On the cohomology of moduli of hyperelliptic curves

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Introduction

- My study is the homotopy theory.

- The results I will talk today were essentially obtained in 2006 (so this is not new), and I gave a talk on it in 2007 at John Hopkins university.

- First I expected to have applications on the stable homotopy theory, however, I have not done it. That is the reason why this work has not been published yet.

In the homotopy theory, one of the most important purpose is to know about the homotopy groups of spaces with finite structure (like CW-complex). In particular, the computations of the stable homotopy groups of spheres is crucial.

To compute the homotopy groups of spaces, J.F.Adams introduced the spectral sequence in 1958.

$$E_2^{*,*} = \mathrm{Ext}_{\mathcal{A}}^{*,*}(H^*(X), \mathbb{Z}/p) \quad \Longrightarrow \quad \pi_*(X) \otimes \mathbb{Z}_p.$$

This is based on the cohomology theory and the action of Steenrod algebra. In 1967 Novikov introduced another spectral sequence based on the cobordims theory

$$E_2^{*,*} = \mathrm{Ext}_{MU^*MU}^{*,*}(MU^*(X), MU^*) \quad \Longrightarrow \quad \pi_*(X).$$

In general, when we have a cohomology theory and the ring of operations, we can construct the similar spectral sequence.
Because of the technical reason, we usually consider the spectral sequences based on the homology theories.

Let E be a ring spectrum (this is a "ring" in the stable homotopy category, each of which has a product called "smash product" $\wedge$). Then we have a homology theory $E_*(-)$ and the associated *E-based spectral sequence*

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{E_*(E)}(E_*, E_*(X)) \quad \Longrightarrow \quad \pi_*(L_E X)$$

where

$$E_* := \pi_*(E) \qquad \text{and} \qquad E_*(X) := \pi_*(E \wedge X),$$

and $L_E X$ is an object which has E-theoretical information of X, called the *Bousfield localization* of X with respect to E.

The key ingredient to compute the E_2-terms of Adams-type spectral sequence is the Hopf algebroid structure of the pair $(E_*, E_*(E))$ and the comodule structure of $E_*(X)$.

Roughly speaking, Hopf algebroid is "Hopf algebra with two unit map".

Definition

A *Hopf algebroid* is a pair (A, Γ) of algebras over a commutative ring which has the structure maps

- left and right unit $\quad \eta_L, \eta_R : A \to \Gamma$
- coproduct $\quad \Delta : \Gamma \to \Gamma \otimes_A \Gamma$
- counit $\quad \varepsilon : \Gamma \to A$
- conjugation $\quad c : \Gamma \to \Gamma$

which fit into some commutative diagrams (similar to those of Hopf algebra).

This is the central player in the today's talk.

Comments

- I was studying the Hopf algebroid arised from the cobordism theory localized at a prime p (this is called the Brown-Peterson theory). It requires a **huge amount of calculations**.

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*(X)) \quad \Longrightarrow \quad \pi_*(X).$$

- The E_2-terms are computed by the chromatic method, conceptually based on the theory of the formal group. The 1st line was computed by Novikov and it is related to the image of J-homomorphism.

- The 2nd line was computed by Miller-Ravenel-Wilson in early 70's.

- We have a little knowledge of the 3rd line (I made some calculations 20 years ago), and the higher lines have not been computed yet.

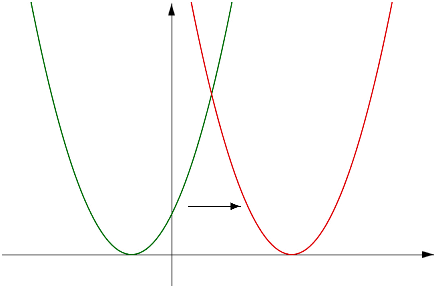
- On the other hand, the Hopf algebroid we will consider today is relatively treatable.

Groupoid and Hopf algebroid

Recall that a *groupoid* is the category in which all morphisms are invertible. A Hopf algebroid is a *co-groupoid object*, which means that for each ring R the pair of ring homomorphisms $(\mathsf{Hom}(A, R), \mathsf{Hom}(\Gamma, R))$ forms objects and morphisms of a groupoid respectively.

Toy example. Let $\mathcal{C}$ be a category defined by

$$\mathrm{Ob}(\mathcal{C}) = (\text{monic quadratic functions})$$
$$\mathrm{Mor}(\mathcal{C}) = (\text{parallel translations})$$



$\mathcal{C}$ is in fact a groupoid, since all morphisms are obviously invertible.

Then we can have an associated Hopf algebroid. Assume that we have appropriate coordinate and consider the monic function $f(x) = x^2 + ax + b$ and the change of variables

$$f(x + r) = x^2 + (a + 2r)x + (b + ar + r^2)$$

If we regard this coordinate change as $x \mapsto x \otimes 1 + 1 \otimes r$ then doing it twice gives

$$\begin{aligned} x \longmapsto x \otimes 1 + 1 \otimes r \longmapsto (x \otimes 1 + 1 \otimes r) \otimes 1 + 1 \otimes 1 \otimes r \\ = x \otimes 1 \otimes 1 + 1 \otimes (r \otimes 1 + 1 \otimes r) \end{aligned}$$

Then we have

Proposition

Set $A = \mathbb{Z}[a, b]$ and $\Gamma = A[r]$. Then the pair (A, Γ) is a Hopf algebroid with the structure maps (η_L and ε is given in the obvious way).

$$\eta_R(a) = a + 2r, \quad \eta_R(b) = b + ar + r^2, \quad \Delta(r) = r \otimes 1 + 1 \otimes r, \quad c(r) = -r$$

Remark. This example is known to give a stack related to bo, the (-1)-connected cover of KO.

Next we need the following definition.

Definition

Given a Hopf algebroid (A, Γ), a Γ-*comodule* M is an A-module M together with A-linear map $\psi_M : M \to M \otimes_A \Gamma$ which is counitary and coassociative.

$$
\begin{tikzcd}
M \arrow[r, "\psi_M"] \arrow[dr, equal] & \Gamma \otimes_A M \arrow[d, "\varepsilon"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[rr, "\varepsilon"] \arrow[d, "\varepsilon"] & & \Gamma \otimes_A M \arrow[d, "\Delta \otimes id_M"] \\
\Gamma \otimes_A M \arrow[rr, "id_\Gamma \otimes \psi_M"] & & \Gamma \otimes_A \Gamma \otimes_A M
\end{tikzcd}
$$

For a Γ-comodule M and N, $\mathrm{Hom}_\Gamma(M, N)$ is the set of comodule maps from M to N. Then, $\mathrm{Ext}^n_\Gamma(A, M)$ is defined as the n-th right derived functor of $\mathrm{Hom}_\Gamma(M, N)$.

The motivating example is the followings.

Example

The homology group $M = E_*(X)$ is a comodule over the Hopf algebroid $(A, \Gamma) = (E_*, E_*(E))$, and

$$\mathrm{Ext}^*_\Gamma(A, M) = \mathrm{Ext}^*_{E_*(E)}(E_*, E_*(X))$$

is the E_2-terms of E-based Adams spectral sequence, as we see before.

Q. How can we compute $\mathrm{Ext}^*_\Gamma(A, M)$?

A. $\mathrm{Ext}^n_\Gamma(A, M)$ is isomorphic to the cohomology of the cobar complex

$$C^n(M) := M \otimes_A \underbrace{\Gamma \otimes_A \cdots \otimes_A \Gamma}_{n\text{-factors}}.$$

The differentials are given by

$$d(m \otimes \gamma_1 \otimes \cdots \gamma_n) = \psi_M(m) \otimes \gamma_1 \otimes \cdots \gamma_n + (-1)^{n+1} m \otimes \gamma_1 \otimes \gamma_n \otimes 1 + \sum_{k=1}^{n} (-1)^k m \otimes \gamma_1 \otimes \cdots \otimes \Delta(\gamma_k) \otimes \cdots \gamma_n$$

The theory of topological modular form

(by Hopkins et al)

I want to recall the theory of topological modular form, which was established by Mike Hopkins and his coworkers.

Recall that the (original) Weierstrass equation was

$$y^2 + (a_1 x + a_3)y = x^3 + a_2 x^2 + a_4 x + a_6.$$

Define elements b_{2i} $(i = 1, 2, 3)$, c_{2j} $(j = 2, 3)$ and the discriminant Δ by

$$b_2 = a_1^2 + 4a_2, \qquad b_4 = 2a_4 + a_1 a_3, \qquad b_6 = a_3^2 + 4a_6,$$

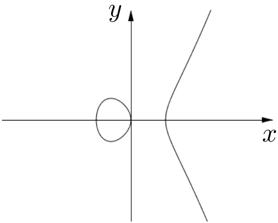
$$c_4 = b_2^2 - 24b_4, \qquad c_6 = -b_2^3 + 36b_2 b_4 - 216b_6,$$

$$\Delta = \frac{c_4^3 - c_6^2}{1728}$$

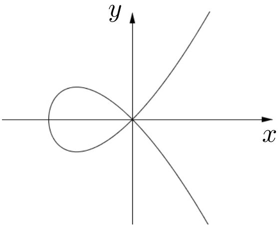
In particular, we can know the type of singularity of curves by Δ and c_4.

It is known that the curve defined by the above equation is

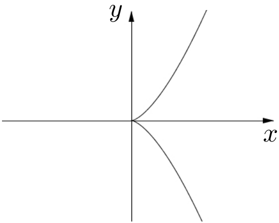
- a smooth elliptic curve when $\Delta \neq 0$
- a curve with nodal singularity when $\Delta = 0$ and $c_4 \neq 0$
- a curve with cuspidal singularity when $\Delta = 0 = c_4$



$y^2 = x^3 - x$
(elliptic)



$y^2 = x^3 + x^2$
(nodal)



$y^2 = x^3$
(cuspidal)

When we consider the coordinate change

$$\begin{cases} x \mapsto x + r \\ y \mapsto y + sx + t \end{cases}$$

Then we have the Hopf algebroid (A, Γ) defined by

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \qquad \text{and} \qquad \Gamma = A[r, s, t].$$

Then, $\mathrm{Ext}^*_\Gamma(A, A)$ is computed by Hopkins et al. In particular, they showed that there is a spectral sequence

$$E_2^{*,*} = \mathrm{Ext}^*_\Gamma(A, A) \quad \Longrightarrow \quad \pi_*(\mathrm{tmf})$$

where tmf is the spectrum representing (the connective version of) the cohomology theory, so-called the topological modular forms. The name came from the fact that the 0-th line

$$\mathrm{Ext}^{0,*}_\Gamma(A, A) \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$$

is isomorphic to the ring of integral modular form.

Comments

- There is a periodic version of tmf, denoted by TMF. This is considered as "the universal elliptic cohomology theory". The E_2-terms of the spectral sequence converging to the coefficient group $\pi_*(TMF)$ can be computed by the Hopf algebroid
$$(A[\Delta^{-1}], \Gamma[\Delta^{-1}]).$$
Then the associated stack $\mathcal{M}_{ell}$ represents smooth elliptic curves.

- There is also another version. If we invert c_4, then we have a Hopf algebroid
$$(A[c_4^{-1}], \Gamma[c_4^{-1}]).$$
The associated stack $\overline{\mathcal{M}}_{ell}$ is the Deligne-Mumford compactification of $\mathcal{M}_{ell}$, and it represent curves which may have the nodal singularity. The corresponding cohomology theory is denoted by Tmf.

Hyperelliptic Hopf algebroid

P.Lockhart considered a model for hyperelliptic curves in his paper :

"On the discriminant of a hyperelliptic curve."
Trans. Amer. Math. Soc. **342** (1994), no. 2, 729–752.

Let C be a hyperelliptic curve over a field k with genus g with a chosen Weierstrass point. Then he showed

Proposition (P.Lockhart, 1994)

We can choose coordinates x and y such that these satisfy

$$y^2 + q(x)y = p(x), \quad \textit{generalized Weierstrass equation (GWE)}$$

where p is monic polynomials of degree $2g+1$ and q is satisfying $\deg(q) \leq g$. Moreover, such an equation is unique up to a change of coordinates of the form

$$x \mapsto \lambda^2 x + r, \qquad y \mapsto \lambda^{2g+1} y + t(x)$$

where $\lambda \in k^\times$, $r \in k$ and t is a polynomial over k of degree $\leq g$.

He also showed

Theorem (P.Lockhart,1994)

The hyperelliptic discriminant for genus g

$$\Delta_g = 2^{4g}\mathrm{Disc}\left(p(x) + \frac{q(x)}{4}\right)$$

is an irreducible polynomial with coefficients in $\mathbb{Z}$*, and the hyperelliptic curve defined by* GWE *is singular if and only if* $\Delta_g = 0$ *for any field* k*.*

Remark. Lockhart also mentioned that this discriminant (over $\mathbb{C}$) is expressed in terms of Siegel modular forms, and he examined a hyperelliptic generalization of a Szpiro conjecture. Today, we will not comment on these.

Express $p(x)$ and $q(x)$ in the equation, and $t(x)$ in the coordinate change:

$$p(x) = x^{2g+1} + \sum_{k=1}^{2g+1} a_{2k}\, x^{2g+1-k}, \qquad q(x) = \sum_{\ell=1}^{g+1} a_{2\ell-1}\, x^{g+1-\ell}$$

$$t(x) = \sum_{n=1}^{g+1} t_{2n-1}\, x^{g+1-n}$$

Then we define the *hyperelliptic Hopf algeboid* (A_g, Γ_g) by

$$\begin{aligned} A_g &:= \mathbb{Z}[a_{2k}, a_{2\ell-1} : 1 \le k \le 2g+1,\ 1 \le \ell \le g+1], \\ \Gamma_g &:= A_g[\lambda, r, t_{2n-1} : 1 \le n \le g+1] \end{aligned}$$

with structure maps obtained by the structure of the corresponding groupoid.

For simplicity we set $\lambda = 1$ hereafter.

The GWE for $g = 2$ is given by

$$y^2 + (a_1x^2 + a_3x + a_5)y = x^5 + a_2x^4 + a_4x^3 + a_6x^2 + a_8x + a_{10}$$

and the coordinate change

$$\begin{cases} x \mapsto x + r \\ y \mapsto y + sx^2 + tx + u \end{cases}$$

Then we have the hyperelliptic Hopf algebroid (A_2, Γ_2) defined by

$$A_2 = \mathbb{Z}[a_1, a_2, a_3, a_4, a_5, a_6, a_8, a_{10}] \qquad \text{and} \qquad \Gamma_2 = A_2[r, s, t, u].$$

The structure maps are very complicated even for this case.

Proposition (N.)

The right unit of the hyperelliptic Hopf algebroid for $g = 2$ is given by

$$\eta_R(a_1) = a_1 + 2s, \quad \eta_R(a_2) = a_2 - a_1 s - s^2 + 5r, \quad \eta_R(a_3) = a_3 + 2a_1 r + 2t$$

$$\eta_R(a_4) = a_4 - a_3 s + 4a_2 r - a_1 t - 2a_1 rs - 2st + 10r^2$$

$$\eta_R(a_5) = a_5 + a_3 r + a_1 r^2 + 2u$$

$$\begin{aligned}\eta_R(a_6) = {} & a_6 - a_5 s + 3a_4 r - a_3 t - a_3 rs + 6a_2 r^2 - a_1 u - 2a_1 rt - a_1 r^2 s \\ & - 2su - t^2 + 10r^3\end{aligned}$$

$$\begin{aligned}\eta_R(a_8) = {} & a_8 + 2a_6 r - a_5 t + 3a_4 r^2 - a_3 u - a_3 rt + 4a_2 r^3 - 2a_1 ru - a_1 r^2 t \\ & - 2tu + 5r^4\end{aligned}$$

$$\eta_R(a_{10}) = a_{10} + a_8 r + a_6 r^2 - a_5 u + a_4 r^3 - a_3 ru + a_2 r^4 - a_1 r^2 u - u^2 + r^5$$

Good news. You don't need to write these down. If you need this formula, then don't hesitate to email me (hnakai@tcu.ac.jp). I will send you this beamer file.

Proposition (N.)

The comultiplication Δ of the hyperelliptic Hopf algebroid for $g=2$ is given by

$$\Delta(r) = 1\otimes r + r\otimes 1, \qquad \Delta(s) = 1\otimes s + s\otimes 1$$
$$\Delta(t) = 1\otimes t + t\otimes 1 + s\otimes 2r$$
$$\Delta(u) = 1\otimes u + u\otimes 1 + s\otimes r^2 + t\otimes r$$

and the conjugation is given by

$$\Delta(r) = 1\otimes r + r\otimes 1, \qquad \Delta(s) = 1\otimes s + s\otimes 1$$
$$\Delta(t) = 1\otimes t + t\otimes 1 + s\otimes 2r$$
$$\Delta(u) = 1\otimes u + u\otimes 1 + s\otimes r^2 + t\otimes r$$

The conjugation and counit is also easily determined.

By these formulas we can compute $\mathrm{Ext}^*_{\Gamma_g}(A_g, A_g)$ using the similar method to tmf (the best reference is T.Bauer's paper in 2008).

In particular, if we tensor the Hopf algebroid with $\mathbb{Q}$, then the situation becomes drastically easy. If we set

$$\widetilde{A}_g = A_g \otimes \mathbb{Q} \qquad \text{and} \qquad \widetilde{\Gamma}_g = \Gamma_g \otimes \mathbb{Q}$$

then we have

Theorem (N.)

The group $\mathrm{Ext}^*_{\widetilde{\Gamma}_g}(\widetilde{A}_g, \widetilde{A}_g)$ is concentrated to the dimension 0, and we have

$$\mathrm{Ext}^0_{\widetilde{\Gamma}_g}(\widetilde{A}_g, \widetilde{A}_g) = \mathbb{Q}[c_{2i} \colon 2 \leq i \leq 2g+1].$$

Remark. In fact, it is enough to invert 2 and $2g+1$. This result corresponds to the fact that if we can invert these then we can reduced the GWE to

$$y^2 = x^{2g+1} + c_4 x^{2g-1} + \cdots + c_{4g} x + c_{4g+2}$$

and that there is no nontrivial coordinate change on it.

If we don't invert 2 and $2g+1$, then the computation is not easy.

Theorem (N.)

The hyperelliptic discriminant Δ_g defined by Lockhart is an invariant in the hyperelliptic Hopf algebroid, i.e.,

$$\Delta_g \in \mathrm{Ext}^{0,*}_{\Gamma_g}(A_g, A_g).$$

It can be shown that there is a positive integer N_g for each Δ_g such that

$$\Delta_g = \frac{(\text{A polynomial in } \mathbb{Z}[c_4, \ldots, c_{4g+2}]\ )}{N_g}.$$

For $g = 1$, we know that

$$N_1 = 1728 = 2^6 3^3 \qquad \text{and} \qquad \Delta_1 = \frac{c_4^3 - c_6^2}{1728}.$$

For $g = 2$, we have

$$\begin{aligned}
N_2 &= 512000000000000000 = 2^{24} 5^{15}, \\
\Delta_2 &= \frac{1}{512000000000000000}(c_{10}^4 - c_8^5 - 10c_4c_8^2c_{10}^2 - 10c_4^2c_8^4 + 20c_6c_8^3c_{10} \\
&\quad - 25c_4^4c_8^3 + 50c_4^3c_6^2c_8^2 + 60c_4c_6c_{10}^3 - 90c_6^2c_8c_{10}^2 - 90c_4^3c_8c_{10}^2 \\
&\quad + 90c_4c_6^2c_8^3 - 135c_6^4c_8^2 + 140c_4^2c_6c_8^2c_{10} - 216c_4^5c_{10}^2 + 360c_4^4c_6c_8c_{10} \\
&\quad - 640c_4^3c_6^3c_{10} + 660c_4^2c_6^2c_{10}^2 - 1260c_4c_6^3c_8c_{10} + 1728c_6^5c_{10}).
\end{aligned}$$

We have an algorithm to express Δ_g by some reduced coefficients c_{2i}, BUT it appears to be really hard.

If we don't invert 2 and p, then we have

$$\mathrm{Ext}^0_{\Gamma_g}(A_g, A_g) = \mathbb{Z}[c_{2i}, \Delta_g, \ldots : 2 \le i \le 2g+1]/(\text{relations})$$

where the part of $\ldots$ are the "baby discriminants" defined by

$$\Delta_8 = \frac{3c_4^2 + c_8}{2^4 \cdot 5^2}, \qquad \Delta_{10} = \frac{c_4 c_6 + c_{10}}{2^3 \cdot 5^2}, \qquad \Delta_{12} = \frac{-c_4 c_8 + c_4^3 + 4c_6^2}{2^6 \cdot 5^3},$$

$$\Delta_{14} = \frac{-c_6 \Delta_8 + c_4 \Delta_{10}}{2^2 \cdot 5}, \qquad \Delta_{16} = \frac{-24 c_4 c_6^2 + 9 c_4^2 c_8 - c_8^2 + 12 c_6 c_{10}}{2^6 \cdot 5^5},$$

$$\Delta_{18} = \frac{-36 c_6^3 + 16 c_4 c_6 c_8 - 9 c_4^2 c_{10} + c_8 c_{10}}{2^6 \cdot 5^5},$$

$$\Delta_{20} = \frac{-c_4 \Delta_8^2 - \Delta_{10}^2 + 60 \Delta_8 \Delta_{12}}{2^3 \cdot 5^2}, \qquad \Delta_{22} = \frac{3\Delta_{10}\Delta_{12} + 2\Delta_8 \Delta_{14}}{5},$$

$$\Delta_{24} = \frac{72 c_6^4 - 59 c_4 c_6^2 c_8 + 17 c_4^2 c_8^2 + 3 c_8^3 - 48 c_4^2 c_6 c_{10} - 38 c_6 c_8 c_{10} + 33 c_4 c_{10}^2}{2^{11} \cdot 5^8},$$

$$\Delta_{26} = \frac{-\Delta_{10}\Delta_{16} + \Delta_8\Delta_{18}}{2^2 \cdot 3 \cdot 5},$$

$$\Delta_{28} = \frac{-c_8\Delta_{20} + 3c_6\Delta_{22} + 60c_4\Delta_{24} + 270\Delta_{12}\Delta_{16} + 190\Delta_8\Delta_{20}}{2^3 \cdot 3 \cdot 5^2 \cdot 7},$$

$$\Delta_{30} = \frac{5\Delta_{14}\Delta_{16} + 2\Delta_{12}\Delta_{18} + 4\Delta_{10}\Delta_{20} + 11c_4\Delta_{26}}{2^3 \cdot 5^2},$$

$$\Delta'_{30} = \frac{-\Delta_{10}\Delta_{20} + \Delta_8\Delta_{22}}{2^2 \cdot 5},$$

$$\Delta_{32} = \frac{3(\Delta_{10}\Delta_{22} + c_4\Delta_8\Delta_{20}) + 20(c_4\Delta_{28} + \Delta_{12}\Delta_{20})}{2^3 \cdot 5^2},$$

$$\Delta_{34} = \frac{-5c_4\Delta'_{30} + 5\Delta_{10}\Delta_{24} + 3\Delta_{12}\Delta_{22} + 12\Delta_{14}\Delta_{20}}{2 \cdot 5^2},$$

$$\Delta_{36} = \frac{-\Delta_8\Delta_{12}\Delta_{16} - 2\Delta_{10}\Delta_{26} + 3\Delta_8^2\Delta_{20}}{2^2 \cdot 5},$$

$$\Delta'_{36} = \frac{\Delta_{18}^2 + c_4\Delta_{16}^2 + 20\Delta_{16}\Delta_{20}}{2^4 \cdot 3 \cdot 5^2}$$

$$\Delta_{38} = \frac{-\Delta_{16}\Delta_{22} + \Delta_{18}\Delta_{20}}{2^2 \cdot 5},$$

$\Delta_{40} =$ (Hyperelliptic discriminant by P.Lockhart),

$$\Delta_{42} = \frac{-9c_6\Delta'_{36} - 14c_4\Delta_{38} - 5\Delta_{18}\Delta_{24} + 20\Delta_{20}\Delta_{22}}{2^3 \cdot 3 \cdot 5^2},$$

$$\Delta_{44} = \frac{3\Delta_8\Delta'_{36} - \Delta_{16}\Delta_{28} - \Delta_{20}\Delta_{24} + \Delta_8\Delta_{16}\Delta_{20} - \Delta_{22}^2}{5}$$

Comments

- Essentially, I obtained these results in 2006 and I gave a talk about the computational part in the conference held at Johns Hopkins University.

- After my talk, Mike Hill came to my place and he taught to me that he made similar calculations in his thesis. His calculations are related to the Hopf algebroid arising from the Artin-Schreier curves, and these gave information on the theory so-called "higher real K-theory".

The language of stack

The following observations are due to Mike Hopkins. The best reference is

"Complex oriented cohomology theories and the language of stacks."
course notes, available on the web.

For a topological space X, we can define a *sheaf* on X

$$\mathcal{F} : (\mathcal{C}_X)^{\mathrm{op}} \longrightarrow Sets$$

In particular, a sheaf of groupoid

$$\mathcal{F} = (X_0, X_1) : (\mathcal{C}_X)^{\mathrm{op}} \longrightarrow Groupoids$$

with some extra condition (i.e., descent condition) is called a *stack* on X.

Recall that a stack is in fact defined over any site $\mathcal{C}$ (i.e., a category with Grothendieck topology) in the similar fashion.

Example. Let $\mathcal{C} = Rings^{\mathrm{op}}$ be a category of affine schemes, with covering maps are the collections $\{\mathrm{Spec}(R_i) \to \mathrm{Spec}(R)\}$ which satisfies

- Each $R \to R_i$ is flat map.
- For each R-module M, if $M \otimes_R R_i = 0$ for all i then $M = 0$.

This is a Grothendieck topology on $Rings^{\mathrm{op}}$, called the *flat topology*.

Remark. There are other popular Grothendieck topologies, for example, étale topology.

For each Hopf algebroid (A, Γ), we can define a "prestack"

$$(X_0, X_1) = (\mathrm{Spec}(A), \mathrm{Spec}(\Gamma)) : Rings^{\mathrm{op}} \to Groupoids$$

and we can "stackify" this into a stack $\mathcal{M} = \mathcal{M}_{(A,\Gamma)}$. This is called an *associated stack* to the Hopf algebroid (A, Γ). The structure sheaf on $\mathcal{M}$ is given by

$$\mathcal{O}_M(x : \mathrm{Spec}(A) \to \mathcal{M}) = A.$$

If we have a flat cover $\mathrm{Spec}(A) \to \mathcal{M}$ then the associated Čech nerve is

$$\begin{tikzcd}
\cdots \mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, equal] & \mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathrm{Spec}(A) \arrow[d, equal] \\
\cdots \mathrm{Spec}(\Gamma \otimes_A \Gamma) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Spec}(\Gamma) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Spec}(A)
\end{tikzcd}$$

where the simplicial object on the bottom is $\mathrm{Spec}(-)$ applied to the cobar construction.

Theorem (M.Hovey, 2002)

Suppose that (A, Γ) is a Hopf algebroid. Then there is an equivalence of categories between (A, Γ)-comodules and quqsi-coherent sheaves over $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$.

For (A, Γ)-comodule M, define a quasi-coherent sheaf $\mathcal{F}_M$ on $\mathcal{M}$ by

$$\mathcal{F}_M(x : \mathrm{Spec}(R) \to \mathcal{M}) = R \otimes_A M.$$

Note that $\mathcal{F}_A$ is identified with the structure sheaf $\mathcal{O}_{\mathcal{M}}$.

The sheaf cohomology of $\mathcal{M}$ with coefficient in $\mathcal{F}_M$ is then defined by applying $\mathcal{F}_M$ to the previous diagram.

$$\begin{tikzcd}
\cdots \mathcal{F}_M(\mathrm{Spec}(\Gamma \otimes_A \Gamma)) \arrow[d, equal] & \mathcal{F}_M(\mathrm{Spec}(\Gamma)) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, equal] & \mathcal{F}_M(\mathrm{Spec}(A)) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, equal] \\
\cdots \Gamma \otimes_A \Gamma \otimes_A M & \Gamma \otimes_A M \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & M \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

The upper sequence gives the sheaf cohomology and the bottom sequence is the cobar complex for the comodule M, and the associated cohomology is $\mathrm{Ext}^*_\Gamma(A, M)$.

We can consider the spectral sequence associated to this Čech nerve

$$H^s(\underbrace{\mathrm{Spec}(A) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathrm{Spec}(A)}_{t\text{-factors}}; \mathcal{F}_M) \quad \Longrightarrow \quad H^{s+t}(\mathcal{M}; \mathcal{F}_M)$$

Since $\mathcal{F}_M$ is quasi-coherent and it has no higher cohomology on affines, this spectral sequence collapses. As we observed in the above, E_1-terms coincides to $\mathrm{Ext}_{(A,\Gamma)}(A, M)$.

Thus we have

Proposition

Let (A, Γ) be a Hopf algebroid, and $\mathcal{M}$ be the associated stack (with flat topology) with the quasi-coherent sheaf $\mathcal{F}_M$. Then we have an isomorphism between flat cohomology of $\mathcal{M}$ and cohomology of Hopf algebroid (A, Γ)

$$H^*(\mathcal{M}; \mathcal{F}_M) \cong \mathrm{Ext}^*_\Gamma(A, M).$$

Thus, we can use our hyperelliptic Hopf algebroid (A_g, Γ_g) to compute the flat cohomology of the stack $\mathcal{M}_g^{GW} = \mathcal{M}_{(A_g, \Gamma_g)}$.

Corollary (N.)

The flat cohomology of the moduli stack $\mathcal{M} = \mathcal{M}_g^{GW}$ with coefficient $\mathcal{O}_\mathcal{M}$ is isomorphic to the Ext groups associated to the hyperelliptic Hopf algebroid for genus g.

$$H^*(\mathcal{M}; \mathcal{O}_\mathcal{M}) \cong \mathrm{Ext}^*_{\Gamma_g}(A_g, A_g).$$

Comments

- The stack $\mathcal{M}_g^{GW}$ is including information of "arbitrary singularities" of curves defined by general Weierstrass equation. The flat cohomology of $\mathcal{M}_g^{GW}$ can be computed by the Hopf algebroid (A_g, Γ_g).

- On the other hand, if we replace A_g and Γ_g with

$$A'_g := A_g[\Delta^{-1}] \qquad \text{and} \qquad \Gamma'_g := \Gamma_g[\Delta^{-1}]$$

  then the associated stack $\mathcal{M}_g^{hyp} := \mathcal{M}_{(A'_g, \Gamma'_g)}$ is the stack representing smooth hyperelliptic curves. Its flat cohomology can also be computed by the Hopf algebroid (A'_g, Γ'_g).

- We may add some information of treatable singularities to the stack $\mathcal{M}_g^{hyp}$ (like nodal singularity) so that we obtain the compactification of it. The geometry of such object may be interesting.

Final observations

In topology it is well-known that

$$B\mathrm{Diff}^0_+ M_g = K(\mathcal{M}_g, 1).$$

The following is an arithmetic analogue.

Theorem (P.Frediani and F.Neumann, 2003)

Let $\mathcal{H}_g$ be a moduli stack of hyperelliptic curves of genus g. There is a weak homotopy equivalence of pro-simplicial sets

$$(\mathcal{H}_g \otimes \overline{\mathbb{Q}})^\wedge_{et} \simeq K(\Gamma_g^{h\wedge}, 1).$$

where Γ_g^h is the hyperelliptic mapping class group.

Note that the left hand side has the information of étale cohomology of $\mathcal{H}_g$.

Comments

- As we mentioned in the above, our Hopf algebroid (A'_g, Γ'_g) is useful to compute the flat cohomology of $\mathcal{M}_g^{hyp}$, which has the information of all smooth hyperelliptic curves (with fixed Weierstrass point).

- Assuming that we find a relation between $\mathcal{M}_g^{hyp}$ and the algebraic stack $\mathcal{H}_g$ and that the change-of-topology spectral sequence (from étale topology to flat topology) collapses, we hope that the flat cohomology of $\mathcal{M}_g^{hyp}$ has information on the cohomology of Γ_g^h.
(This observation came from the discussion with Andrew Salch.)

Thank you so much.