Beals-type characterization of Fourier integral operators

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Conic Lagrangian manifolds:

* $\Lambda \subset T^*\mathbb{R}^m$: a smooth m-dimensional submanifold.

* L is called Lagrangian if the pull back of the standard canonical form vanishes on Λ, i.e., $i^*(dx \wedge d\xi) = 0$ on $T^*\Lambda$.

* Λ is called conic if $(x, \xi) \in \Lambda$ implies $(x, \lambda\xi) \in \Lambda$ for $\lambda > 0$.

Besov space $B_2^{\sigma,\infty}(\mathbb{R}^m)$:

Let $\sigma \in \mathbb{R}$ and let $u \in \mathcal{S}'(\mathbb{R}^m)$ such that $\hat{u} \in L^2_{loc}(\mathbb{R}^m)$. Then we set

$$\|u\|_{B_2^{\sigma,\infty}} = \left(\int_{|\xi|\le 1} |\hat{u}(\xi)|^2 d\xi\right)^{1/2} + \sup_{j\ge 0}\left(\int_{2^j \le |\xi| \le 2^{j+1}} \left|2^{\sigma j}\hat{u}(\xi)\right|^2 d\xi\right)^{1/2}$$

and the Besov space is defined by

$$B_2^{\sigma,\infty}(\mathbb{R}^m) = \left\{ u \in \mathcal{S}'(\mathbb{R}^m) \;\middle|\; \|u\|_{B_2^{\sigma,\infty}} < \infty \right\}$$

Symbol class:

We write $a \in S^m_{\rho,\delta}(\mathbb{R}^n)$ if $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and for any $\alpha, \beta \in \mathbb{Z}^n_+$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta} \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|}, \quad x, \xi \in \mathbb{R}^n,$$

with some $C_{\alpha\beta} > 0$. We write $a \in S^m_{cl}(\mathbb{R}^n)$ if $a \in S^m_{1,0}$ and a has an asymptotic expansion:

$$a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi), \quad \text{as } |\xi| \to \infty,$$

where $a_j(x,\xi)$ are homogeneous of order $(m-j)$ in ξ.

Lagrangian distribution:

Let $\Lambda \subset T^*\mathbb{R}^m \setminus 0$ be a conic Lagrangian submanifold, $u \in \mathcal{S}'(\mathbb{R}^m)$ and $\sigma \in \mathbb{R}$. u is called Lagrangian distribution associated to Λ of order σ if for any $p_1, \ldots, p_N \in S^1_{cl}(\mathbb{R}^m)$ such that the principal symbol of p_j vanish on Λ $(j = 1, 2, \ldots, N)$,

$$p_1(x, D_x) p_2(x, D_x) \cdots p_N(x, D_x) u \in B^{-\sigma - m/4, \infty}_{2,loc}(\mathbb{R}^m),$$

and we denote $u \in I^\sigma(\mathbb{R}^m, \Lambda)$.

Fourier integral operators:
Let S be a canonical transform from $T^*\mathbb{R}^n$ to $T^*\mathbb{R}^n$, and suppose S is homogeneous of order 1 with respect to ξ. Let

$$\Lambda_S = \Big\{(y,x,\eta,-\xi) \,\Big|\, (y,\eta) = S(x,\xi), (x,\xi) \in T^*\mathbb{R}^n \setminus 0\Big\} \subset T^*\mathbb{R}^{2n} \setminus 0.$$

Let $U \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ and let $u \in \mathcal{S}'(\mathbb{R}^{2n})$ be its distribution kernel. Then U is called a Fourier integral operator associated to S if $u \in I^\sigma(\mathbb{R}^{2n}, \Lambda_S)$.

Remark: If U is a Fourier integral operator, $m \leq 2n$, a homogeneous phase function $\Psi(x,\theta,y)$ ($x, y \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$), and a symbol $a(x,\theta,y) \in S^{\sigma+n/2-m/2}_{1,0}$ such that

$$U\varphi(x) = (2\pi)^{-n/2+m/2} \int_{\mathbb{R}^m} e^{i\Psi(x,\theta,y)} a(x,\theta,y)\varphi(y)dyd\theta.$$

Let $a \in S^m_{1,0}(\mathbb{R}^n)$ ($m \in \mathbb{R}$) such that

$$\left\{x \,\middle|\, a(x,\xi) \neq 0 \text{ for some } \xi\right\} \Subset \mathbb{R}^n, \quad \text{supp}\, a \cap (\mathbb{R}^n \times \{0\}) = \emptyset.$$

For such a, we set

$$Ad_S(a)U = (a \circ S^{-1})(x, D_x)U - Ua(x, D_x) \;:\; \mathcal{S} \to \mathcal{S}'.$$

Theorem: Let S as above, and let $U \in \mathcal{L}(\mathcal{S}, \mathcal{S}')$. U is an FIO of order 0 associated to S if and only if for any $a_1, a_2, \ldots, a_N \in S^1_{cl}(\mathbb{R}^n)$ satisfying (2.1),

$$Ad_S(a_1)Ad_S(a_2)\cdots Ad_S(a_N)U \in \mathcal{L}(L^2_{cpt}(\mathbb{R}^n), L^2_{loc}(\mathbb{R}^n)).$$

Corollary:

Let S and U as above, and suppose U is invertible. If for any $a \in S^1_{cl}(\mathbb{R}^n)$ satisfying (2.1) there is $b \in S^0_{cl}(\mathbb{R}^n)$ such that

$$Ua(x, D_x)U^{-1} = (a \circ S^{-1})(x, D_x) + b(x, D_x),$$

then U is an FIO of order 0 associated to S.

* Thus we learn that the Egorov theorem implies the operator U is an FIO.

In order to prove Theorem 2.1, we first notice that the L^2_{cpt}-L^2_{loc} boundedness implies the distribution kernel is locally $B_2^{-n/2,\infty}$.

Lemma: Suppose $U \in \mathcal{L}(L^2_{cpt}(\mathbb{R}^n), L^2_{loc}(\mathbb{R}^n))$, and let $u \in \mathcal{S}'(\mathbb{R}^n)$ be its distribution kernel. Then $u \in B_{2,loc}^{-n/2,\infty}(\mathbb{R}^n)$.

Proof: Suppose u is the distribution kernel of U. Let $\chi, \psi \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ (χ: even function), and we consider

$$I = \int_{\mathbb{R}^{2n}} \Big| \psi(\xi)\psi(\eta)\mathcal{F}[\chi(x)\chi(y)u(x,y)](\xi,\eta) \Big|^2 d\xi\, d\eta.$$

We choose $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ so that $\chi_1(x) = 1$ on supp χ. We note I can be expressed in terms of the Hilbert-Schmidt norm:

$$I = \left\| \psi \mathcal{F} \chi U \chi \mathcal{F}^{-1} \psi \right\|_{HS}^2$$

where $\psi = \psi(\xi)$ and $\chi = \chi(x)$ denote the multiplication operators on $L^2(\mathbb{R}^n_x)$ and $L^2(\mathbb{R}^n_\xi)$, respectively. $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm. Then we represent the Hilbert-Schmidt norm by a trace:

$$\begin{aligned} I &= \mathrm{Tr}\Big[(\psi \mathcal{F} \chi U \chi \mathcal{F}^{-1} \psi)^* (\psi \mathcal{F} \chi U \chi \mathcal{F}^{-1} \psi) \Big] \\ &= \mathrm{Tr}\Big[\psi \mathcal{F} \chi U^* \chi \mathcal{F}^{-1} \psi^2 \mathcal{F} \chi U \chi \mathcal{F}^{-1} \psi \Big] \\ &= \mathrm{Tr}\Big[((\chi_1 U \chi_1)^* \chi \mathcal{F}^{-1} \psi^2 \mathcal{F} \chi)((\chi_1 U \chi_1) \chi \mathcal{F}^{-1} \psi^2 \mathcal{F} \chi) \Big] \\ &= \mathrm{Tr}\Big[((\chi_1 U \chi_1)^* \chi \psi^2(D_x) \chi)((\chi_1 U \chi_1) \chi \psi^2(D_x) \chi) \Big] \end{aligned}$$

We use the Hörder inequality for the trace to obtain

$$
\begin{aligned}
I &\leq \left\|(\chi_1 U \chi_1)^* \chi \psi^2(D_x)\chi\right\|_{HS} \left\|(\chi_1 U \chi_1)\chi \psi^2(D_x)\chi\right\|_{HS} \\
&\leq \|\chi_1 U \chi_1\|^2_{\mathcal{L}(L^2)} \left\|\chi \psi^2(D_x)\chi\right\|^2_{HS} \\
&= (2\pi)^{-n} \|\chi_1 U \chi_1\|^2_{\mathcal{L}(L^2)} \|\psi\|^4_{L^4} \|\chi\|^4_{L^4}.
\end{aligned}
$$

Here we have used the well-known properties: $\|AB\|_{HS} \leq \|A\| \, \|B\|_{HS}$ and $\|a(x)b(D_x)\|_{HS} = (2\pi)^{-n/2} \|a\|_{L^2} \|b\|_{L^2}$.

Now we choose $\psi \in C_0^\infty(\mathbb{R}^n)$ so that $\psi(\xi) = 1$ for $1 \le |\xi| \le 2$, and we set

$$\psi_N(\xi) = \psi(2^{-N}\xi), \quad N = 1, 2, \ldots, \xi \in \mathbb{R}^n.$$

We note $\|\psi_N\|_{L^4}^4 = 2^{nN}\|\psi\|_{L^4}^4$. Then, by the above estimate, we have

$$\begin{aligned}
&\iint_{2^N \le |\xi|, |\eta| \le 2^{N+1}} \Big| \mathcal{F}[\chi(x)\chi(y)u(x,y)](\xi,\eta) \Big|^2 d\xi d\eta \\
&\qquad \le \iint \Big| \psi_N(\xi)\psi_N(\eta)\mathcal{F}[\chi(x)\chi(y)u(x,y)](\xi,\eta) \Big|^2 d\xi d\eta \\
&\qquad \le (2\pi)^{-n} \|\chi_1 U \chi_1\|_{\mathcal{L}(L^2)}^2 \|\psi\|_{L^4}^4 \|\chi\|_{L^4}^4 \times 2^{nN},
\end{aligned}$$

and this implies $\chi(x)\chi(y)u(x,y) \in B_2^{-n/2,\infty}(\mathbb{R}^{2n})$ for any $\chi \in C_0^\infty(\mathbb{R}^n)$. $\square$

We set

$$\tilde{\Lambda}_S = \Big\{(y,\eta,x,\xi) \,\Big|\, (y,\eta) = S(x,\xi)\Big\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n.$$

Lemma: Let $p \in S^1_{cl}(\mathbb{R}^{2n})$ such that the principal symbol of p vanishes on $\tilde{\Lambda}_S$, and suppose p is supported in a convex conic neighborhood of $(S(x_0,\xi_0),x_0,\xi_0) \in \tilde{\Lambda}_S$. Then there exist $b_j \in S^0_{cl}(\mathbb{R}^{2n})$, $f_j \in S^1_{cl}(\mathbb{R}^n)$ $(j = 1,2,\ldots 2n)$, and $r \in S^0_{cl}(\mathbb{R}^{2n})$ such that

$$p(y,\eta,x,\xi) = \sum_{j=1}^{2n} b_j(y,\eta,x,\xi)\Big((f_j \circ S^{-1})(y,\eta) - f_j(x,\xi)\Big) + r(y,\eta,x,\xi).$$

Proof: Elementary calculus. □

Proof of Theorem: Symbol calculus using above lemmas. □