Remarks on scattering theory for Schrödinger operators on scattering manifolds

Kenichi Ito (Tsukuba U) and Shu Nakamura (U Tokyo)

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Contents:

1. Schrödinger operators on scattering manifolds (Model)

2. Spectral propertirs of the operator

3. Construction of the scattering theory

4. Microlocal properties of the scattering matrix
   (Melrose-Zworski theorem)

5. Idea of proof (1):
   The classical mechanical scattering for conic manifolds

6. Idea of proof (2):
   Egorov type estimates in the *scattering symbol class*

1. Schrödinger operators on scattering manifolds (Model)

Suppose $M = M_c \cup M_\infty$, where M_c is relatively compact, $M_\infty \cong (0,\infty)\times\partial M$, ∂M: a closed manifold. We consider a formally self-adjoint operator on M such that P is expressed as

$$P = -\frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G\begin{pmatrix} a_1 & a_2 \\ {}^t a_2 & a_3 \end{pmatrix}\Big(\partial_r, \partial_\theta/r\Big) + V$$

on M_∞. We consider P as an operator on $\mathcal{H} = L^2(M, Gdx)$, where $a_k, V \in C^\infty(M)$ such that for any ℓ, α,

$$|\partial_r^\ell \partial_\theta^\alpha (a_1 - 1)| \le C r^{-1-\mu-\ell}, \quad |\partial_r^\ell \partial_\theta^\alpha a_2| \le C r^{-\mu-\ell},$$

$$|\partial_r^\ell \partial_\theta^\alpha (a_3 - \hat{h})| \le C r^{-\mu-\ell}, \quad |\partial_r^\ell \partial_\theta^\alpha V| \le C r^{-1-\mu-\ell}.$$

Here $G(r,\theta) = r^{n-1}H(\theta)$ with $H(\theta)$: some density on ∂M, $\hat{h}$ is a positive 2-tensor on ∂M. We suppose $\mu > 0$.

Example: $M = \mathbb{R}^n$, $H = -\triangle + V$. In this case, we set $\partial M = S^{n-1}$, $\hat{h}$ is the standard metric on S^{n-1}, $a_1 = 1$, $a_2 = 0$, $a_3 = \hat{h}$, and V satisfies the short-range condition of the usual scattering theory. Even if $M = \mathbb{R}^n$, our assumption is weaker than the usual short-range condition (with respect to a_2, a_3).

Remark: Our model is somewhat more general than the *scattering metric* in the sense of Melrose. In his setting, $\mu = 1$ and all coefficients (a_1, a_2, a_3, V) have asymptotic expansions in r^{-1} as $r \to \infty$.

2. Spectral propertirs of the operator

(cf. e.g. Froese-Hislop-Perry 1989, etc.)

Theorem 0: (1) P is essentially self-adjoint on $C_0^\infty(M)$.
(2) $\sigma_{ess}(P) = [0, \infty)$ and $\sigma_d(P) \subset (-\infty, 0]$ is bounded.
(3) $\sigma_{pp}(P)$ is discrete except for 0, $\sigma_{ac}(P) = [0, \infty)$,
and $\sigma_{sc}(P) = \emptyset$.

Proof: Use the Weyl sequence, the Mourre estimate. $\square$

3. Construction of the scattering theory

Reference system: We set

$$M_f = \mathbb{R} \times \partial M, \quad \mathcal{H}_f = L^2(M_f, H(\theta)drd\theta), \quad P_f = -\frac{1}{2}\frac{\partial^2}{\partial r^2}$$

as the "free" system. We let $j(r) \in C^\infty(\mathbb{R})$ such that $j(r) = \begin{cases} 1, & (r \geq 1) \\ 0, & (r \leq 0) \end{cases}$. We set $\mathcal{J}$: $\mathcal{H}_f \to \mathcal{H}$ by

$$\mathcal{J}\phi(r,\theta) = r^{-(n-1)/2} j(r)\phi(r,\theta) \qquad \text{if } (r,\theta) \in M_\infty$$

for $\phi \in \mathcal{H}_f$. Then we define the wave operators by

$$W_\pm = W_\pm(P, P_f, \mathcal{J}) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itP}\mathcal{J}e^{-itP_f}.$$

We can easily show $W_\pm$ exist, but it is *not* isometry.

We write

$\mathcal{F}$: Fourier transform in r,

We decompose $\mathcal{H}_0 = \mathcal{H}_{0,+} \oplus \mathcal{H}_{0,-}$ where

$$\mathcal{H}_{0,\pm} = \{\phi \in \mathcal{H}_0 \,|\, \mathrm{supp}(\mathcal{F}\phi) \subset \overline{\mathbb{R}_\pm} \times \partial M\}.$$

Theorem 1: (1) The wave operators $W_\pm$ exist. Moreover, $W_\pm \mathcal{H}_{0,\mp} = 0$ and $W_\pm$ are isometry from $\mathcal{H}_{0,\pm}$ into $\mathcal{H}$.
(2) $W_\pm$ are complete, i.e., Ran $W_\pm = \mathcal{H}_c(P)$.

Thus we can approximate the solution $\psi(t) = e^{-itP}\psi_0$ $(\psi_0 \in \mathcal{H}_c(P))$ by

$$\psi(t) \sim \mathcal{J}\, e^{-itP_f}\varphi_\pm \quad \text{as } t \to \pm\infty$$

where P_f is the 1-dim Laplacian, and $\varphi_\pm = W_\pm^* \psi_0$.

Remarks: (1) Even for $P = -\triangle + V$ on $\mathbb{R}^n$, the wave operators $W_\pm$ are <u>not</u> the same as the standard wave operators, since P_f is not the same as the free Laplacian.
(2) Scattering theory for such manifolds were studied by, e.g., De Biévre-Hislop-Sigal (1992), but the construction and hence the wave operators are quite different.
(3) P is <u>not</u> a small perturbation of P_f. In fact, $P - P_f$ is not P_f-bounded, and not even P-bounded. So, standard theorems of the scattering theory do not directly apply.
(4) Still, the idea is functional analytic, and we follow the essential ideas of the scattering theory very closely (Birman, Kato, Kuroda, etc.). We use a generalized formalism of the abstract 2-space stationary scattering theory.

Scattering matrix: By the asymptotic completeness, the scattering operator:

$$S = W_+^* W_- \ : \ \mathcal{H}_{0,-} \to \mathcal{H}_{0,+}$$

is a unitary operator. We set

$$(F_{0,\pm}(\lambda)\phi)(\theta) = (2\lambda)^{-1/4}(\mathcal{F}\phi)(\pm\sqrt{2\lambda}, \theta)$$

for $\phi \in \mathcal{H}_{0,\pm}$. Then there exists the scattering matrix: $\{S(\lambda) \in \mathcal{L}(\mathcal{H}_b) \,|\, \lambda > 0\}$ (where $\mathcal{H}_b = L^2(\partial M, H d\theta)$) such that

$$F_{0,+}(\lambda) S\phi = S(\lambda) F_{0,-}(\lambda)\phi$$

for $\phi \in \mathcal{H}_0^-$. ($S(\lambda)$ is unitary for a.e. λ.)

Theorem 2: For $\phi \in \mathcal{D}(Q)$ and $\lambda > 0$, there exists a λ-generalized eigenfunction Ψ such that

$$\left\| \Psi(r, \cdot) - r^{-(n-1)/2}(e^{-ikr}\phi + e^{ikr}(S(\lambda)\phi)) \right\|_{\mathcal{H}_b} = O(r^{-(n-1)/2-\delta})$$

as $r \to \infty$ with some $\delta > 0$, where $k = \sqrt{2\lambda}$.

Thus $S(\lambda)$ is the same scattering matrix as defined using asymptotic expansion of generalized eigenfunctions (Melrose).

4. Microlocal properties of the scattering matrix

We write $q(\theta,\omega) = \dfrac{1}{2}\sum_{j,k} \widehat{h}^{jk}(\theta)\omega_j\omega_k$ be the symbol of the free operator on ∂M. We denote the Hamilton flow generated by a symbol b by exp(tH_b), $t \in \mathbb{R}$.

Theorem 3.(Melrose-Zworski) Suppose P and S as above with $\mu = 1$. Then $S(\lambda)$ is an FIO (Fourier integral operator) associated to exp$(\pi H_{\sqrt{2q}})$.

Remark: (1) exp$(tH_{\sqrt{2q}})$ is the geodesic flow on ∂M corresponding to the (co)metric $\widehat{h}$.
(2) The above result is true for $\mu > 0$ if we generalize the definition of FIOs.

Theorem 3 implies

$$(*) \qquad WF(S(\lambda)\phi) = \exp(\pi H_{\sqrt{2q}})(WF(\phi))$$

for $\phi \in L^2(\partial M, H d\theta)$. This holds more generally:

Theorem 4. $(*)$ holds under the assumption $\mu > 0$.

Example: If $M = \mathbb{R}^n$, and if $\hat{h}$ is the standard metric on S^{n-1}, then $\exp(\pi H_{\sqrt{2q}})$ is the antipodal map. This result is a generalization and refinement of the well-known result: the integral kernel of $S(\lambda)$ is smooth except for the diagonal set.

5. Idea of proof (1): The classical mechanical scattering for conic manifolds

We set

$$p_c(r,\rho,\theta,\omega)=\frac{1}{2}\Big(\rho^2+\frac{q(\theta,\omega)}{r^2}\Big),\qquad r>0,\rho\in\mathbb{R},(\theta,\omega)\in T^*\partial M,$$

be the classical hamilton function corresponding to the *conic* metric, and we consider the scattering for the classical mechanics. If we set

$$(r(t),\rho(t),\theta(t),\omega(t))=\exp(tH_{p_c})(r_0,\rho_0,\theta_0,\omega_0),\qquad t\in\mathbb{R}.$$

We can solve the Hamilton equation explicitly, and

$$r(t)=\sqrt{2E_0t^2+2r_0\rho_0+r_0^2},\qquad \rho(t)=\frac{2E_0t+r_0\rho_0}{\sqrt{2E_0t^2+2r_0\rho_0+r_0^2}},$$

where $E_0=p_c(r_0,\rho_0,\theta_0,\omega_0)=p_c(r(t),\rho(t),\theta(t),\omega(t))$, and

$$(\theta(t), \omega(t)) = \exp(\tau(t) H_q)(\theta_0, \omega_0),$$

where

$$\tau(t) = \int_0^t \frac{ds}{r(s)^2} = \frac{1}{\sqrt{2q_0}} \left\{ \tan^{-1}\left(\frac{2E_0 t + \rho_0 r_0}{\sqrt{2q_0}} \right) - \tan^{-1}\left(\frac{\rho_0 r_0}{\sqrt{2q_0}} \right) \right\}$$

with $q_0 = q(\theta_0, \omega_0) = q(\theta(t), \omega(t))$. Noting

$$\frac{1}{\sqrt{2q}} H_q = H_{\sqrt{2q}},$$

we can also write

$$(\theta(t), \omega(t)) = \exp(\sigma(t) H_{\sqrt{2q}})(\theta_0, \omega_0),$$

where $\sigma(t) = \tan^{-1}\left(\frac{2E_0 t + \rho_0 r_0}{\sqrt{2q_0}} \right) - \tan^{-1}\left(\frac{\rho_0 r_0}{\sqrt{2q_0}} \right)$.

Now we consider the behavior as $t \to \pm\infty$. Then

$$r_\pm = \lim_{t\to\pm\infty}(r(t) - t\rho(t)) = \pm\frac{r_0\rho_0}{\sqrt{2E_0}},$$

$$\rho_\pm = \lim_{t\to\pm\infty}\rho(t) = \pm\sqrt{2E_0},$$

$$(\theta_\pm, \omega_\pm) = \lim_{t\to\pm\infty}(\theta(t), \omega(t)) = \exp(\sigma_\pm H_{\sqrt{2q}})(\theta_0, \omega_0)$$

where $\sigma_\pm = \pm\frac{\pi}{2} - \tan^{-1}\left(\frac{\rho_0 r_0}{\sqrt{2q_0}}\right)$.

We write this map (the classical wave operator) by

$$w_\pm : (r_0, \rho_0, \theta_0, \omega_0) \mapsto (r_\pm, \rho_\pm, \theta_\pm, \omega_\pm)$$

and $w_\pm$ are diffeomorphism from $\mathbb{R}_+ \times \mathbb{R} \times (T^*\partial M \setminus 0)$ to $\mathbb{R} \times \mathbb{R}_\pm \times (T^*\partial M \setminus 0)$. ($w_\pm^{-1}$ is also easily computed.)

If we write $p_f = \frac{1}{2}\rho^2$, then we may write

$$w_\pm(r_0, \rho_0, \theta_0, \omega_0) = \lim_{t \to \pm\infty} \exp(-tH_{p_f}) \circ \exp(tH_{p_c})(r_0, \rho_0, \theta_0, \omega_0)$$

Then the classical scattering operator:

$$s_c = w_+ \circ w_-^{-1} : \mathbb{R} \times \mathbb{R}_- \times (T^*\partial M \setminus 0) \ \to \ \mathbb{R} \times \mathbb{R}_+ \times (T^*\partial M \setminus 0)$$

is a diffeomorphism and easily computed:

$$s_c : (r, \rho, \theta, \omega) \mapsto (-r, -\rho, \exp(\pi H_{\sqrt{2q_0}}(\theta, \omega)).$$

This is the quantity we saw in Theorems 3 and 4.

6. Idea of proof (2): Egorov type estimates in the scattering symbol class

Let $a \in C^\infty(T^*M)$ (or $\in C^\infty(T^*M_f)$). We denote $a \in S^m_{sc}(M)$ (or $a \in S^m_{sc}(M_f)$, resp.) if for any $\alpha, \beta, \gamma, \delta$,

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta a(r, \rho, \theta, \omega)| \le C\langle r; \omega\rangle^{m-|\alpha|-|\delta|} \quad \text{in } M_\infty \text{ (or } M_f\text{)}$$

where $\langle r; \omega\rangle = \sqrt{1 + |r|^2 + q(\theta, \omega)}$.

When we say <u>conic</u>, it means conic with respect to (r, ω). We generally suppose symbols are supported in conic sets away from $\{\omega = 0\}$, and away from M_c and $\{r = 0\}$ if $a \in S^m_{sc}(M)$, and away from $\{\rho = 0\}$ if $a \in S^m_{sc}(M_f)$.

Remark: Our symbol class is analogous to the class $S\Big(\langle x\rangle^m, \frac{dx^2}{\langle x\rangle^2} + d\xi^2\Big)$ used by Isozaki, Kitada, and others.

For a symbol $a \in S_{sc}^{m}$, we denote the quantization by

$$Op(a) = a(r, D_r, \theta, D_\theta).$$

Let us fix $a \in S_{sc}^{0}(M)$. Analogously to the proof of Egorov theorem (but in our symbol class), we solve the Heisenberg equation to show the following:

Step 1. There is $a(t) \in S_{sc}^{0}(M)$ such that

$$e^{-itP} Op(a) e^{itP} = Op(a(t)).$$

Moreover, $a(t) - a \circ (\exp(tH_{p_c}))^{-1} \in S_{sc}^{-\mu}(M)$, and $a(t)$ is supported in $\exp(tH_{p_c})(\text{supp }(a))$ modulo $S_{sc}^{-\infty}(M)$.

Step 2. There is $b(t) \in S^0_{sc}(M_f)$ such that

$$e^{itP_f}\mathcal{J}^* e^{-itP} Op(a) e^{itP} \mathcal{J} e^{itP_f} = Op(b(t)).$$

Moreover, $b(t) - a \circ (\exp(-tH_{p_f}) \circ \exp(tH_{p_c}))^{-1} \in S^{-\mu}_{sc}(M_f)$, and $b(t)$ is supported in $\exp(-tH_{p_f}) \circ \exp(tH_{p_c})(\mathrm{supp}\ (a))$ modulo $S^{-\infty}_{sc}(M_f)$.

Then we take the limit $t \to \pm\infty$:

Step 3. There is $b_\pm \in S^0_{sc}(M_f)$ such that

$$W_\pm^* Op(a) W_\pm = Op(b_\pm).$$

Moreover, $b_\pm - a \circ w_\pm^{-1} \in S^{-\mu}_{sc}(M_f)$, and $b_\pm$ are supported in $w_\pm(\mathrm{supp}\ (a))$ modulo $S^{-\infty}_{sc}(M_f)$.

Step 3'. Let $b \in S^0_{sc}(M_f)$ and suppose b is supported in $\{\pm\rho > 0\}$. Then there is $a \in S^0_{sc}(M_f)$ such that

$$W_\pm Op(b) W_\pm^* = Op(a).$$

Moreover, $a - b \circ w_\pm \in S^{-\mu}_{sc}(M)$, and a is supported in $w_\pm^{-1}(\text{supp }(b))$ modulo $S^{-\infty}_{sc}(M)$.

Combining these, we have

Step 4. Let $a \in S^0_{sc}(M_f)$ and suppose a is supported in $\{\rho < 0\}$. Then there is $b \in S^0_{sc}(M_f)$ which is supported in $\{\rho > 0\}$ such that

$$S\, Op(a)\, S^* = Op(b).$$

Moreover, $b - a \circ s_c^{-1} \in S^{-\mu}_{sc}(M_f)$ and b is supported in $s_c(\text{supp }(a))$ modulo $S^{-\infty}_{sc}(M_f)$.

Step 5. Let $\lambda > 0$ and $a \in S^0(\partial M)$. Then there is $b \in S^0_{1,0}(\partial M)$ such that

$$S(\lambda)Op(a)S(\lambda)^* = Op(b).$$

Moreover, $b - a \circ (\exp(\pi H_{\sqrt{2q}}))^{-1} \in S^{-\mu}_{1,0}(\partial M)$, and b is supported in $\exp(\pi H_{\sqrt{2q}})(\text{supp } a)$ modulo $S^{-\infty}(\partial M)$.

Now if $\mu = 1$, then by a Beals type characterization of FIOs (cf. Ref.4.), we have Theorem 3. Even if $\mu < 1$, we can use the the usual characterization of the wave front set using pseudodifferential operators to show Theorem 4.

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