Microlocal Singularities and Scattering Theory for Schrödinger Equations on Manifolds

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Plan of Talk

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2. Scattering for the Geodesic Flow on Asymptotically Conic Manifolds

3. Quantization (1) — Analysis of Singularities

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1. Very brief introduction to scattering theory

1.1 Scattering for the Newton particle (cf. [Reed-Simon] XI-2)

Let \( x = x(t) \in \mathbb{R}^d \) be the position of a particle, and let \( V(x) \) be the potential function. The motion is described by the Newton equation:

\[
x''(t) = -\nabla V(x(t)), \quad x(0) = x_0, \quad x'(0) = p_0.
\]

We suppose \( \text{Supp}[V] : \text{compact} \).

- If the particle is \textit{not trapped}, then the trajectory converges to the straight motion (= free motion) as \( t \to \pm \infty \), i.e., there are

\[
p_{\pm} = \lim_{t \to \pm \infty} x'(t), \quad x_{\pm} = \lim_{t \to \pm \infty} (x(t) - tp_{\pm}).
\]

In other words, the motion is asymptotically \( x(t) \sim x_{\pm} + tp_{\pm} \ (t \to \pm \infty) \).

\( (x_0, p_0) \mapsto (x_{\pm}, p_{\pm}) \) (the scattering data)

**Remark:** The above statement holds if \( |\nabla V(x)| = O(|x|^{-2-\varepsilon}) \) (\( \varepsilon > 0, |x| \to \infty \)).
### 1.2 Scattering in the quantum mechanics
(cf. [Reed-Simon] Vol.3, [Yafaev], etc.)

Let $\mathcal{H} = L^2(\mathbb{R}^d)$ and suppose $\text{Supp}[V]$ : compact. Let

$$ H = -\frac{1}{2} \triangle + V(x), \quad H_0 = -\frac{1}{2} \triangle \quad \text{on } \mathcal{H} $$

be Schrödinger operators. The solution to the Schrödinger equation:

$$ i \frac{\partial}{\partial t} u(t) = Hu(t), \quad u(0) = u_0 \in \mathcal{H} $$

is given by $u(t) = e^{-itH}u_0 \in C(\mathbb{R}; \mathcal{H})$.

If $u_0$ is orthogonal to all the eigenfunctions, then $u(t)$ converges to a free motion as $t \to \pm \infty$, i.e.,

$$ \exists u_\pm \in \mathcal{H}, \quad ||u(t) - e^{-itH_0}u_\pm|| \to 0 \quad (t \to \pm \infty). $$

The scattering data are: $u_0 \mapsto u_\pm$.

$\mathcal{W}_\pm : u_\pm \mapsto u_0$ (wave operators), $\mathcal{S} : u_- \mapsto u_+$ (scattering operator)

**Remarks:**
1. Spectral properties of $H$ follows from these.
2. We only need to assume $V(x) = O(|x|^{-1-\varepsilon}) \ (\varepsilon > 0, \ |x| \to \infty)$. 
2. Scattering for the geodesic flow on asymptotically conic manifolds

2.1 Hamilton flow on asymptotically conic manifolds

Let $M : d$-dim, non-compact manifold, s.t. $M = M_c \cup M_\infty$, $M_c :$ precompact,

$$M_\infty \cong (0, \infty) \times \partial M, \quad \partial M : \text{a compact manifold.}$$

In the following, we always use the coordinate:

$$\mathbf{(r, \theta)} \in (0, \infty) \times \partial M \cong M_\infty$$
on $M_\infty$. We let $g :$ an asymptotically conic Riemannian metric on $M$, i.e., $g$ has the form:

$$g \sim dr^2 + r^2 h, \quad \mathbf{(r, \theta)} \in M_\infty, \quad r \to \infty.$$

Here $h$ is a Riemannian metric on $\partial M$. The corresponding energy function is:

$$k(x, \xi) = \frac{1}{2} \sum_{i,j} g_{ij}(x) \xi^i \xi^j, \quad (x, \xi) \in T^*M$$

$$\sim \frac{1}{2} \left( \rho^2 + \frac{1}{r^2} \sum_{j,k} h_{jk}(\theta) \omega^j \omega^k \right), \quad (r, \rho, \theta, \omega) \in T^*M_\infty, \quad r \to \infty.$$

Here we use the identification: $T^*M_\infty \cong T^*\mathbb{R}_+ \times T^*\partial M$. 
Assumption : We write

\[ k(x, \xi) = \frac{1}{2} \left( a_1(r, \theta) \rho^2 + \frac{2}{r} \sum_{j=1}^{d-1} a_{2,j}(r, \theta) \rho \omega^j + \frac{1}{r^2} \sum_{j,k=1}^{d-1} a_{3,jk}(r, \theta) \omega^j \omega^k \right) \]

on \( T^*M_\infty \). There exists \( \mu > 0 \) such that for any indeces \( \ell \in \mathbb{Z}_+, \alpha \in \mathbb{Z}^{d-1}_+ \),

\[
|\partial_r^\ell \partial_\theta^\alpha (a_1(r, \theta) - 1)| \leq C_{\ell \alpha} r^{-1-\mu-\ell}, \quad |\partial_r^\ell \partial_\theta^\alpha a_2(r, \theta)| \leq C_{\ell \alpha} r^{-\mu-\ell}, \quad |\partial_r^\ell \partial_\theta^\alpha (a_3(r, \theta) - h(\theta))| \leq C_{\ell \alpha} r^{-\mu-\ell}.
\]

We write the Hamilton flow generated by \( k(x, \xi) \):

\[ \exp(tH_k) : T^*M \to T^*M \quad (t \in \mathbb{R}) \]

(Note this is the geodesic flow up to parameterizations.)
2.2 Nontrapping condition and the existence of scattering

Nontrapping Condition: Let \((x_0, \xi_0) \in T^*M, \xi \neq 0\) and we denote 
\((x(t), \xi(t)) = \exp(tH_k)(x_0, \xi_0)\). Then \((x_0, \xi_0)\) is called nontrapping if for \(\forall K \subseteq M\), \(\exists T > 0\) such that \(|t| \geq T \Rightarrow x(t) \notin K\).

If \((x_0, \xi_0)\) is nontrapping, we may suppose \((x(t), \xi(t)) \in T^*M_\infty\) for large \(|t|\). Thus we write

\[\exp(tH_k)(x_0, \xi_0) = (r(t), \rho(t), \theta(t), \omega(t)) \in T^*\mathbb{R}_+ \times T^*\partial M\]

for \(|t| \gg 0\). Then the limits:

\[r_\pm = \lim_{t \to \pm \infty} (r(t) - t\rho(t)), \quad \rho_\pm = \lim_{t \to \pm \infty} \rho(t),\]

\[\theta_\pm = \lim_{t \to \pm \infty} \theta(t), \quad \omega_\pm = \lim_{t \to \pm \infty} \omega(t)\]

exist, i.e., \(\rho(t), \theta(t), \omega(t)\) have limits, and \(r(t) \sim r_\pm + t\rho_\pm (t \to \pm \infty)\).
**The idea of Proof:**

1. We denote $q(\theta, \omega) = \frac{1}{2} \sum h_{jk}(\theta) \omega^j \omega^k$. Then $k \sim \frac{1}{2} \rho^2 + \frac{q(\theta, \omega)}{r^2}$ as $r \to \infty$. Hence $\rho(t), \frac{\omega(t)}{r(t)}$ are bounded by the energy conservation law.

2. Since $\frac{d^2}{dt^2} (r(t)^2) \sim 8k > 0$ (as $r \to \infty$), $r(t) > c|t| - C$ as $|t| \to \infty$.

3. $q(\theta(t), \omega(t))$ is uniformly bounded.

4. Combining these, we show the derivative of these scattering quantities $(r(t) - t\rho(t), \rho(t), \theta(t), \omega(t))$ are integrable, and hence the limits exist.
2.3 Wave operators; Scattering operator

We set $M_f = \mathbb{R} \times \partial \mathcal{M}$, $T^*M_f \cong T^*\mathbb{R} \times T^*\partial \mathcal{M}$. We call the correspondence:

$$w_\pm : (r_\pm, \rho_\pm, \theta_\pm, \omega_\pm) \in T^*M_f \mapsto (x_0, \xi_0) \in T^*\mathcal{M}$$

the (classical mechanical) wave operators.

**Remark:** Actually, we always have $\pm \rho_\pm > 0$, $w_\pm$ is not defined on $(\mathbb{R} \times \mathbb{R}_{\mp}) \times T^*\partial \mathcal{M}$. Hence, $w_\pm : (\mathbb{R} \times \mathbb{R}_{\pm}) \times T^*\partial \mathcal{M} \to T^*\mathcal{M}$.

If we define $k_f(\rho) = \frac{1}{2} \rho^2$, the free energy with respect to $r$, then

$$\exp(tH_{k_f})(r, \rho, \theta, \omega) = (r + t\rho, \rho, \theta, \omega), \quad (r, \rho, \theta, \omega) \in T^*\mathcal{M}_0.$$ 

Thus (with suitable identifications),

$$w_\pm^{-1} = \lim_{t \to \pm \infty} \exp(-tH_{k_f}) \circ \exp(tH_k), \quad w_\pm = \lim_{t \to \pm \infty} \exp(-tH_k) \circ \exp(tH_{k_f}).$$

**Remark:** $w_\pm$ is a homogeneous canonical map: $w_\pm(x, \lambda \xi) = (r_\pm, \lambda \rho_\pm, \theta_\pm, \lambda \omega_\pm)$ $(\forall \lambda > 0)$.

We define the scattering operator by

$$s = w_+^{-1} \circ w_- : (r_-, \rho_-, \theta_-, \omega_-) \mapsto (r_+, \rho_+, \theta_+, \omega_+)$$

Note $s : (\mathbb{R} \times \mathbb{R}_-) \times T^*\partial \mathcal{M} \to (\mathbb{R} \times \mathbb{R}_+) \times T^*\partial \mathcal{M}$.
2.4 Euclidean space – The scattering theory in the polar coordinate

Let $M = \mathbb{R}^d$, $\partial M = S^{d-1}$, $\mathbb{R}^d = \{|x| < 2\} \cup (1, \infty) \times S^{d-1}$, and can we apply our formulation. For $(x_0, \xi_0) \in T^*\mathbb{R}^d$, we write

$$\xi_0 = |\xi_0| \hat{\xi}_0, \quad \hat{\xi}_0 = \frac{\xi_0}{|\xi_0|} \in S^{d-1}, \quad x_0 = x_0^\perp + s_0 \hat{\xi}_0, \quad x_0^\perp \perp \hat{\xi}_0.$$ 

It is easy to see

$$\rho_\pm = \pm |\xi_0|, \quad \theta_\pm = \pm \hat{\xi}_0, \quad r_\pm = \pm s_0, \quad \omega_\pm = \mp |\xi_0| x_0^\perp.$$ 

The free motion is $x(t) = x_0^\perp + (t + s_0/|\xi_0|) \hat{\xi}_0$, and we may consider

- $\rho_\pm$: length of the momentum,
- $\theta_\pm$: direction of the momentum,
- $r_\pm$: time-delay (× momentum),
- $\omega_\pm$: impact parameter (/ momentum).

The scattering operator $s$ is: $(r, \rho, \theta, \omega) \mapsto (-r, -\rho, -\theta, -\omega)$.
2.5 Scattering Matrix

We can show, if we write $s(r_-, \rho_-, \theta_-, \omega_-) = (r_+, \rho_+, \theta_+, \omega_+)$, then $\rho_+ = -\rho_-,

\[ s(r_- + s, \rho_-, \theta_-, \omega_-) = (r_+ - s, \rho_+, \theta_+, \omega_+), \quad \forall s \in \mathbb{R}. \]

Hence, we can set

\[ s(\lambda) : (\theta_-, \omega_-) \mapsto (\theta_+, \omega_+), \quad \lambda = \frac{1}{2} \rho^2 \]

with $\rho = \rho_+$ as the parameter. $s(\lambda)$ is called the scattering matrix.

Remark: $s(\lambda) : T^*\partial M \to T^*\partial M$ is a homogeneous canonical map.

Remark: $t(\lambda; \theta, \omega) = r_- + r_+$ is the time delay.
2.6 Conic manifolds – The geodesic flow in the boundary manifold

We consider an important special case, i.e., completely conical metric:

\[ k(r, \rho, \theta, \omega) = \frac{1}{2} \left( \rho^2 + \frac{1}{r^2} \sum_{j,k} h_{jk}(\theta) \omega^j \omega^k \right), \quad (r, \rho, \theta, \omega) \in T^*\mathbb{R}_+ \times T^*\partial M. \]

(Note \( k \) has a singularity at \( r = 0 \), but we consider trajectories which do not hit \( r = 0 \).)

- \( q(\theta, \omega) = \frac{1}{2} \sum h_{jk}(\theta) \omega^j \omega^k \) is an invariant.
- By setting \( q_0 = q(\theta_0, \omega_0) \), we can solve \((r(t), \rho(t))\) in terms of \((r_0, \rho_0)\) and \( q_0 \). In particular,
  \[ r(t) = \sqrt{2E_0 t^2 + 2r_0 \rho_0 t + r_0^2}, \quad E_0 = \frac{1}{2} \rho_0^2 + \frac{q_0}{r_0^2}. \]

- By changing the time variable as \( \sigma(t) = \sqrt{2q_0} \int_0^t r(s)^{-2} ds \), we have
  \[ (\theta(t), \omega(t)) = \exp(\sigma(t)H_{\sqrt{2q}})(\theta_0, \omega_0). \]
  The right hand side is the geodesic flow on \( T^*\partial M \).
- We have \( \sigma(\infty) - \sigma(-\infty) = \pi \), and hence \( s(\lambda) = \exp(\pi H_{\sqrt{2q}}) \).

Remark: \( s(\lambda) \) does not depend on \( \lambda > 0 \), but it is specific to the conic case. If \( g \) is asymptotically conic, there may be \( \lambda \)-dependent lower order terms.
3. Quantization (1) — Analysis of Singularities

3.1 Function space and Schrödinger operators

▷ Let $G(x)dx$ be the standard density on $M$, and set

$$\mathcal{H} = L^2(M, Gdx)$$

be our Hilbert space.

▷ Let $P$ be the quantization of $k(x, \xi)$, i.e., the Laplace-Beltrami operator. On $M_\infty$, $P$ can be written:

$$P = -\frac{1}{2} G^{-1} \left( \partial_r, \partial_\theta / r \right) G \begin{pmatrix} a_1 & a_2 \\ t a_2 & a_3 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta / r \end{pmatrix}$$

Here $a_1, a_2, a_3$ satisfy the above assumptions.

▷ $P$ is essentially self-adjoint on $C_0^\infty(M)$.

▷ $\sigma_{\text{ess}}(P) = \sigma_{\text{ac}}(P) = [0, \infty)$. No positive eigenvalues. (Froese-Hislop, Donnelly, etc...)

▷ We study the time-evolution generated by $P$:

$$U(t) = e^{-itP} : \mathcal{H} \to \mathcal{H}.$$

Remark: We can add the potential function $V(x)$, but we omit it for simplicity here.
3.2 Construction of free quantum system

▻ We need a *free system* to construct a scattering theory. In order that, we quantize 
\[ k_f(\rho) = \frac{1}{2} \rho^2 \] 
(as in the classical mechanics case). We set
\[ \mathcal{H}_f = L^2(\mathbb{R} \times \partial M; dr \wedge H(\theta) d\theta), \]
where \( H(\theta) d\theta \) is the standard density on \( (\partial M, h) \).

▻ The free motion is
\[ P_f = -\frac{1}{2} \frac{\partial^2}{\partial r^2} \text{ on } \mathcal{H}_f. \]

▻ We set \( j(r) \in C^\infty(\mathbb{R}) \) s.t. \( j(r) = \begin{cases} 0, & (r \leq 1), \\ 1, & (r \geq 2). \end{cases} \)

The identification operator: \( \mathcal{I} : \mathcal{H}_f \rightarrow \mathcal{H} \) is defined by
\[ \mathcal{I}\varphi(r, \theta) = (H(\theta)/G(r, \theta))^{1/2} j(r) \varphi(r, \theta), \quad \varphi \in \mathcal{H}_f. \]
3.3 Construction of the fundamental solution

**Theorem 1:** (Ito-N) We suppose all the geodesics are nontrapping on $T^*M$. Then for each $\pm t > 0$,

$$W(t) = e^{itP} \mathcal{I}^* e^{-itP}$$

is a Fourier integral operator (FIO) (asymptotically) corresponding to $w^{-1}_{\pm}$.

**Remark:** $W(t)$ may be considered as a quantization of $w^{-1}_{\pm}$, though it is *finite time* evolution, since:

- FIOs describe (microlocal) singularities of an operator, i.e., behavior at high Fourier variables.
- A trajectories with high Fourier variable (momentum) move to far away, even for a fixed time. By a scaling, they corresponds to long-time motions.
- Long-time behavior of trajectories can be described by the (classical) scattering theory, $w_{\pm}$ in particular.

**Remark:** If $\mu = 1$ Theorem 1 holds with *usual* FIO, but if $0 < \mu < 1$, we need slightly generalized notion of FIOs.
**Remark:** From Theorem 1 and *microlocal smoothing effect*, we can show

$$e^{-itP} = \mathcal{I}e^{-itP_f}W(t) + K, \quad \text{(with } K \text{ an smoothing operator)}. $$

Since $e^{-itP_f}$ can be explicitly known, $e^{-itP}$ can be written as a product of a known operator and an FIO. (cf. Hassell-Wunsch 2004)

**Remark:** We can also show the wave operator $W_\pm = \lim_{t \to \pm \infty} e^{itP} \mathcal{I}e^{-itP_f}$ is an FIO corresponding to $w_\pm$ by letting $t \to \pm \infty$. 
As an application of Theorem 1, we can determine the wave front sets of solutions to Schrödinger equations.

**Wave front set (definition):** Let \( u \in \mathcal{S}'(\mathbb{R}^d) \). \( (x_0, \xi_0) \not\in \text{WF}(u) \) (\( \xi_0 \neq 0 \)) if and only if \( \exists \varphi \in C_0^\infty(\mathbb{R}^d), \varphi(x_0) \neq 0 \), \( \exists \Gamma \subset \mathbb{R}^d \): a conic neighborhood of \( \xi_0 \) and moreover, \( \forall N, \exists C_N > 0 \) s.t.

\[
\mathcal{F}(\varphi u)(\xi) \leq C_N (1 + |\xi|)^{-N}, \quad \xi \in \Gamma.
\]

The wave front set of solutions to the Schrödinger equation is given by:

\[
\text{WF}(e^{-itP}u) = w_{\mp}(\text{WF}(e^{-itP}I^*u)), \quad \pm t > 0.
\]


**Remark:** If there are trapped trajectories, the characterization of \( \text{WF}(u(t)) \) is not well-understood. (However, see Burq-Guillarmou-Hassell for Strichartz estimate)

**Remark:** We denote the distribution kernel of \( W(t) \) by the same symbol. Then

\[
\text{WF}(W(t)) = \{(w_{\pm}^{-1}(x, \xi), x, -\xi) \mid (x, \xi) \in T^*M\}, \quad \pm t > 0.
\]

follows from Theorem 1. The above result follows from this.
3.5 The idea of the proof — Standard quantization

▷ For a symbol (function) \( a(x, \xi) \) on \( T^*M \) (or \( T^*M_f \)), we quantize it by

\[
\text{Op}_\hbar(a) = a(x, \hbar D_x), \quad \hbar > 0.
\]

(\( \text{Op}_\hbar(a) \) is an \( \hbar \)-pseudodifferential operator.)

▷ Then, in particular, \((x_0, \xi_0) \not\in \text{WF}(u)\) if and only if \( \exists a \in C_0^\infty(T^*M) \), such that

\[a(x_0, \xi_0) \neq 0\] and

\[
\|\text{Op}_\hbar(a)u\| = O(\hbar^\infty), \quad \hbar \to 0.
\]

▷ \( A(t) = W(t)\text{Op}_\hbar(a)W(t)^* \) satisfied the Heisenberg equation:

\[
\frac{d}{dt}A(t) = -i[L(t), A(t)] + (\text{small errors}), \quad A(0) = \text{Op}_\hbar(a). \quad (\text{HEq})
\]

Here \( L(t) \) is a self-adjoint operators such that

\[
L(t) \sim e^{iP_f}Pe^{-iP_f} - P_f
\]

ignoring the identification operator \( \mathcal{I} \).
The principal symbol of $L(t)$ is given by

$$\ell(t; r, \rho, \theta, \omega) \sim k(r + t\rho, \rho, \theta, \omega) - k_f(\rho) \sim \frac{q(\theta, \omega)}{(r + t\rho)^2},$$

which is the generator of flow: $w(t) = \exp(-tH_{k_0}) \circ \exp(tH_k)$.

We construct an asymptotic solution to (HEq), analogously to Egorov Theorem. Then the principal symbol of $A(t)$ is given by

$$\text{Sym}(A(t)) \sim a \circ w(t) \sim a \circ w_{\pm} \quad \text{if} \quad \pm t > 0.$$  

Here we use a scaling property:

$$w(t; x, \lambda \xi) = (z(\lambda t), \lambda \xi(\lambda t)), \quad \text{where} \quad (z(t), \xi(t)) = w(t; x, \xi).$$

and hence $w(t; x, \xi) \sim w_{\pm}(x, \xi)$ when $|\xi| \rightarrow \infty$.

We then use a Beals type characterization for FIOs: Let $\Sigma: T^*\mathbb{R}^d \rightarrow T^*\mathbb{R}^d$ be a homogeneous canonical diffeomorphism. Let $U$: bounded in $L^2(\mathbb{R}^d)$, such that for any $a(x, \xi) \in C_0^\infty(T^*\mathbb{R}^d \setminus 0),

$$\text{Op}\, \hat{a} (a \circ \Sigma^{-1})U - U\text{Op}\, \hat{a} = \text{Op}\, \hat{b} U + O(\hat{h}^\infty)$$

with $b = b(\hat{h}; x, \xi)$, $b = O(\hat{h})$. Then $U$ is an FIO (of order 0) associated to $\Sigma$.

$W(t)$ satisfies the characterization with $\Sigma = w_{\pm}$, and Theorem 1 follows.
3. Quantization (2) — Microlocal analysis of scattering matrix

3.1 Construction of scattering theory — Wave operators and the completeness

▷ We consider the quantization of the geodesic flow and the quantum scattering theory.
▷ We first construct a quantum scattering theory. Let $\mathcal{H}_f$, $P_f$, $I : \mathcal{H}_f \to \mathcal{H}$ as in the last section. We set $F$ be the Fourier transform in $r$-variable; and function spaces $\mathcal{H}_{f,\pm}$ by

$$
(Ff)(\rho) = (2\pi)^{-1/2} \int e^{-ir\rho} f(r) dr,
$$

$$
\mathcal{H}_f = \mathcal{H}_{f,+} \oplus \mathcal{H}_{f,-}, \quad \mathcal{H}_{f,\pm} = \{ u \in \mathcal{H} \mid \text{supp}(Fu) \subset \mathbb{R}_\pm \}.
$$

▷ The wave operators are defined by

$$
W_{\pm} \varphi = \lim_{t \to \pm \infty} e^{itP} I e^{-itP_f} \varphi, \quad \varphi \in \mathcal{H}_f.
$$

Then $W_{\pm}$ exist, $W_{\pm} = 0$ on $\mathcal{H}_{f,\mp}$, $W_{\pm} : \mathcal{H}_{f,\pm} \to \mathcal{H}$ are isometries.

▷ Asymptotic completeness: $\text{Ran}(W_{\pm}) = \mathcal{H}_c(H)$. In particular, the scattering operator: $S = W^*_+ W_- : \mathcal{H}_{f,-} \to \mathcal{H}_{f,+}$, is unitary. (Ito-N 2009. See also De Bievre-Hislop-Sigal 1992, Ito-Skibsted 2011, Hempel-Post-Wedder 2012, etc.)
4.2 Singularities of the scattering matrix— Melrose-Zworski theorem

▷ **Scattering matrix:** Thanks to the energy conservation: ([S, Pf] = 0), S is decomposed as follows:

\[ \text{FSF}^{-1} \varphi(\rho, \cdot) = (S(\lambda)\varphi)(-\rho, \cdot), \quad \varphi \in L^2(\mathbb{R} \times \partial M), \]

where \( \lambda = \frac{1}{2} \rho^2 \), \( S(\lambda) \): \( L^2(\partial M, H_d\theta) \to L^2(\partial M, H_d\theta) \). S(\lambda) is the scattering matrix.

**Theorem 2:** (Melrose-Zworski, Ito-N) \( S(\lambda) \) is an FIO (asymptotically) corresponding to \( \exp(\pi H_{\sqrt{2q}}) \).

**Remark:** Melrose-Zworski (1996) proved Theorem 2 when \( \mu = 1 \), and the formulation is completely different. When \( 0 < \mu < 1 \) we need a slightly generalized definition of FIOs.

**Remark:** We note \( \exp(\pi H_{\sqrt{2q}}) \) is the classical mechanical scattering matrix for the conic metric case. Thus this theorem may be considered as its quantization.

**Remark:** It may be considered as a refinement of the off-diagonal smoothness of the scattering matrix (cf. Isozaki-Kitada 1985, Yafaev (textbooks), etc.)
4.3 Idea of proof — Nonstandard quantization

- Since $S$ is an operator on $\partial M$, the singularities corresponds to its behavior as $|\omega| \to \infty$ on $T^*\partial M$. (with $(\theta, \omega) \in T^*\partial M$)

- Since $\omega$ is the impact parameter, a particle with large $\omega$ cannot enter the area with small $r$. In other words, since $E \sim \frac{1}{2} \rho^2 + O((\omega/r)^2)$, we have to have $r = O(\omega)$ as long as the energy is fixed.

- Hence, we need to localize in $(\rho, \theta)$, and carry on the asymptotic analysis for large $(r, \omega)$. In the following, we always work in $M_\infty \cong \mathbb{R}_+ \times \partial M$.

- For a symbol: $a = a(r, \rho, \theta, \omega) \in C^\infty(T^*(\mathbb{R}_+ \times \partial M))$, we quantize $a$ by

$$\text{Op}^\hbar(a) = a(\hbar r, D_r, \theta, \hbar D_\theta), \quad \hbar > 0,$$

(This is analogous to Isozaki-Kitada calculus, or the scattering calculus of Melrose.)

- We construct an asymptotic solution of $W(t)^*\text{Op}^\hbar(a)W(t)$ as $\hbar \to 0$, and then analyze the properties as before.
5. Conclusion

- We approximate the long-time behavior of classical trajectories (geodesic flow) by a free motion which is defined suitably.
- The relationship between the classical trajectories and free motion is given by wave operators/scattering operators.
- By quantizations (semiclassical pseudodifferential operator calculus), we can analyze behaviors of solutions to Schrödinger equations.
- By usual quantization, we can characterize the microlocal singularities of solutions.
- By scattering-type quantization, we can analyze the microlocal properties of scattering matrix.
- The strategy is very simple that various generalizations are possible:
  - more general manifolds, e.g., polynomially growing ends (Itozaki)
  - long-range perturbations (N, Itozaki)
  - the operators with potentials (including unbounded potentials)
  - perturbation of harmonic oscillators (Wunsch, Mao-N)
  - microlocal analytic singularities (Robbiano-Zuily, Martinez-N-Sordoni)
  - etc...