

**Localization for the random displacement model.
II. The Lifshitz tail and the Wegner estimate**

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0. Purpose of the talk:

2 key lemmas for the Anderson localization at the bottom of the spectrum:

1. Lifshitz tail
2. Wegner estimate

+ Multiscale analysis (Fröhlich-Spencer, von Drefus-Klein, Germinet-Klein, or textbooks by Stollmann, Kirsch), we have the localization.

We discuss the idea of proof of these estimates for the RDM (random displacement model).

1. Technical differences from the “standard” Anderson model

(1) **Non monotonous:** The perturbation is not monotone with respect to parameters.

(2) **Non multiplicative:** The perturbation is not linear in parameters.

(3) **Multiple minima:** The parameters on which the local operator take minimal ground state energy are not unique.

– We first discuss how to address the points (1) and (2) using a simpler one-parameter random operator model.

– Then we discuss how to address the point (3) using a recent result by Klopp-N.

Contents:

1. Technical difference from the “standard” Anderson model
2. A simple one-parameter random model
3. Lifshitz tail for the simple model
- 4 .Wegner estimate for the simple model
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2. A simple one-parameter random model:

Let $I = [0, 1]$, and suppose

$$V_\lambda : \lambda \in I \mapsto V_\lambda \in C_0(\Lambda_1(0))$$

is a C^2 -class function, where

$$\Lambda_r(x) = \{y \mid |y_j - x_j| < r/2, j = 1, \dots, d\}.$$

Let $\{\omega_i \mid i \in \mathbb{Z}^d\}$ be i.i.d.'s with values in I , and we consider

$$H^\omega = -\Delta + V^\omega, \quad V^\omega = \sum_{i \in \mathbb{Z}^d} V_{\omega_i}(\cdot - i).$$

Neumann decoupling: Let Λ be a cube, and we set

$$H_\Lambda^{\omega, N} = -\Delta + V^\omega \text{ on } L^2(\Lambda)$$

with Neumann boundary conditions, i.e., $\mathcal{Q}(H_\Lambda^{\omega, N}) = H^1(\Lambda)$.

We also set , for $i \in \mathbb{Z}^d$,

$$H_{\Lambda_1(i)}^N(\lambda) = -\Delta + V_\lambda(\cdot - i) \quad \text{on } L^2(\Lambda_1(i)).$$

It is well-known:

$$H^\omega \geq \bigoplus_{i \in \mathbb{Z}^d} H_{\Lambda_1(i)}^N(\omega_i) \quad \text{on } L^2(\mathbb{R}^d) \cong \bigoplus_{i \in \mathbb{Z}^d} L^2(\Lambda_1(i)).$$

For $L \gg 0$ we write $\Lambda_L = \Lambda_{2L+1}(0)$.

We consider $H_{\Lambda_L}^{\omega, N}$ on $L^2(\Lambda_L)$, and we also have

$$H_{\Lambda_L}^{\omega, N} \geq \bigoplus_{i \in \Lambda'_L} H_{\Lambda_1(i)}^N(\omega_i) \quad \text{on } L^2(\Lambda_L) \cong \bigoplus_{i \in \Lambda'_L} L^2(\Lambda_1(i))$$

where $\Lambda'_L = \Lambda_{2L+1}(0) \cap \mathbb{Z}^d$.

We set

$$e(\lambda) = \inf \sigma \left(H_{\Lambda_1(0)}^N(\lambda) \right) \quad \text{for } \lambda \in I.$$

Assumption (S1):

$$E_0 = e(0) = \min_{\lambda \in I} e(\lambda), \quad e(\lambda) > E_0 \text{ for } \lambda > 0,$$

and $e'(0) > 0$.

Remark: (S1) implies $H^\omega \geq E_0$, but not necessarily $\inf \sigma(H^\omega) = E_0$. If we assume $V(0)$ is symmetric with respect to reflections, then it holds.

Assumption (S2): The distribution of $\omega(0)$ has a density $g(\lambda)$ which is absolutely continuous on $(0, 1)$. Moreover, $0 \in \text{supp } g$.

Theorem S1:(Lifshitz tail) Let $N(E)$ be the IDS (integrated density of states) for H^ω , then

$$\limsup_{E \rightarrow E_0 + 0} \frac{\log |\log N(E)|}{\log(E - E_0)} \leq -\frac{d}{2}.$$

Theorem S2:(Wegner estimate) Let $\alpha \in (0, 1)$. There exist $\delta > 0$ and $C_\alpha > 0$ such that

$$\mathbb{P}\left(\sigma\left(H_{\Lambda_L}^{\omega, N}\right) \cap J \neq \emptyset\right) \leq C_\alpha |J|^\alpha |\Lambda_L|$$

for any interval $J \subset [E_0, E_0 + \delta]$. Moreover, $N(E)$ is Hölder continuous of order α .

Theorem S3:(Anderson localization) There is $\delta > 0$ such that H^ω has pure point spectrum in $[E_0, E_0 + \delta]$ almost surely, and the eigenfunctions decay exponentially.

3. Lifshitz tail for the simple model:

(cf. Klopp-N: Commun. Math. Phys. 2009)

Let $h(\lambda) = H_{\Lambda_1(0)}^N(\lambda)$. We note

$$h(\lambda) = h(0) + \lambda h'(0) + O(\lambda^2) \quad (\text{in operator norm})$$

for small λ . We don't know if $h'(0) \geq 0$, but we can show:

Lemma S1:

$$h(\lambda) - E_0 \geq \frac{1}{C}(h(0) - E_0 + \lambda), \quad \lambda \in I$$

with some $C > 0$ in the operator sense.

Proof: Since

$$h(\lambda) \geq \inf \sigma(h(\lambda)) = e(0) + \lambda e'(0) + O(\lambda^2) \geq E_0 + c_0 \lambda$$

for small λ , we have

$$h(0) + \lambda h'(0) = h(\lambda) + O(\lambda^2) \geq E_0 + c_1 \lambda$$

for $\lambda \in [0, \lambda_0]$ with some $\lambda_0, c_1 > 0$. Then we have

$$\begin{aligned} h(0) + \lambda h'(0) - E_0 &= \frac{1}{2}(h(0) - E_0) + \frac{1}{2}(h(0) - E_0 + 2\lambda h'(0)) \\ &\geq \frac{1}{2}(h(0) - E_0) + c_1 \lambda \\ &= \frac{1}{2}(h(0) - E_0 + 2c_1 \lambda) \end{aligned}$$

provided $2\lambda \in [0, \lambda_0]$. For $\lambda \geq \lambda_0/2$, it is easy to show the estimate because $h(\lambda) \geq \varepsilon > 0$ and $h(0) - h(\lambda)$ is uniformly bounded. \square

Proof of Theorem S1: Lemma S1 implies

$$H^\omega - E_0 \geq \bigoplus_{i \in \mathbb{Z}^d} \left(H_{\Lambda_1(i)}^N(\omega_i) - E_0 \right) \geq \frac{1}{C} \bigoplus_{i \in \mathbb{Z}^d} \left(H_{\Lambda_1(i)}^N(0) + \omega_i - E_0 \right)$$

Since $H^1(\mathbb{R}^d) \subset \bigoplus_{i \in \mathbb{Z}^d} H^1(\Lambda_1(i))$, we have

$$\langle \psi, (H^\omega - E_0)\psi \rangle \geq \frac{1}{C} \langle \psi, (\tilde{H}^\omega - E_0)\psi \rangle$$

for $\psi \in H^1(\mathbb{R}^d)$, where

$$\tilde{H}^\omega = -\Delta + \sum_{i \in \mathbb{Z}^d} \left(V_0(\cdot - i) + \omega_i \cdot \mathbf{1}_{\Lambda_1(i)} \right).$$

\tilde{H}^ω is usual Anderson model with a periodic background potential, so the Lifshitz tail is well-known for \tilde{H}^ω (Kirsch-Simon).

Since $N(E) \leq \tilde{N}(E_0 + C(E - E_0))$ (where $\tilde{N}(E)$ is the IDS for \tilde{H}^ω), Theorem S1 follows immediately. \square

4. Wegner estimate for the simple model:

We first note $H_{\Lambda_L}^{\omega, N}$ is a C^2 -function of $\{\omega_i \mid i \in \Lambda'_L\} \cong I^{(2L+1)^d}$. We consider derivatives of the operator in ω_i 's.

Proposition S1: There are $\delta_1, \delta_2 > 0$ such that

$$\sum_{i \in \Lambda'_L} \left\langle \psi, \left(\frac{\partial}{\partial \omega_i} H_{\Lambda_L}^{\omega, N} \right) \psi \right\rangle \geq \delta_1 \|\psi\|^2$$

provided $\psi \in H^1(\Lambda_L)$ satisfies

$$\left\langle \psi, \left(H_{\Lambda_L}^{\omega, N} - E_0 \right) \psi \right\rangle \leq \delta_2 \|\psi\|^2.$$

(This implies, if the state ψ has very small energy, then by pushing all ω_i 's up, we can increase the total energy.)

Outline of Proof: (Similar to the argument used in Klopp-N-Nakano-Nomura (Ann. IHP, 2003).) Let

P_i : the eigenprojection to the ground state of $H_{\Lambda_1(i)}^N(\omega_i)$,

$\bar{P}_i = 1 - P_i$ (in $L^2(\Lambda_1(i))$),

$E_1 = \inf_{\lambda \in I} \inf(\sigma(H_{\Lambda_1(0)}^N(\lambda) \setminus e(\lambda)) > E_0$.

Lemma S2: If $\langle \psi, (H_{\Lambda_L}^{\omega, N} - E_0)\psi \rangle \leq \delta_2 \|\psi\|^2$, then

$$\sum_{i \in \Lambda'_L} \|\bar{P}_i \psi_i\|^2 \leq \frac{\delta_2}{E_1 - E_0} \|\psi\|^2,$$

where ψ_i is the restriction of ψ to $\Lambda_1(i)$.

Lemma S3: Let $\delta_3 > 0$. If $\langle \psi, (H_{\Lambda_L}^{\omega, N} - E_0)\psi \rangle \leq \delta_2 \|\psi\|^2$, then

$$\sum_{i \in \Lambda_L''} \|\psi_i\|^2 \leq \frac{\delta_2}{\delta_3} \|\psi\|^2$$

where $\Lambda_L'' = \{i \in \Lambda_L' \mid e(\omega_i) - E_0 > \delta_3\}$.

– Proof of these lemmas is simple energy estimate.

– These lemmas imply, if ψ has sufficiently small energy, then the main part of $\sum \langle \psi, (\partial H_{\Lambda_L}^{\omega, N} / \partial \omega_i) \psi \rangle$ comes from $e'(\omega_i) P_i$ with ω_i very close to 0. We have positivity of $e'(\omega_i)$ for such i 's, and Proposition follows by direct computations. \square

Proof of Theorem S2: Use the argument of Hislop-Klopp (JFA 2002), based on ideas from Klopp (CMP 1995) and Combes-Hislop-N (CMP 2001).

We need the differentiability of the density function since we use integration by parts, and we cannot use the original Wegner's method nor computation based on Stieltjes integral.

5. Random displacement model:

(1) Wegner estimate: Essentially the same proof as the simple model works (though more involved due to more parameters).

(2) Lifshitz tail: The energy minimum is attained at 2^d corners.

—→ Straightforward generalization does not work.

—→ We use a recent result by Klopp-N (Analysis and PDE).

Reduction to a different model: Let

$$H_{\Lambda_1}^\omega(i) = -\Delta + \sum_{i \in \mathbb{Z}^d} q(\cdot - \omega_i)$$

be our RDM, where $q \in C_0^\infty$ is supported in $[-r, r]^d$ with $r \in (0, 1/4)$, and $\omega_i \in G = [-d_{max}, d_{max}]^d$ with $d_{max} = 1/2 - r$.

— For $\omega_i \in G$, we set $\bar{\omega}_i \in \{\pm d_{max}\}^d$ be the nearest corner of G . (If there are several, we choose one arbitrarily.)

Lemma RD1: There is $C > 0$ such that

$$H^\omega - E_0 \geq \frac{1}{C}(H^{\bar{\omega}} - E_0) \quad (\text{in the operator sense}).$$

The proof of Lemma RD1 uses the same argument as Lemma S1. In fact, we can show

$$H^\omega - E_0 \geq \frac{1}{C}(H^{\bar{\omega}} - E_0 + W) \geq \frac{1}{C}(H^{\bar{\omega}} - E_0)$$

where

$$W = \sum_{i \in \mathbb{Z}^d} |\omega_i - \bar{\omega}_i| \cdot \mathbf{1}_{\Lambda_1(i)}$$

but we do **not** (cannot) take advantage of the positivity of W . Since (as before) it follows:

$$N(E) \leq \bar{N}(E_0 + C(E - E_0))$$

where $\bar{N}(E)$ denotes the IDS for $H^{\bar{\omega}}$, the Lifshitz tail follows from that for $H^{\bar{\omega}}$.

6. Lifshitz tail for generalized alloy type models:

Let $V_1, \dots, V_N \in C_0(\Lambda_1(0))$, and let $\{\omega_i \mid i \in \mathbb{Z}^d\}$ be i.i.d.'s with values in $\{1, \dots, N\}$. (We suppose $\mathbb{P}(\omega(0) = j) > 0$ for $j = 1, \dots, N$.) We consider the random Schrödinger operator:

$$H^\omega = -\Delta + \sum_{i \in \mathbb{Z}^d} V_{\omega_i}(\cdot - i)$$

on $L^2(\mathbb{R}^d)$. We write

$$H_k^N = -\Delta + V_k \quad \text{on } L^2(\Lambda_1(0))$$

with Neumann boundary condition. Let

$$E_0 = \min_{k=1, \dots, N} \inf \sigma(H_k^N).$$

Theorem GA1: Suppose there is k such that $\inf \sigma(H_k^N) > E_0$. Suppose moreover that if $\inf \sigma(H_k^N) = E_0$ then V_k is symmetric with respect to reflections. Then $\inf \sigma(H^\omega) = E_0$ almost surely, and

$$\limsup_{E \rightarrow E_0 + 0} \frac{\log |\log(N(E))|}{\log(E - E_0)} < -\frac{1}{2}.$$

— Application to the RDM: $(H^{\bar{\omega}})$, we combine 2^d connected cubes together, and decompose $\mathbb{R}^d = \bigoplus \Lambda_2(2i + e_0)$ ($e_0 = (1/2, \dots, 1/2)$). Then the minimum energy E_0 is attained by 2^d -symmetric configurations, and the other $((2^d)^{2^d} - 2^d)$ -configurations have higher ground state energies. Thus we can apply Theorem GA1 to obtain the Lifshitz tail (with the exponent $1/2$) for $H^{\bar{\omega}}$, and hence that for H^ω . \square

Outline of proof of Theorem GA1:

- We decompose Λ_L into $(2L + 1)^{d-1}$ quasi-1D-strips (by Neumann decoupling). We show the Lifshitz tail for each quasi-1D-strip. (Hence 1D type bound.)
- The worst case (lowest ground state energy) is attained by putting all V_k in the strip with the lowest ground state energy. Such probability is exponentially small. (We don't even need large deviation.)
- If there are at least one potential with higher ground state energy in the strip, then the ground state energy is bounded from below by $E_0 + cL^{-2}$. We use “ground state transform”, “a variation of Poincaré's inequality”, and “positivity of the Dirichlet-Neumann map” to prove this.