Johnson homomorphisms and symplectic representation theory

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Mapping class groups

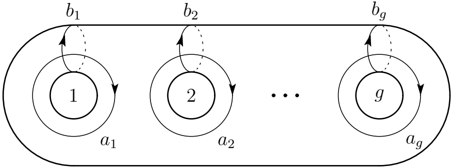
- Σ_g: a closed oriented connected surface of genus g
- $\mathcal{M}_g := \mathsf{Diff}_+\Sigma_g/(\mathsf{isotopy}) = \pi_0(\mathsf{Diff}_+\Sigma_g)$
  : the mapping class group of Σ_g
- $H_{\mathbb{Z}} := H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$
- Intersection form on $H_{\mathbb{Z}}$:

$$\mu : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \longrightarrow \mathbb{Z} \qquad \left(\begin{array}{l} \text{non-degenerate} \\ \text{skew-symmetric} \end{array} \right)$$

- Poincaré duality:

$$H_{\mathbb{Z}} := H_1(\Sigma_g; \mathbb{Z}) = H_1(\Sigma_g; \mathbb{Z})^* = H^1(\Sigma_g; \mathbb{Z}) = H_{\mathbb{Z}}^*.$$

- Fix a symplectic basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of $H_{\mathbb{Z}}$ w.r.t. μ:



- symplectic element (class):

$$\begin{aligned} \omega_0 &= \sum_{i=1}^{g} (a_i \otimes b_i - b_i \otimes a_i) \in H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \\ &= \sum_{i=1}^{g} a_i \wedge b_i \in \wedge^2 H_{\mathbb{Z}}. \end{aligned}$$

- $\mathrm{Sp}(H_{\mathbb{Z}}) \cong \mathrm{Sp}(2g, \mathbb{Z})$: symplectic group,

$$\mathrm{Sp}(H_{\mathbb{Z}}) \curvearrowright H_{\mathbb{Z}} \qquad \mu\text{-preserving } (\omega_0\text{-preserving}) \text{ action.}$$

- $\mathcal{M}_g$ acts on $H_{\mathbb{Z}}$ with preserving μ. This gives

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{M}_g \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1 \qquad \text{(exact)}$$

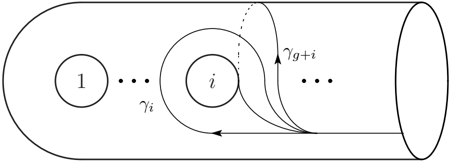
where $\mathcal{I}_g$ is called the Torelli group.

We also consider

- $\Sigma_{g,1}$: a compact oriented connected surface of genus g w/ one boundary component
- $\mathcal{M}_{g,1} := \mathrm{Diff}(\Sigma_{g,1} \operatorname{rel} \partial\Sigma_{g,1})/(\mathrm{isotopy})$: the mapping class group of $\Sigma_{g,1}$
- $H_1(\Sigma_{g,1}, \mathbb{Z}) = H_{\mathbb{Z}} \cong \mathbb{Z}^{2g}$
- Corresponding Torelli group:

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1 \qquad \text{(exact)}$$

- $\pi_1(\Sigma_{g,1}) = \langle \gamma_1, \gamma_2, \ldots, \gamma_{2g} \rangle = F_{2g}$, where



$$\zeta := \prod_{i=1}^{g} [\gamma_i, \gamma_{g+i}]$$ is the boundary loop.

- $\pi_1(\Sigma_{g,1}) \twoheadrightarrow \pi_1(\Sigma_g) = \langle \gamma_1, \gamma_2, \ldots, \gamma_{2g} \rangle / \langle \zeta \rangle$

- $\mathcal{M}_{g,1}$ acts naturally on $\pi_1(\Sigma_{g,1})$:

$$\sigma : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}\,(\pi_1(\Sigma_{g,1})),$$
$$\overline{\sigma} : \mathcal{M}_g \longrightarrow \mathrm{Out}\,(\pi_1(\Sigma_g)) := \mathrm{Aut}\,(\pi_1(\Sigma_g))/\mathrm{Inn}\,(\pi_1(\Sigma_g))$$

Theorem [Dehn, Nielsen, Baer, Epstein, Zieschang et al.]

The homomorphisms σ and $\overline{\sigma}$ are injective and

$$\mathrm{Im}\,\sigma = \{\varphi \in \mathrm{Aut}\,(\pi_1(\Sigma_{g,1})) \mid \varphi(\zeta) = \zeta\},$$
$$\mathrm{Im}\,\overline{\sigma} = \mathrm{Out}_+(\pi_1(\Sigma_g)) : \text{ (orientation-preserving).}$$

Johnson homomorphisms

In the following, we mainly focus on the $\mathcal{M}_{g,1}$-case.

- $\mathcal{I}_{g,1}$ measures the gap between $\mathcal{M}_{g,1}$ and $\mathrm{Sp}(2g,\mathbb{Z})$.
- It is known that

$$H_1(\mathcal{M}_{g,1}) = \mathcal{M}_{g,1}/[\mathcal{M}_{g,1},\mathcal{M}_{g,1}] = 0 \quad \text{for } g \geq 3.$$

  $\rightsquigarrow$ It is not easy to make an "approximation" of $\mathcal{M}_{g,1}$ without looking the structure of $\mathcal{I}_{g,1}$.

- The structure of $\mathcal{I}_{g,1}$ is more complicated than that of $\mathcal{M}_{g,1}$.

In a series of papers, Dennis Johnson showed:

Theorem [Johnson]

1. $\mathcal{I}_{g,1}$ is finitely generated for $g \geq 3$.
2. (The first Johnson homomorphism)
   There exists an $\mathcal{M}_{g,1}$-equivariant homomorphism
   $$\tau_{g,1}(1) : \mathcal{I}_{g,1} \twoheadrightarrow \wedge^3 H_{\mathbb{Z}}.$$
   Dehn twists along BSCC form a generating system of $\mathrm{Ker}\,\tau_{g,1}(1)$.
3. $\tau_{g,1}(1)$ gives the abelianization $H_1(\mathcal{I}_{g,1}) = \mathcal{I}_{g,1}/[\mathcal{I}_{g,1}, \mathcal{I}_{g,1}]$ modulo 2-torsions.

   (The torsion part is given by Birman-Craggs homormophisms.)

Morita's generalization

- $\pi := \pi_1(\Sigma_{g,1}) = \langle \gamma_1, \gamma_2, \ldots, \gamma_{2g} \rangle$.
- $\pi = \Gamma_1(\pi) \supset \Gamma_2(\pi) \supset \Gamma_3(\pi) \supset \cdots$
  : The lower central series of π defined by
$$\Gamma_{i+1}(\pi) = [\Gamma, \Gamma_i(\pi)] \qquad \text{for } i \geq 1.$$
- $\mathcal{L}(H_{\mathbb{Z}}) = \bigoplus_{i=1}^{\infty} \mathcal{L}_i(H_{\mathbb{Z}})$: the free Lie algebra generated by $H_{\mathbb{Z}}$

$$\begin{aligned} a &\in \mathcal{L}_1(H_{\mathbb{Z}}) = H_{\mathbb{Z}}, \\ [a,b] &\in \mathcal{L}_2(H_{\mathbb{Z}}) \cong \wedge^2 H_{\mathbb{Z}}, \\ [a,[b,c]] &\in \mathcal{L}_3(H_{\mathbb{Z}}) \cong (H_{\mathbb{Z}} \otimes (\wedge^2 H_{\mathbb{Z}}))/ \wedge^3 H_{\mathbb{Z}}, \\ &\vdots \end{aligned}$$

Fact

There exists an $\mathcal{M}_{g,1}$-equivariant isomorphism

$$\begin{array}{ccc}
\Gamma_i(\pi)/\Gamma_{i+1}(\pi) & \xrightarrow{\cong} & \mathcal{L}_i(H_{\mathbb{Z}}) \\
\Cup & & \Cup \\
[\alpha_1,[\alpha_2,\cdots,\alpha_i]]\cdots] & \longmapsto & [\overline{\alpha_1},[\overline{\alpha_2},\cdots,\overline{\alpha_i}]]\cdots]
\end{array}$$

where $\pi \ni \alpha_j \longmapsto \overline{\alpha_j} \in H_{\mathbb{Z}}$.

- Iterating expansion

$$[X,Y] \longmapsto X \otimes Y - Y \otimes X$$

gives an (degree preserving) embedding $\mathcal{L}(H_{\mathbb{Z}}) \hookrightarrow \bigoplus_{i=1}^{\infty} H_{\mathbb{Z}}^{\otimes i}$.

- $\mathcal{M}_{g,1} \subset \mathrm{Aut}(\pi) \curvearrowright \Gamma_i(\pi)$ for $i \geq 1$.

  $\rightsquigarrow \ \mathcal{M}_{g,1} \curvearrowright \pi/\Gamma_i(\pi) \qquad (\pi/\Gamma_2(\pi) = H_{\mathbb{Z}})$

Definition (Johnson filtration)

$$\mathcal{M}_{g,1}[0] = \mathcal{M}_{g,1} \supset \mathcal{M}_{g,1}[1] = \mathcal{I}_{g,1} \supset \mathcal{M}_{g,1}[2] \supset \mathcal{M}_{g,1}[3] \supset \cdots,$$

where

$$\mathcal{M}_{g,1}[k] := \mathrm{Ker}\left(\sigma_k : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}(\pi/\Gamma_{k+1}(\pi))\right).$$

Definition (The k-th Johnson homomorphism)

We have an $\mathcal{M}_{g,1}$-equivariant homomorphism defined by

$$\begin{array}{rccc} \tau_{g,1}(k): & \mathcal{M}_{g,1}[k] & \longrightarrow & \mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}(H_{\mathbb{Z}})) \\ & \Uplus & & \Uplus \\ & f & \longmapsto & \left(\overline{\gamma} \mapsto [f(\gamma)\gamma^{-1}]\right) \end{array}$$

where $[f(\gamma)\gamma^{-1}] \in \Gamma_{k+1}(\pi)/\Gamma_{k+2}(\pi) = \mathcal{L}_{k+1}(H_{\mathbb{Z}})$.

- By definition,

$$\begin{aligned} \operatorname{Ker} \tau_{g,1}(k) &= \mathcal{M}_{g,1}[k+1], \\ \operatorname{Im} \tau_{g,1}(k) &= \mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1]. \end{aligned}$$

- $\mathrm{Hom}(H_{\mathbb{Z}}, \mathcal{L}_{k+1}(H_{\mathbb{Z}})) = H_{\mathbb{Z}}^* \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}}) \overset{\mathrm{PD}}{=\!=} H_{\mathbb{Z}} \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$.

Theorem [Morita]

1. The image of $\tau_k : \mathcal{M}_{g,1}[k] \to H_{\mathbb{Z}} \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}})$ is included in

$$\mathfrak{h}_{g,1}(k) := \mathrm{Ker}\left(H_{\mathbb{Z}} \otimes \mathcal{L}_{k+1}(H_{\mathbb{Z}}) \xrightarrow{[\cdot,\cdot]} \mathcal{L}_{k+2}(H_{\mathbb{Z}}) \right).$$

2. The direct sums

$$\mathrm{Im}\,\tau_{g,1} := \bigoplus_{k=1}^{\infty} \mathrm{Im}\,\tau_{g,1}(k) \quad \text{and} \quad \mathfrak{h}_{g,1}^{+} := \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$$

have natural positively graded Lie algebra structures and

$$\tau_{g,1} := \bigoplus_{k=1}^{\infty} \tau_{g,1}(k) : \mathrm{Im}\,\tau_{g,1} \longrightarrow \mathfrak{h}_{g,1}^{+}$$

is a Lie algebra embedding.

Problem

Determine:

(I) the Lie subalgebra $\operatorname{Im}\tau_{g,1} = \bigoplus_{k=1}^{\infty} \operatorname{Im}\tau_{g,1}(k)$ of $\mathfrak{h}_{g,1}^{+}$.

(II) the abelianization

$$H_1(\mathfrak{h}_{g,1}^{+}) = \mathfrak{h}_{g,1}^{+}/[\mathfrak{h}_{g,1}^{+}, \mathfrak{h}_{g,1}^{+}] = \bigoplus_{k=1}^{\infty} H_1(\mathfrak{h}_{g,1}^{+})_k.$$

of $\mathfrak{h}_{g,1}^{+}$, where

$$\begin{cases} H_1(\mathfrak{h}_{g,1}^{+})_1 = \mathfrak{h}_{g,1}(1) \\ H_1(\mathfrak{h}_{g,1}^{+})_k = \mathfrak{h}_{g,1}(k) \Big/ \sum_{\substack{i+j=k \\ i,j \geq 1}} [\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(j)] \qquad (k \geq 2). \end{cases}$$

Remarks

- In the following, we consider the rational ($\mathbb{Q}$-)version:

$$H := H_1(\Sigma_g; \mathbb{Q}) = H_{\mathbb{Z}} \otimes \mathbb{Q}$$
$$\tau_{g,1} \otimes \mathbb{Q} : \mathrm{Im}\, \tau_{g,1} \otimes \mathbb{Q} \longrightarrow \mathfrak{h}_{g,1}^{+} \otimes \mathbb{Q}$$

For simplicity, we omit "$\otimes\mathbb{Q}$".

- $\mathfrak{h}_{g,1}^{+} = \mathrm{Der}^{+}(\mathcal{L}(H), \omega_0)$, the positive symplectic derivations.

- There are related theories in

$\mathrm{Aut}\, F_n$, Link theory, Number theory.

In this workshop, we shall see the relationship among them!

Johnson homomorphims up to degree 6

Tools I: Representation theory of $\mathrm{Sp}(2g,\mathbb{Q})$

- The actions of $\mathcal{M}_{g,1}$ on $\mathrm{Im}\,\tau_{g,1}$ and $\mathfrak{h}_{g,1}^+$ descend to those of $\mathrm{Sp}(2g,\mathbb{Z}) = \mathcal{M}_{g,1}/\mathcal{I}_{g,1} = \mathcal{M}_{g,1}[0]/\mathcal{M}_{g,1}[1]$.

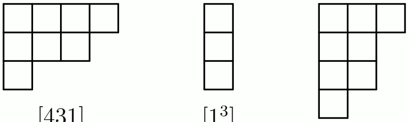
  $\rightsquigarrow$ We have an $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant embedding

$$\tau_{g,1} : \mathrm{Im}\,\tau_{g,1} \longrightarrow \mathfrak{h}_{g,1}^+.$$

- $\mathrm{Im}\,\tau_{g,1}(k)$ and $\mathfrak{h}_{g,1}(k)$ are finite dimensional $\mathrm{Sp}(2g,\mathbb{Q})$-module.
- As pointed out by Asada-Nakamura, $\tau_{g,1}$ is in fact an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant embedding.

Fact (Representations of $\mathrm{Sp}(2g,\mathbb{Q})$)

$$\left\{ \begin{array}{l} \text{Finite dimensional irreducible} \\ \text{polynomial representations} \\ \text{of } \mathrm{Sp}(2g,\mathbb{Q}) \end{array} \right\} \overset{\cong}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Young diagrams} \\ \text{w/ } \sharp(\text{rows}) \leq g \end{array} \right\}$$



$[431]$ $[1^3]$ $[32^21]$

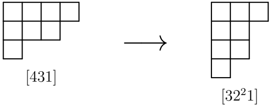
Example

$$\mathbb{Q} = [0] \quad \text{(trivial representation)},$$
$$H = [1] \quad \text{(fundamental representation)},$$
$$S^k H = [k],$$
$$\wedge^{2k} H = [1^{2k}] + [1^{2k-2}] + \cdots + [0],$$
$$\wedge^{2k+1} H = [1^{2k+1}] + [1^{2k-1}] + \cdots + [1].$$

Irreducible representation V_λ for the Young diagram λ.

Example For $\lambda = [431]$,

1. Take the transpose $\lambda' = [32^21]$:



2. V_λ is the minimum $\mathrm{Sp}(2g, \mathbb{Q})$-module containing

$$v_\lambda := (a_1 \wedge a_2 \wedge a_3) \otimes (a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes a_1$$

in

$$(\wedge^3 H) \otimes (\wedge^2 H) \otimes (\wedge^2 H) \otimes (\wedge^1 H).$$

v_λ is called the *highest weight vector* of V_λ.

Irreducible decomposition of $H^{\otimes k}$

Fact

Any irreducible subrepresentation V_λ in $H^{\otimes k}$ can be detected by a combination of

1. contractions $\mu_{i,j} : H^{\otimes n} \longrightarrow H^{\otimes(n-2)}$,
2. projections $\wedge^n : H^{\otimes n} \longrightarrow \wedge^n H$

as a quotient representation of $H^{\otimes k}$.

(Just detect the highest weight vector v_λ.)

In our setting $\mathfrak{h}_{g,1}^{+} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}(k)$,

- $\mathfrak{h}_{g,1}(k)$ is a finite dimensional $\mathrm{Sp}(2g,\mathbb{Q})$-module.
  $\Longrightarrow$ $\mathfrak{h}_{g,1}(k)$ has the irreducible decomposition.

- $\mathfrak{h}_{g,1}(k) \subset H \otimes \mathcal{L}_{k+1}(H) \subset H^{\otimes(k+2)}$: $\mathrm{Sp}(2g,\mathbb{Q})$-submodules.
  $\Longrightarrow$ The irreducible decomposition of $\mathfrak{h}_{g,1}(k)$ is obtained by combinations of contractions and projections in $H^{\otimes(k+2)}$.

- We may assume that g is sufficiently large ($g \geq 3k$).
  $\Longrightarrow$ The irreducible decomposition stabilizes.

Tools II: Hain's theory

Hain gave an infinitesimal presentation of $\mathcal{I}_g$ by using the Hodge theory (Mixed Hodge Structures). From this,

Theorem [Hain]

1. The Lie subalgebra $\operatorname{Im}\tau_{g,1}$ is generated by its degree 1 part $\operatorname{Im}\tau_{g,1}(1) = \mathfrak{h}_{g,1}(1) = \wedge^3 H$.

2. There exists an ideal $\mathfrak{j}_{g,1} = \bigoplus_{k=1}^{\infty} \mathfrak{j}_{g,1}(k)$ in $\mathfrak{h}_{g,1}^{+}$ such that

$$\mathfrak{j}_{g,1}(k) \cap \operatorname{Im}\tau_{g,1}(k) = \{0\} \qquad \text{for all } k \geq 3.$$

Precisely speaking,

$$\begin{aligned}\mathfrak{j}_{g,1}(k) &:= \operatorname{Ker}(\mathfrak{h}_{g,1}(k) \twoheadrightarrow \mathfrak{h}_{g,*}(k)) \\ &= \operatorname{Ker}\left(H \otimes (\mathcal{L}_{k+1}(H)/\langle \omega_0 \rangle_{k+1}) \xrightarrow{[\cdot\,,\cdot]} (\mathcal{L}_{k+2}(H)/\langle \omega_0 \rangle_{k+2}) \right).\end{aligned}$$

Remarks

- Our problem (I) is equivalent to:

Problem

(I') Determine the Lie subalgebra of $\mathfrak{h}_{g,1}^+$ generated by its degree 1 part $\mathfrak{h}_{g,1}(1) = \operatorname{Im}\tau_{g,1}(1) = \wedge^3 H$.

- $\operatorname{Im}\tau_{g,1}(k) \subset \operatorname{Ker}\big(\mathfrak{h}_{g,1}(k) \to H_1(\mathfrak{h}_{g,1}^+)_k\big)$ for $k \geq 2$.

  (i.e. $H_1(\mathfrak{h}_{g,1}^+)_k \subset \mathfrak{h}_{g,1}(k)/\operatorname{Im}\tau_{g,1}(k)$ as $\mathrm{Sp}(2g,\mathbb{Q})$-module.)

Tools III: Trace maps and Enomoto-Satoh's obstruction

Theorem [Morita] (trace map)

For $k \geq 2$, the composition

$$\mathrm{Tr}_{2k-1} : \mathfrak{h}_{g,1}(2k-1) \subset H \otimes \mathcal{L}_{2k}(H) \hookrightarrow H^{\otimes(2k+1)} \xrightarrow{\mu_{1,2}} H^{\otimes(2k-1)} \xrightarrow{\text{proj}} S^{2k-1}H$$

gives

$$S^{2k-1}H = [2k-1] \subset H_1(\mathfrak{h}_{g,1}^{+})_{2k-1}.$$

(i.e. Tr_{2k-1} is a non-trivial homomorphism vanishing on brackets.)

In particular, $\mathrm{Im}\,\tau_{g,1}(2k-1) \subset \mathrm{Ker}\,\mathrm{Tr}_{2k-1}$.

Enomoto-Satoh's obstruction

Theorem [Enomoto-Satoh]

For $k \geq 2$, consider the composition

$$\mathrm{ES}_k : \mathfrak{h}_{g,1}(k) \subset H \otimes \mathcal{L}_{k+1}(H) \hookrightarrow H^{\otimes(k+2)} \xrightarrow{\mu_{1,2}} H^{\otimes k} \xrightarrow{\mathrm{proj}} \left(H^{\otimes k}\right)_{\mathbb{Z}/k\mathbb{Z}},$$

where $\mathbb{Z}/k\mathbb{Z} \curvearrowright H^{\otimes k}$ is given by the cyclic permutation. Then

$$\operatorname{Im} \tau_{g,1}(k) \subset \operatorname{Ker} \mathrm{ES}_k.$$

$\rightsquigarrow \quad \operatorname{Im} \mathrm{ES}_k \subset \mathfrak{h}_{g,1}(k) / \operatorname{Im} \tau_{g,1}(k).$

We call the map ES_k the ES-obstruction.

It is essentially the same as the divergence cocycle by Alekseev-Torossian.

Tools IV: Relation with number theory

In 1980's, Oda predicted:

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ should "appear" in $(\mathrm{Coker}\,\tau_g)^{\mathrm{Sp}} \otimes \mathbb{Z}_p$ (p:prime).

Nakamura, Matsumoto: proof and related many works.

"Encounter with the Galois obstruction!"
(The first one appears in $\tau_g(6)$.)

Problem

Describe the Galois image explicitly.

- Earlier foundational works for $g = 0$: Ihara, Deligne.
- More recent works for $g = 1$: Hain-Matsumoto, Nakamura.

Johnson homomorphisms up to degree 6

(I) Previously known facts on $\mathrm{Im}\,\tau_{g,1} \subset \mathfrak{h}_{g,1}^{+}$ (up to degree 4):

Fact

- $\mathrm{Im}\,\tau_{g,1}(1) = \mathfrak{h}_{g,1}(1) = \wedge^3 H = [1^3] + [1]$ (Johnson),
- $\mathrm{Im}\,\tau_{g,1}(2) = \mathfrak{h}_{g,1}(2) = [2^2] + [1^2] + [0]$ (Hain, Morita),
- $\mathrm{Im}\,\tau_{g,1}(3) = [31^2] + [21] \subsetneqq \mathfrak{h}_{g,1}(3) = [31^2] + [21] + [3]$ (Hain, Asada-Nakamura),
- $\mathrm{Im}\,\tau_{g,1}(4) = [42] + [31^3] + 2[31] + [2^3] + [21^2] + 2[2]$

  $\subsetneqq \mathfrak{h}_{g,1}(4) = [42] + [31^3] + 2[31] + [2^3] + 2[21^2] + 3[2]$ (Morita).

(II) Previously known facts on $H_1(\mathfrak{h}_{g,1}^+)_k$ (up to degree 4):

Fact

- By definition $H_1(\mathfrak{h}_{g,1}^+)_1 = \mathfrak{h}_{g,1}(1) = [1^3] + [1]$.
- Arguments using Trace map give

$$H_1(\mathfrak{h}_{g,1}^+)_2 = 0, \quad H_1(\mathfrak{h}_{g,1}^+)_3 \cong S^3H = [3], \quad H_1(\mathfrak{h}_{g,1}^+)_4 = 0.$$

Degree 5

Theorem 1. [Morita-Suzuki-S. 2011] w/ a correction by Enomoto

- $$\begin{aligned}\mathrm{Im}\,\tau_{g,1}(5) = &([51^2]+[421]+[3^21]+[321^2]+[2^21^3]) \\ &+(2[41]+2[32]+2[31^2]+2[2^21]+2[21^3]) \\ &+([3]+3[21]+2[1^3])+[1].\end{aligned}$$
- $$\begin{aligned}\mathfrak{h}_{g,1}(5)/\,\mathrm{Im}\,\tau_{g,1}(5) = &([5]+[32]+[2^21]+[1^5]) \\ &+(2[21]+2[1^3])+2[1].\end{aligned}$$
  (completely detected by ES-obstruction)
- $H_1(\mathfrak{h}_{g,1}^+)_5 \cong S^5H = [5]$. (only the trace component)

Proof: Computer calculation + ES-obstruction + trace map.

Degree 6

Theorem 2. [MSS. 2011]

- $$\begin{aligned}\mathrm{Im}\,\tau_{g,1}(6) = {} & ([62]+[521]+[51^3]+[4^2]+[431]+2[42^2]+[421^2] \\ & +[41^4]+2[3^21^2]+[32^21]+[321^3]+[2^4]+[2^21^4]) \\ & +(3[51]+3[42]+4[41^2]+3[3^2]+7[321]+3[31^3] \\ & +[2^3]+5[2^21^2]+2[21^4]+[1^6]) \\ & +(4[4]+6[31]+9[2^2]+6[21^2]+4[1^4]) \\ & +(3[2]+6[1^2])+2[0].\end{aligned}$$

Theorem 2 (continue).

- $$\mathfrak{h}_{g,1}(6)/\operatorname{Im}\tau_{g,1}(6) = (2[41^2]+[3^2]+[321]+[31^3]+[2^21^2])$$
$$+(2[4]+3[31]+3[2^2]+3[21^2]+2[1^4])$$
$$+([2]+5[1^2])+3[0],$$

  in which the ES-obstruction cannot detect $[1^4]+[1^2]+[0]$.

Proof: Theoretical consideration + computer calculations

- $[1^4]+[1^2]$: Two proofs by
  (1) Checking all patterns of brackets.
  (2) Finding a component in the ideal $\mathfrak{j}_{g,1}(6)$ outside of $\operatorname{Ker}\mathrm{ES}_6$.
- $[0]$: The Galois obstruction w/ explicit description.
  (Find a normalizer of $\operatorname{Im}\tau_{g,1}$ outside of $\operatorname{Ker}\mathrm{ES}_6$.)

Abelianization of $H_1(\mathfrak{h}_{g,1}^+)$ (in progress)

Problem (bis)

(II) Determine the abelianization $H_1(\mathfrak{h}_{g,1}^+) = \bigoplus_{k=1}^{\infty} H_1(\mathfrak{h}_{g,1}^+)_k$ of $\mathfrak{h}_{g,1}^+$.

Background of (II): Kontsevich's theorem says:

Theorem [Kontsevich]

For any $n \geq 1$ and $k \geq 1$, there exists an isomorphism

$$PH_n\big(\mathfrak{h}_{\infty,1}^+\big)_{2k}^{\mathrm{Sp}} \cong H^{2k-n}(\mathrm{Out}(F_{k+1});\mathbb{Q}),$$

where $\mathfrak{h}_{\infty,1}^+ := \lim_{g\to\infty} \mathfrak{h}_{g,1}^+$.

$\rightsquigarrow$ $H_1\big(\mathfrak{h}_{\infty,1}^+\big)_{2k}^{\mathrm{Sp}} \cong H^{2k-1}(\mathrm{Out}(F_{k+1});\mathbb{Q})$ for any $k \geq 1$.

Morita once conjectured that

$$\text{The trace components } \bigoplus_{k=1}^{\infty}[2k+1] \text{ gave } H_1(\mathfrak{h}_{g,1}^{+}).$$

However, in 2011, Conant-Kassabov-Vogtmann disproved it:

Theorem [Conant-Kassabov-Vogtmann]

There exist much more components other than the trace components $\bigoplus_{k=1}^{\infty}[2k+1]$ in $H_1(\mathfrak{h}_{g,1}^{+})$:

1-loop part (=trace components), 2-loops part, 3-loops part, …

They described the 2-loops part in terms of the Eichler-Shimura isomorphism in the theory of modular forms.

Conant showed that the 3-loops part is non-trivial.

Motivated by their results, we obtained explicit descriptions for (a part of) their new components of $H_1(\mathfrak{h}_{g,1}^+)$:

Theorem 3. [MSS. 2011]

1. $H_1(\mathfrak{h}_{g,1}^+)_6 = [31]$. ($\supset$ was first proved by CKV)

2. For $k \geq 3$, the composition

$$\begin{aligned} H \otimes \mathcal{L}_{2k+1}(H) \hookrightarrow H^{\otimes(2k+2)} &\xrightarrow{\mu_{1,3} \circ \mu_{4,2k+1}} H^{\otimes(2k-2)} \\ &\xrightarrow{\wedge_{1,(2k-2)}} H^{\otimes(2k-4)} \otimes \wedge^2 H \\ &\xrightarrow{\mathrm{proj} \otimes \mathrm{id}} S^{2k-4} H \otimes \wedge^2 H \end{aligned}$$

gives

$$[(2k-3)\,1] \subset H_1(\mathfrak{h}_{g,1}^+)_{2k}.$$

Proof: Combinatorial argument w/o using computer.

$\mathrm{Out}(F_7)$ and $H_1(\mathfrak{h}_{g,1}^+)_{12}^{\mathrm{Sp}}$

- Bartholdi (2015) showed

$$H^p(\mathrm{Out}(F_7);\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & (p = 0, 8, 11) \\ 0 & (\text{otherwise}) \end{cases}$$

  with the aid of computers.
  (Need to compute the rank of a 2038511×536647 matrix)

- $H^{11}(\mathrm{Out}(F_7);\mathbb{Q}) \cong \mathbb{Q}$ is remarkable because it is the first non-trivial odd and (virtually) top rational cohomology group which is explicitly described.

- By theorems of Kontsevich and Bartholdi, we have

$$H_1\left(\mathfrak{h}_{\infty,1}^+\right)_{12}^{\mathrm{Sp}} \cong H^{11}(\mathrm{Out}(F_7);\mathbb{Q}) \cong \mathbb{Q}.$$

Theorem 4. [MSS. 2016] Direct computation of $H_1(\mathfrak{h}_{\infty,1}^+)_{12}^{\mathrm{Sp}}$

There exists an $\mathrm{Sp}(2g,\mathbb{Q})$-invariant linear map

$$C : \mathfrak{h}_{g,1}(12) \longrightarrow \mathbb{Q}$$

satisfying that

- C is non-trivial for any $g \geq 2$,
- the restriction of C to $\displaystyle\sum_{i=1}^{11}[\mathfrak{h}_{g,1}(i), \mathfrak{h}_{g,1}(12-i)]$ is trivial.

That is, the cocycle C gives a surjection

$$\widetilde{C} : H_1(\mathfrak{h}_{g,1}^+)_{12}^{\mathrm{Sp}} \twoheadrightarrow \mathbb{Q}$$

for every $g \geq 2$. Moreover $\widetilde{C}$ is an isomorphism for $g \geq 8$.

- Since $H_1(\mathfrak{h}_{1,1}^+)_{12}^{\mathrm{Sp}} = 0$, our bound of genus for the non-triviality of $H_1(\mathfrak{h}_{g,1}^+)_{12}^{\mathrm{Sp}}$ is best possible.

Method for computation of $H_1(\mathfrak{h}_{g,1}^+)_{12}^{\mathrm{Sp}}$

Our computation also uses computers.

1. Find a coordinate system of $\mathfrak{h}_{g,1}(12)^{\mathrm{Sp}} \cong \mathbb{Q}^{650}$.
2. Compute the bracket map

$$[\,\cdot\,,\,\cdot\,] : \left(\bigoplus_{i=1}^{6} \big(\mathfrak{h}_{g,1}(i) \otimes \mathfrak{h}_{g,1}(12-i)\big)\right)^{\mathrm{Sp}} \longrightarrow \mathfrak{h}_{g,1}(12)^{\mathrm{Sp}}.$$

   We see that the image includes a subspace $W \cong \mathbb{Q}^{649}$.

3. Find a linear map $C : \mathfrak{h}_{g,1}(12)^{\mathrm{Sp}} \twoheadrightarrow \mathbb{Q}$ which annihilates W.
4. Check that C is trivial on the image of the bracket map.

Relationship with the Enomoto-Satoh map

Proposition 5. [MSS. 2016]

For $g \geq 6$, the $\mathrm{Sp}(2g,\mathbb{Q})$-invariant cocycle $C : \mathfrak{h}_{g,1}(12) \to \mathbb{Q}$ factors through the Enomoto-Satoh map

$$ES_{12} : \mathfrak{h}_{g,1}(12) \hookrightarrow H \otimes \mathcal{L}_{13}(H) \hookrightarrow H^{\otimes 14} \xrightarrow{\mu\otimes(\mathrm{id}^{\otimes 12})} H^{\otimes 12} \longrightarrow \left(H^{\otimes 12}\right)_{\mathbb{Z}/12\mathbb{Z}}.$$

- This theorem provides another description of the map C in the form
$$C = C' \circ ES_{12}$$
with C' described by chord diagrams with 6 chords, which serve as coordinate functions of $\left(H^{\otimes 12}\right)^{\mathrm{Sp}}_{\mathbb{Z}/12\mathbb{Z}} \cong \mathbb{Q}^{897}$.

Topological setting

- Σ_g: a connected oriented closed surface of genus $g \geq 3$
- $\mathcal{M}_g$: the mapping class group of Σ_g
- $H^{2i}(\mathcal{M}_g;\mathbb{Q}) \ni e_i$: the i-th MMM tautological class
- $\mathcal{R}^*(\mathcal{M}_g) =$ subalgebra of $H^*(\mathcal{M}_g;\mathbb{Q})$ generated by e_i's
  : the tautological algebra in cohomology of $\mathcal{M}_g$
- Stably, $H^*(\mathcal{M}_\infty;\mathbb{Q}) \cong \mathcal{R}^*(\mathcal{M}_\infty) \cong \mathbb{Q}[e_1, e_2, \ldots]$
  (by Madsen-Weiss)
- When g is in the unstable range, there are many relations among e_i's.

From Algebraic geometry

- $\mathbf{M}_g$: the moduli space of Riemann surfaces of genus g
- $H^*(\mathbf{M}_g;\mathbb{Q}) \cong H^*(\mathcal{M}_g;\mathbb{Q})$
- $\mathcal{A}^*(\mathbf{M}_g)$: the Chow algebra of $\mathbf{M}_g$
- $\mathcal{A}^i(\mathbf{M}_g) \ni \kappa_i$: the i-th Mumford kappa class
- $\mathcal{R}^*(\mathbf{M}_g) =$ subalgebra of $\mathcal{A}^*(\mathbf{M}_g)$ generated by κ_i's

  : the tautological algebra of $\mathbf{M}_g$
- We have a canonical surjection

$$\mathcal{R}^*(\mathbf{M}_g) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{R}^{2*}(\mathcal{M}_g) \quad (\kappa_i \longmapsto (-1)^{i+1} e_i)$$

as the restriction of $\mathcal{A}^*(\mathbf{M}_g) \to H^{2*}(\mathbf{M}_g;\mathbb{Q}) \cong H^{2*}(\mathcal{M}_g;\mathbb{Q})$.

Faber's conjecture

- In 1993, Faber gave a series of conjectures concerning the structure of $\mathcal{R}^*(\mathbf{M}_g)$.
- After that, many results have been given by many people.
  - $\mathcal{R}^{g-2}(\mathbf{M}_g) \cong \mathbb{Q}$ (Looijenga + Faber)
  - $\mathbb{Q}[\kappa_1, \ldots, \kappa_{\lfloor g/3 \rfloor}] \twoheadrightarrow \mathcal{R}^*(\mathbf{M}_g)$ (Morita for $\mathcal{R}^*(\mathcal{M}_g)$, Ionel) and no relations in $\mathcal{R}^{\leq \lfloor g/3 \rfloor}(\mathbf{M}_g)$ (Harer, Ivanov, et.al).
  - An explicit formula for the intersection numbers (Givental, Liu-Xu, Buryak-Shadrin)
- The following remains open:

Conjecture (Faber's Gorenstein conjecture)

$$\mathcal{R}^*(\mathbf{M}_g) \cong H^*(\text{smooth projective variety of } \dim = g-2\,; \mathbb{Q}),$$

in particular, Poincaré duality holds? (verified for $g \leq 23$ by Faber)

Johnson homomorphisms and tautological algebra

- $\mathrm{Sp} := \mathrm{Sp}(2g, \mathbb{Q}) \curvearrowright H = H_1(\Sigma_g; \mathbb{Q})$ preserving μ
- $U := \wedge^3 H/H =$ irrep. $[1^3]_{\mathrm{Sp}} \cong H_1(\mathcal{I}_g; \mathbb{Q})$
- The extended Johnson homomorphism (by Morita)

$$\rho_1 : \mathcal{M}_g \longrightarrow U \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

induces $\Phi := \rho_1^* : \left(\wedge^* U/([2^2]_{\mathrm{Sp}})\right)^{\mathrm{Sp}} \to H^*(\mathcal{M}_g; \mathbb{Q})$.

- Sp-invariant tensors in $(\wedge^* U)^{\mathrm{Sp}}$ can be described by trivalent graphs and $[2^2]_{\mathrm{Sp}}$ corresponds to Whitehead moves.

Theorem [Kawazumi-M. 1996]

$$\mathrm{Im}\,\Phi = \mathcal{R}^*(\mathcal{M}_g) = \mathbb{Q}[\text{MMM-classes}]/\text{relations}$$

- A similar result holds for $\rho_1 : \mathcal{M}_{g,1} \to \wedge^3 H \rtimes \mathrm{Sp}(2g,\mathbb{Q})$, where $\wedge^3 H = U \oplus H = [1^3]_{\mathrm{Sp}} + [1]_{\mathrm{Sp}} = [1^3]_{\mathrm{GL}}$.
$$(\mathrm{GL} := \mathrm{GL}(2g,\mathbb{Q}))$$

- We have

$$\Phi : \left(\wedge^* U/([2^2]_{\mathrm{Sp}})\right)^{\mathrm{Sp}} \longrightarrow\!\!\!\!\rightarrow \mathcal{R}^*(\mathcal{M}_g),$$
$$\Phi : \left(\wedge^* (\wedge^3 H)\right)^{\mathrm{Sp}} \longrightarrow\!\!\!\!\rightarrow \mathcal{R}^*(\mathcal{M}_{g,1}).$$

- In the unstable range (i.e. g is small), we have many relations in $\left(\wedge^* U/([2^2]_{\mathrm{Sp}})\right)^{\mathrm{Sp}}$ and $\left(\wedge^* (\wedge^3 H)\right)^{\mathrm{Sp}}$ as degenerations of Sp-invariant tensors.

- Using the first degenerations, Morita proved that

$$\mathbb{Q}[e_1, \ldots, e_{\lfloor g/3 \rfloor}] \longrightarrow\!\!\!\!\rightarrow \mathcal{R}^*(\mathcal{M}_g).$$

We want to understand the structures of

$$\wedge^* U = \wedge^*[1^3]_{\mathrm{Sp}}, \quad \wedge^*[1^3]_{\mathrm{Sp}}/([2^2]_{\mathrm{Sp}}), \quad \wedge^*(\wedge^3 H) = \wedge^*[1^3]_{\mathrm{GL}}, \quad \cdots$$

Plethysm: composition of two Schur functors

- Littlewood determined, by explicit formulas, the plethysms

$$S^*(S^2 H), \ \ \wedge^*(S^2 H), \ \ S^*(\wedge^2 H), \ \ \wedge^*(\wedge^2 H).$$

- Determination of plethysm is very difficult in general.

Theorem [Manivel, –1994]

Plethysm $S^k(S^l H)$ "stabilizes" (M-stabilizes) as $k \to \infty$, in particular the M-stable decomposition of $S^\infty(S^3 H)$ is given by

$$S^*(S^2 H \oplus S^3 H).$$

We apply involution on symmetric polynomials: $H_k H_3 \overset{\text{dual}}{\Longleftrightarrow} E_k E_3$

Proposition 6. [MSS, 2014]

Let

$$\wedge^k(\wedge^3 H) = \wedge^k[1^3]_{\mathrm{GL}} = \bigoplus_{\lambda, |\lambda|=3k} m_\lambda \lambda_{\mathrm{GL}}$$

be the stable irreducible GL-decomposition. Then, for any k, the mapping

$$\wedge^k(\wedge^3 H)_{\text{irrep.}} \longrightarrow \wedge^{k+1}(\wedge^3 H)_{\text{irrep.}}$$

induced by the operation $\lambda \mapsto \lambda^+ = [\lambda 1^3]$ is injective and bijective for the part λ^+_{GL} with $2k \leq h(\lambda) \leq 3k$, namely

$$m_\lambda \begin{cases} \leq m_{\lambda^+} & \\ = m_{\lambda^+} & (2k \leq h(\lambda) \leq 3k) \end{cases}$$

Theorem 7. [MSS, 2014]

We have determined the M-stable irreducible decomposition of $\wedge^\infty[1^3]_{\mathrm{GL}}$ and its Sp-invariant part $\left(\wedge^\infty[1^3]_{\mathrm{GL}}\right)^{\mathrm{Sp}}$ up to codimension 30.

Table : M-stable irreducible decomposition of $\wedge^\infty[1^3]_{\mathrm{GL}}$

cod.	irreducible decomposition
0	$[1^*]$
1	$[21^*]$
2	$[2^21^*]$
3	$[2^31^*]$
4	$[2^41^*][3^21^*]$
5	$[2^51^*][32^31^*][3^221^*]$
6	$2[2^61^*]2[3^22^21^*][4^21^*]$
7	$[2^71^*][32^51^*]2[3^22^31^*][3^321^*][432^21^*][4^221^*]$

Number of relations in $\mathcal{R}^*(\mathbf{M}_g)$

$$\mathcal{R}^*(\mathbf{M}_g) \to \mathcal{R}^{2*}(\mathcal{M}_g) \to G^*(\mathbf{M}_g) \text{ (Gorenstein quotient)}$$

Expectation [Faber-Zagier, based on Faber-Zagier relations]

The number

$$p(k) - \dim G^k(\mathbf{M}_g) = \text{number of relations of codimension } k$$

depends only on $\ell = 3k - 1 - g$ in the range $2k \leq g - 2$ (i.e. $k \geq \ell + 3$), and is given by

$$a(\ell) := \sharp \left\{ \begin{array}{l} \text{Partitions of } \ell \text{ with parts:} \\ 1, 2, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, \ldots \\ (n \neq 2 \text{ is excluded if } n \equiv 2 \bmod 3) \end{array} \right\}.$$

Bergvall, Faber, Yin gave similar expectations for $\mathbf{M}_{g,*}$.

We have the following theorem which might serve as a supporting evidence for the above expectation.

Theorem 8. [MSS, 2014]

The number

$$\tilde{a}(\ell) := p(k) - \dim\left(\wedge^{2k}U/([2^2]_{\mathrm{Sp}})\right)^{\mathrm{Sp}}$$

depends only on $\ell = 3k-1-g$ in the same range

$$2k \leq g-2 \qquad \text{(i.e. } k \geq \ell+3\text{)}.$$

- We have a similar result for $\mathbf{M}_{g,*}$.
- More precise results by using a canonical metric on $\left(\wedge^{2k}U\right)^{\mathrm{Sp}}$ are given.

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Fin.