{E\to E_-} \frac{\log|\log N(E)|}{\log(E-E_-)} \le -\frac{d}{2},$$

where $N(E)$ is the integrated density of states for H_ω.

The main step of the proof employs a simple operator inequality:

$$H_\omega \ge c\big[-\triangle + V_p + \sum_{\gamma\in\mathbb{Z}^d} (\omega_\gamma - a)\big] \quad \text{on } L^2(\mathbb{R}^d)$$

with some $c > 0$, where we suppose $E_- = E(a)$. This in turn follows from another operator inequality:

$$H_\lambda^N \ge c\big[-\triangle + V_p + (\lambda - a)\big]^N \quad \text{on } L^2(\Lambda_1(0)),$$

where both sides are Neumann operators. This (simple but rather surprising) argument relies on the fact:

$$H^1(\mathbb{R}^d) \subset \bigoplus_{\gamma\in\mathbb{Z}^d} H^1(\Lambda_1(\gamma))$$

and that the form domain of the Neumann operator on $\Omega \subset \mathbb{R}^d$ is $H^1(\Omega)$.

If $E(a) = E(b)$ then we have the following somewhat weaker result:

Theorem 2 ([2]) If $E(a) = E(b)$ and if μ is not Bernoulli, then the Lifshitz tail holds at the bottom of the spectrum, i.e., $\inf \sigma(H_\omega) = E_-$ almost surely, and

$$\limsup_{E \to E_-} \frac{\log |\log N(E)|}{\log(E - E_-)} \leq -\frac{1}{2},$$

where $N(E)$ is the integrated density of states for H_ω.

In this case, the above comparison theorem does not hold, and we need to use completely different method. In particular, we cannot use the argument involving the Temple inequality. We use, instead:

- Neumann decomposition to long pseudo 1D domains.
- Poincaré type inequality for long pseudo 1D domains with periodic background potential.
- The positivity of the Dirichlet-to-Neumann operator for positive Schrödinger operators on small domains.

Combining these, we can obtain the necessary lower bound of lowest eigenvalues for H_ω restricted to large boxes to show the Lifshitz singularities.

Note that in [2] we consider Schrödinger operators with generalized alloy-type random potentials, which takes finitely many forms randomly at each $\gamma \in \mathbb{Z}^d$. We then combine the result with concavity argument to show Theorem 2. Our general result applies also to some random displacement models discussed in a talk by Günter Stolz.

References

[1] Klopp, F., Nakamura, S.: Spectral extrema and Lifshitz tails for non monotonous alloy type models. Commun. Math. Phys. **287**, 1133-1143 (2009).

[2] Klopp, F., Nakamura, S.: Lifshitz tails for generalized alloy type random Schrödinger operators. Preprint, March 2009 (`http://arxiv.org/abs/0903.2105`)

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