Remarks on Scattering Theory on Scattering Manifolds

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Idea: Construct a time-dependent scattering theory for Schrödinger operators on manifolds with asymptotically conic structure using a simple ”free system” and the old 2-space abstract scattering theory (Kato, Birman, Kuroda, ...).

Model: Suppose $M = M_c \cup M_\infty$, where M_c is relatively compact, $M_\infty \cong (0,\infty)\times\partial M$, ∂M: a closed manifold. We consider a formally self-adjoint operator on M such that P is expressed as

$$P = -\frac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G\begin{pmatrix} a_1 & a_2 \\ {}^t a_2 & a_3 \end{pmatrix}\Big(\partial_r, \partial_\theta/r\Big) + V$$

on M_∞. We consider P as an operator on $\mathcal{H} = L^2(M, Gdx)$, where $a_k, V \in C^2(M)$ such that for $\ell + |\alpha| \le 2$,

$$|\partial_r^\ell\partial_\theta^\alpha(a_1 - 1)| \le Cr^{-\mu_1-\ell}, \quad |\partial_r^\ell\partial_\theta^\alpha a_2| \le Cr^{-\mu_2-\ell},$$

$$|\partial_r^\ell\partial_\theta^\alpha(a_3 - \hat{h})| \le Cr^{-\mu_3-\ell}, \quad |\partial_r^\ell\partial_\theta^\alpha V| \le Cr^{-\mu_4-\ell}.$$

Here $G(r,\theta) = r^{n-1}H(\theta)$ with $H(\theta)$: some density on ∂M, $\hat{h}$ is a positive 2-tensor on ∂M. We suppose $\mu_1, \mu_4 > 1$, $\mu_2, \mu_3 > 0$, and we call M a *scattering manifold of short-range type* if M is equipped with such P and G.

Example: $M = \mathbb{R}^n$, $H = -\triangle + V$. In this case, we set $\partial M = S^{n-1}$, $\widehat{h}$ is the metric on T^*S^{n-1}, $a_1 = 1$, $a_2 = 0$, $a_3 = \widehat{h}$, and V satisfies the "short-range" condition in the usual scattering theory.
If $a_2, a_3 \not\equiv 0$ but $\mu_2, \mu_3 > 1$, then the operator is also a short-range perturbation of the flat Laplacian, i.e., the coefficients minus the flat metric is $O(|x|^{-\mu_2 \wedge \mu_3})$ as $|x| \to \infty$. Our assumption above is weaker.

Remark: Our model is somewhat more general than the *scattering metric* in the sense of Melrose. In this case, roughly, $\mu_1 = 2$, $\mu_2 = \mu_3 = 1$, $V \equiv 0$, and each coefficients have asymptotic expansions in r^{-1} as $r \to \infty$.

Spectral properties: (cf. e.g. Froese-Hislop-Perry 1989)

Theorem 0: (1) P is essentially self-adjoint on $C_0^\infty(M)$.
(2) $\sigma_{ess}(P) = [0, \infty)$ and $\sigma_d(P) \subset (-\infty, 0]$ is bounded.
(3) $\sigma_{pp}(P)$ is discrete except for 0, $\sigma_{ac}(P) = [0, \infty)$,
and $\sigma_{sc}(P) = \emptyset$.

(In fact, we need only $\mu_1, \mu_2, \mu_3, \mu_4 > 0$ for this result.)

Proof: Use the Weyl sequence, the Mourre estimate. $\square$

Scattering Theory (a very short introduction)

We consider a pair of self-adjoint operators H on a Hilbert space $\mathcal{H}$ and $H_{0,\pm}$ on $\mathcal{H}_{0,\pm}$. Let $\mathcal{J}_\pm : \mathcal{H}_{0,\pm} \to \mathcal{H}$ be bounded operators.

We are interested in the long time behavior of the solution to

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t), \quad \psi(0) = \psi_0 \in \mathcal{H}, \quad \text{i.e., } \psi(t) = e^{-itH}\psi_0.$$

If $e^{-itH_{0,\pm}}$ is easy to describe, and if $\exists \varphi_\pm \in \mathcal{H}_{0,\pm}$ such that

$$\left\| e^{-itH}\psi_0 - \mathcal{J}_\pm e^{-itH_{0,\pm}}\varphi_\pm \right\| \to 0 \quad \text{as } t \to \pm\infty,$$

these give a nice approximation of the long time behavior of $\psi(t)$. This is equivalent to

$$\psi_0 = \lim_{t\to\pm\infty} e^{itH}\mathcal{J}e^{-itH_{0,\pm}}\varphi_\pm$$

We suppose

$$W_\pm(H, H_{0,\pm}, \mathcal{J}) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}\mathcal{J}_\pm e^{-itH_{0,\pm}} P_{ac}(H_{0,\pm}).$$

exist, where $P_{ac}(A)$ is the spectral projection to the absolutely continuous subspace of A, and suppose

$$\lim_{t\to\pm\infty} \left\| \mathcal{J}_\pm e^{-itH_{0,\pm}} \varphi \right\| = \|\varphi\| \quad \text{for } \varphi \in \mathcal{H}_{0,\pm}.$$

$W_\pm(H, H_{0,\pm}, \mathcal{J}_\pm)$ are called the wave operators and they are said to be asymptotically complete if

$$\mathrm{Ran}[W_\pm(H, H_{0,\pm}, \mathcal{J})] = \mathcal{H}_c(H),$$

where $\mathcal{H}_c(H)$ is the continuous spectral subspace of H. The completeness implies the above approximation is valid for all $\psi_0 \in \mathcal{H}_c(H)$.

Moreover, the asymptotic completeness implies

> H on $\mathcal{H}_c(H)$ is unitarily equivalent to $H_{0,\pm}$ on $\mathcal{H}_{ac}(H_{0,\pm})$.
> In particular, $\sigma_{sc}(H) = \emptyset$.

If $H_{0,\pm}$ is simple operators, and if the completeness holds, then the spectral behavior of H is described nicely by $W_\pm(H, H_{0,\pm}, \mathcal{J}_\pm)$.

Example: $\mathcal{H} = \mathcal{H}_{0,\pm} = L^2(\mathbb{R}^n)$, $\mathcal{J}_\pm = I$, $H = -\triangle + V(x)$, and $H_{0,\pm} = -\triangle$, where $V(x) = O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$. Then the wave operators exists and the asymptotic completeness holds.
(Birman, Kato, Kuroda, Agmon, Enss, ...)

Classical mechanical scattering for M.

— For our operator H, the choice of $H_{0,\pm}$ is not obvious. We may take $H_0 = -\frac{1}{2}(\partial_r^2 + r^{-2}\partial_\theta \widehat{h} \partial_\theta)$ on $L^2(\mathbb{R}_+ \times \partial M)$ (with some boundary condition at $r = 0$), but this H_0 is not very simple. —

In order to motivate our choice of the "free" system, we look at the classical mechanics generated by the corresponding symbol $p(x, \xi)$ on T^*M, which has the form:

$$p(r, \theta, \rho, \omega) = -\frac{1}{2}(\rho, \omega/r) \begin{pmatrix} a_1(r, \theta) & a_2(r, \theta) \\ {}^t a_2(r, \theta) & a_3(r, \theta) \end{pmatrix} (\rho, \omega/r) + V(r, \theta)$$

on M_∞.

We consider the classical flow: $(x(t), \xi(t)) = \exp tH_p(x_0, \xi_0)$ for some $(x_0, \xi_0) \in T^*M$. Suppose (x_0, ξ_0) is forward nontrapping, i.e.

$$x(t) \in M_\infty \text{ for large } t, \text{ and } r(t) \to \infty \text{ as } t \to \infty.$$

where we denote

$$(x(t), \xi(t)) = (r(t), \theta(t), \rho(t), \omega(t)) \in T^*(\mathbb{R}_+ \times \partial M) \quad \text{if } x(t) \in M_\infty.$$

(backward nontrapping is defined similarly.)
Then we can easily show

$$\rho_\pm = \lim_{t \to \pm\infty} \rho(t) \quad \text{(asymptotic radial momentum)}$$

$$\theta_\pm = \lim_{t \to \pm\infty} \theta(t) \quad \text{(asymptotic direction)}$$

exist (under the assumption: $\mu_1, \mu_2, \mu_3, \mu_4 > 0$).

Moreover, under our “short-range” assumptions ($\mu_1, \mu_4 > 1$), we can show

$$\omega_\pm = \lim_{t\to\pm\infty} \omega(t) \quad (\text{“impact parameter”}\,(\times\rho_\pm));$$

$$z_\pm = \lim_{t\to\pm\infty} (r(t) - t\rho(t)) \quad (\text{“time delay”}\,(\times\rho_\pm))$$

exist. Thus, we have “scattering data” $(z_\pm, \theta_\pm, \rho_\pm, \omega_\pm)$, and only $z_\pm$ (or $r(t)$) requires *modifications*. The evolution: $r(t) \mapsto r(t) - t\rho(t)$ is generated by $p_0(\rho) = \frac{1}{2}\rho^2$, i.e., $\exp tH_{p_0}(r, \theta, \rho, \omega) = (r + t\rho, \theta, \rho, \omega)$. Thus we can write

$$(z_\pm, \theta_\pm, \rho_\pm, \omega_\pm) = \lim_{t\to\pm\infty} \exp(-tH_{p_0}) \circ \exp tH_p(x_0, \xi_0)$$

This suggests that the “natural” choice of the free hamiltonian is $p_0(\rho) = \frac{1}{2}\rho^2$.

Reference system: We set

$$M_f = \mathbb{R} \times \partial M, \quad \mathcal{H}_0 = L^2(M_f, H(\theta)drd\theta), \quad P_0 = -\frac{1}{2}\frac{\partial^2}{\partial r^2}$$

as the "free" system.

We let $j(r) \in C^\infty(\mathbb{R})$ such that $j(r) = \begin{cases} 1, & (r \geq 1) \\ 0, & (r \leq 0) \end{cases}$.

We set $\mathcal{J}$: $\mathcal{H}_0 \to \mathcal{H}$ by

$$\mathcal{J}\phi(r,\theta) = r^{-(n-1)/2} j(r)\phi(r,\theta) \qquad \text{if } (r,\theta) \in M_\infty$$

for $\phi \in \mathcal{H}_0$. Then we define the wave operators by

$$W_\pm = W_\pm(P, P_0, \mathcal{J}) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itP}\mathcal{J}e^{-itP_0}.$$

We can easily show $W_\pm$ exist, but it is *not* isometry.

We write

$\mathcal{F}$: Fourier transform in r,

We decompose $\mathcal{H}_0 = \mathcal{H}_{0,+} \oplus \mathcal{H}_{0,-}$ where

$$\mathcal{H}_{0,\pm} = \{\phi \in \mathcal{H}_0 \,|\, \mathrm{supp}(\mathcal{F}\phi) \subset \overline{\mathbb{R}_\pm} \times \partial M\}.$$

Theorem 1: (1) The wave operators $W_\pm$ exist. Moreover, $W_\pm \mathcal{H}_{0,\pm} = 0$ and $W_\pm$ are isometry from $\mathcal{H}_{0,\pm}$ into $\mathcal{H}$.
(2) $W_\pm$ are complete, i.e., Ran $W_\pm = \mathcal{H}_{ac}(P)$.

Thus we can approximate the solution $\psi(t) = e^{-itP}\psi_0$ ($\psi_0 \in \mathcal{H}_c(P)$) by

$$\psi(t) \sim \mathcal{J}\, e^{-itP_0}\varphi_\pm \quad \text{as } t \to \pm\infty$$

where P_0 is the 1-dm Laplacian, and $\varphi_\pm = W_\pm^* \psi_0$.

Remarks: (1) Even for $P = -\triangle + V$ on $\mathbb{R}^n$, the wave operators $W_\pm$ are not the same as the standard wave operators, since P_0 is not the same as the free Laplacian.
(2) Scattering theory for such manifolds were studied by, e.g., De Biévre-Hislop-Sigal (1992), but the construction and hence the wave operators are quite different.
(3) P is not a small perturbation of P_0. In fact, $P - P_0$ is not P_0-bounded, and not even P-bounded. So, standard theorems of the scattering theory do not directly apply.
(4) Still, the idea is functional analytic, and we follow the essential ideas of the scattering theory very closely (Birman, Kato, Kuroda, etc.). We use a generalized formalism of the abstract 2-space stationary scattering theory. (cf. Kato-Kuroda, Yafaev, Derezinski-Gérard, or Isozaki (in Japanese))

Small trick of the proof : We have the limiting absorption principle:

$$\langle r\rangle^{-\nu}(P-\lambda \pm i0)^{-1}\langle r\rangle^{-\nu} \in \mathcal{L}(\mathcal{H})$$

if $\lambda > 0$ and $\nu > 1/2$ (not sufficient for the completeness). We set

$$T = P\mathcal{J} - \mathcal{J}P_0 \ : \ \mathcal{H}_0 \to \mathcal{H}$$

Then

$$T \sim O(r^{-\mu_1})\partial_r^2 + O(r^{-\mu_2})\partial_r(\frac{\partial_\theta}{r}) + O(r^{-\mu_3})(\frac{\partial_\theta}{r})^2 + O(r^{-\mu_4})$$

as $r \to \infty$. Since $\mu_2, \mu_3 > 0$, and ∂_θ is not P-bounded, it is not clear if

$$\langle r\rangle^{\nu}T^*(P-\lambda \pm i0)^{-1}\langle r\rangle^{-\nu} \in \mathcal{L}(\mathcal{H})$$

with some $\nu > 1/2$. (Maybe not bounded unless $\mu_2, \mu_3 > 1$.)

But we can show

$$\langle r\rangle^{-\nu}Q(P-\lambda\pm i0)^{-1}(1+Q)^{-1}\langle r\rangle^{-\nu}\in\mathcal{L}(\mathcal{H})$$

if $\nu>1/2$, where

$$Q=-\frac{1}{2}H^{-1}\sum\partial_{\theta_j}H\hat{h}_{jk}\partial_{\theta_k}\quad\text{(self-adjoint on }\partial M\text{)}.$$

(This can be proved by commutator estimates and a simple bootstrap argument.)
Using this, we can show

$$\langle r\rangle^{\nu}T^*(P-\lambda\pm i0)^{-1}(1+Q)^{-1}\langle r\rangle^{-\nu}\in\mathcal{L}(\mathcal{H})$$

with some $\nu>1/2$. Then, with the help of the abstract 2-space stationary scattering theory, we can prove the completeness.

Scattering matrix: By the asymptotic completeness, the scattering operator:

$$S = W_+^* W_- \ : \ \mathcal{H}_{0,-} \to \mathcal{H}_{0,+}$$

is a unitary operator. We set

$$(F_{0,\pm}(\lambda)\phi)(\theta) = (2\lambda)^{-1/4}(\mathcal{F}\phi)(\pm\sqrt{2\lambda}, \theta)$$

for $\phi \in \mathcal{H}_{0,\pm}$. Then there exists the scattering matrix: $\{S(\lambda) \in \mathcal{L}(\mathcal{H}_b) \,|\, \lambda > 0\}$ (where $\mathcal{H}_b = L^2(\partial M, H d\theta)$) such that

$$F_{0,+}(\lambda) S\phi = S(\lambda) F_{0,-}(\lambda)\phi$$

for $\phi \in \mathcal{H}_0^-$. ($S(\lambda)$ is unitary for a.e. λ.)

Theorem 2: For $\phi \in \mathcal{D}(Q)$ and $\lambda > 0$, there exists a λ-generalized eigenfunction Ψ such that

$$\left\| \Psi(r,\cdot) - r^{-(n-1)/2}(e^{-ikr}\phi + e^{ikr}(S(\lambda)\phi)) \right\|_{\mathcal{H}_b} = O(r^{-(n-1)/2-\delta})$$

as $r \to \infty$ with some $\delta > 0$, where $k = \sqrt{2\lambda}$.

Thus $S(\lambda)$ is the same scattering matrix as defined using asymptotic expansion of generalized eigenfunctions (Melrose).

Idea of proof: We have

$$S(\lambda) = -2\pi i F_{0,+}(\lambda)(\mathcal{J}^*T - T^*(P-\lambda-i0)^{-1}T)F_{0,-}(\lambda)^*$$

and generalized eigenfunction (related to ϕ by W_-) is given by

$$\Psi = (\mathcal{J} - (P-\lambda-i0)^{-1}T)F_{0,-}(\lambda)^*\phi,$$

Then by the resolvent equation

$$\begin{aligned} J^*\Psi = {} & \mathcal{J}^*\mathcal{J}F_{0,-}(\lambda)^*\phi \\ & - (P_0-\lambda-i0)^{-1}(\mathcal{J}^*T - T^*(P-\lambda-i0)^{-1}T)F_{0,-}(\lambda)^*\phi \end{aligned}$$

The first term is $(2\pi k)^{-1/2}e^{-ikr}\phi(\theta)$, and we can easily show the next term is asymptotically $\sim (2\pi k)^{-1/2}e^{ikr}S(\lambda)\phi(\theta)$ by comparing the above formulas, and using the explicit representation of $(P_0-z)^{-1}$.

Addendum: scattering for wave-like equations

Here we consider the properties of the solutions to the equation:

$$i\frac{\partial}{\partial t}\psi(t,x) = \sqrt{2P}\,\psi(t,x), \quad \psi(0,x) = \psi_0(x) \in \mathcal{H} = L^2(M, G dx)$$

i.e., the properties of the evolution group: $e^{-it\sqrt{2P}}$. Here M, G and P satisfy above assumptions, and we suppose $V(x) \equiv 0$ so that $P \geq 0$ in the operator sense.

By the Birman-Kato invariance principle, we have

$$W_\pm(P, P_{0,\pm}; \mathcal{I}) = W_\pm\Big(\sqrt{2P}, \sqrt{2P_{0,\pm}}; \mathcal{I}\Big) \ : \ \mathcal{H}_{0,\pm} \to \mathcal{H}$$

and hence we also have the same asymptotic completeness for the pairs $\sqrt{2P}$ and $\sqrt{2P_{0,\pm}}$.

On the other hand, $\sqrt{2P_{0,\pm}} = \sqrt{-\partial_r^2} = |D_r|$ and hence

$$\sqrt{2P_{0,+}} = D_r \quad \text{on } \mathcal{H}_{0,+}; \qquad \sqrt{2P_{0,-}} = -D_r \quad \text{on } \mathcal{H}_{0,-},$$

i.e., $\exp(-it\sqrt{2P_{0,\pm}})$ are simple translations in the r-variable:

$$e^{-it\sqrt{2P_{0,+}}}\psi(r,\theta) = \psi(r-t,\theta); \qquad e^{-it\sqrt{2P_{0,-}}}\psi(r,\theta) = \psi(r+t,\theta).$$

Hence we learn any $\psi_0 \in \mathcal{H}_c(P)$,

$$\psi(t) \sim \mathcal{I}(\varphi_\pm(r \mp t,\theta)) \quad \text{as } t \to \pm\infty$$

where $\varphi_\pm = W_\pm^* \psi_0$. This is similar to, but different from the *translation representation* of Lax-Phillips.

Other topcs:

*Long-range perturbation?

*Other type of manifolds, for example hyperbolic manifolds, polynomially diverging metric?

*Scattering matrix under our setting (cf. Melrose-Zworski, 1996)?