The Kontsevich integral for bottom tangles in handlebodies: algebraic aspects

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Johnson homomorphisms and related topics
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The functor $Z_q^\varphi \colon \mathcal{B}_q \longrightarrow \widehat{\mathbf{A}}^\varphi$

Theorem (Massuyeau–H)

For each Drinfeld associator $\varphi = \varphi(X, Y) \in \mathbb{Q}\langle\langle X, Y\rangle\rangle$, there is a braided monoidal functor

$$Z_q^\varphi \colon \mathcal{B}_q \longrightarrow \widehat{\mathbf{A}}_q^\varphi.$$

Here

- $\mathcal{B}_q$ is the non-strictification of the *category $\mathcal{B}$ of bottom tangles in handlebodies*,
- $\widehat{\mathbf{A}}_q^\varphi$ is the non-strictification of the degree-completion $\widehat{\mathbf{A}}$ of the *category $\mathbf{A}$ of chord diagrams in handlebodies*, equipped with a braided monoidal structure associated to φ,
- Z_q^φ is constructed by using the Kontsevich integral of (bottom) tangles in handlebodies.

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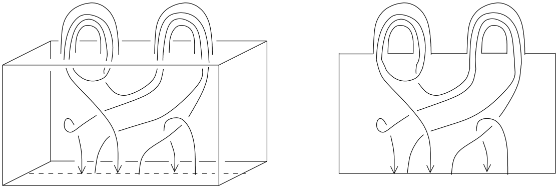
By ignoring subtleties, we have a functor $Z \colon \mathcal{B} \longrightarrow \widehat{\mathbf{A}}$.

The category $\mathcal{B}$

The category $\mathcal{B}$ of bottom tangles in handlebodies

- $\mathrm{Ob}(\mathcal{B}) = \mathbb{N} = \{0, 1, 2, \ldots\}$.
- $\mathcal{B}(m, n) = \{n\text{-component bottom tangles in } V_m\}/\text{isotopy}$.



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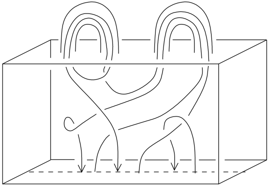

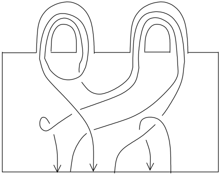

Composition: Regard a morphism $m \longrightarrow n$ as a cobordism between $\Sigma_{m,1}$ and $\Sigma_{n,1}$, compose, and regard the result as a bottom tangle in a handlebody.
(Thus $\mathcal{B}$ may be regarded as a subcategory of a cobordism category.)

The Vassiliev filtration on $\mathbb{Q}\mathcal{B}$

Let $\mathbb{Q}\mathcal{B}$ be the $\mathbb{Q}$-linearization of $\mathcal{B}$.

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The Vassiliev filtration on $\mathbb{Q}\mathcal{B}(m,n)$:

$$\mathbb{Q}\mathcal{B}(m,n) = \mathcal{V}^0(m,n) \supset \mathcal{V}^1(m,n) \supset \dots,$$

where $\mathcal{V}^d(m,n)$ is $\mathbb{Q}$-spanned by all the alternating sums

$$\sum_{S \subset \{1,\dots,d\}} (-1)^{|S|} T_S$$

of d independent crossing/framing changes on bottom tangles T.

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Then $\mathbb{Q}\mathcal{B}$ with $\mathcal{V}^d$, $d \geq 0$, is a filtered linear braided monoidal category, i.e.,

$$\mathcal{V}^d(n, p) \circ \mathcal{V}^{d'}(m, n) \subset \mathcal{V}^{d+d'}(m, p),$$

$$\mathcal{V}^d(m, n) \otimes \mathcal{V}^{d'}(m', n') \subset \mathcal{V}^{d+d'}(m + m', n + n').$$

The associated graded $\mathrm{gr}(\mathbb{Q}\mathcal{B})$ of $\mathbb{Q}\mathcal{B}$

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- $\mathrm{gr}^d(\mathbb{Q}\mathcal{B})(m,n) = \mathcal{V}^d(m,n)/\mathcal{V}^{d+1}(m,n)$.

$\mathrm{gr}(\mathbb{Q}\mathcal{B})$ is a graded $\mathbb{Q}$-linear *symmetric* monoidal category.

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Theorem (Massuyeau–H)

The functor $Z\colon \mathcal{B} \longrightarrow \widehat{\mathbf{A}}$ induces an isomorphism of a graded $\mathbb{Q}$-linear symmetric monoidal categories

$$\mathrm{gr}\, Z\colon\ \mathrm{gr}(\mathbb{Q}\mathcal{B}) \xrightarrow{\simeq} \mathrm{gr}(\widehat{\mathbf{A}}) = \mathbf{A},$$

The category **A**

The *category* **A** *of chord diagrams in handlebodies*:

- $\mathrm{Ob}(\mathbf{A}) = \mathbb{N}$,
- $\mathbf{A}(m, n) = \dfrac{\mathbb{Q}\{\text{chord diagrams on bottom } n\text{-strands in } V_m\}}{\text{homotopy, 4T}}$

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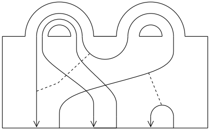
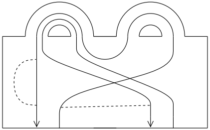


$: 2 \longrightarrow 3$

and

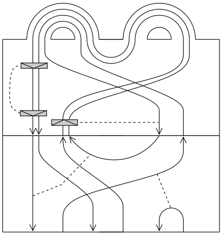
$D' =$



$: 2 \longrightarrow 2,$

then

$D \circ D' =$



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The category $\mathbf{A}$ is a *graded, $\mathbb{Q}$-linear, symmetric, (strict) monoidal category*.
Here the *degree* of a chord diagram is the number of chords.

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Remark

The category $\mathbf{A}$ of chord diagrams in handlebodies, and the category of *Jacobi diagrams* (i.e., vertex-oriented unitrivalent graphs) in handlebodies are the same.

Coalgebra enrichment of **A**

The space $\mathbf{A}(m,n)$ admits a coalgebra structure

$$\Delta\colon\ \mathbf{A}(m,n) \longrightarrow \mathbf{A}(m,n)\otimes\mathbf{A}(m,n), \quad \epsilon\colon\ \mathbf{A}(m,n)\longrightarrow\mathbb{Q}$$

defined by

$$\Delta(X,D) = \sum_{D'\sqcup D''=D}(X,D')\otimes(X,D''), \quad \epsilon(X,D)=\delta_{|D|,0}.$$

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The symmetric monoidal category $\mathbf{A}$ *is enriched over cocommutative coalgebras; i.e., the following are coalgebra maps*

$$\circ = \circ_{m,n,p} : \mathbf{A}(n,p) \otimes \mathbf{A}(m,n) \longrightarrow \mathbf{A}(m,p) \quad (m,n,p \geq 0),$$
$$\mathbb{Q} \longrightarrow \mathbf{A}(m,m), \quad 1 \longmapsto \mathrm{id}_m \quad (m \geq 0),$$
$$\otimes : \mathbf{A}(m,n) \otimes \mathbf{A}(m',n') \longrightarrow \mathbf{A}(m+m', n+n') \quad (m,n,m',n' \geq 0),$$
$$\mathbb{Q} \longrightarrow \mathbf{A}(m+n, n+m), \quad 1 \longmapsto P_{m,n} \quad (m,n \geq 0).$$

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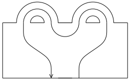
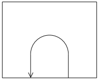
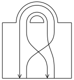
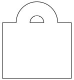
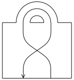
The category $\widehat{\mathbf{A}}$ *admits a (symmetric monoidal) subcategory* $\mathbf{A}^{\mathrm{grp}}$ *whose hom spaces* $\mathbf{A}^{\mathrm{grp}}(m,n)$ *consist of grouplike elements, in which* $Z\colon\ \mathcal{B} \longrightarrow \widehat{\mathbf{A}}$ *takes values.*

Hopf algebra in **A**

Define morphisms in **A** by

$\mu =$  $: 2 \to 1,$ $\quad \eta =$  $: 0 \to 1,$

$\Delta =$  $: 1 \to 2,$ $\quad \epsilon =$  $: 1 \to 0,$ $\quad S =$  $: 1 \to 1.$

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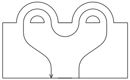
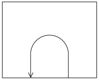
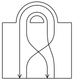
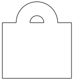
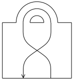
$\mu =$  $: 2 \to 1, \quad \eta =$  $: 0 \to 1,$

$\Delta =$  $: 1 \to 2, \quad \epsilon =$  $: 1 \to 0, \quad S =$  $: 1 \to 1.$

Proposition (Massuyeau–H)

$(1, \mu, \eta, \Delta, \epsilon, S)$ *form a cocommutative Hopf algebra in the symmetric monoidal category* **A**.

Casimir 2-tensor

Definition

A *Casimir 2-tensor* for a cocommutative Hopf algebra H in a linear symmetric monoidal category $\mathcal{C}$ is a morphism $c : I \to H^{\otimes 2}$ s.t.

$$\begin{aligned} P_{H,H} c &= c && \text{(symmetric)} \\ (\Delta \otimes \mathrm{id}_H) c &= c_{13} + c_{23} && \text{(left primitive)}, \\ (\mathrm{ad} \otimes \mathrm{ad})(\mathrm{id}_H \otimes P_{H,H} \otimes \mathrm{id}_H)(\Delta \otimes c) &= c\epsilon && \text{(ad-invariant)}. \end{aligned}$$

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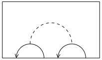
$c =$  $: 0 \to 2$ *is a Casimir 2-tensor in* $\mathbf{A}$.

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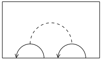
$c =$  $: 0 \to 2$ *is a Casimir 2-tensor in* $\mathbf{A}$.

Theorem (Massuyeau–H)

As a $\mathbb{Q}$-linear symmetric strict monoidal category, $\mathbf{A}$ *is freely generated by a Casimir Hopf algebra.*

Convolution algebra structure of $\mathbf{A}(m,n)$

$\mathbf{A}(m,n)$ is an algebra with multiplication given by *convolution*

$$*\colon\ \mathbf{A}(m,n)\otimes\mathbf{A}(m,n)\longrightarrow\mathbf{A}(m,n)$$

with unit $\eta^{\otimes n}\epsilon^{\otimes m}$.

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Let $\mathbf{A}(m, n)_{\mathrm{triv}} \subset \mathbf{A}(m, n)$ be spanned by chord diagrams (X, D) with X "trivial".
Then we have a linear isomorphism

$$\mathbf{A}(m, n)_{\mathrm{triv}} \otimes \mathbf{A}_0(m, n) \xrightarrow[\simeq]{*} \mathbf{A}(m, n).$$

Convolution algebra structure of $\mathbf{A}(m, n)$

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Ribbon quasi-Hopf algebra

A *quasi-Hopf algebra* is a generalization of a Hopf algebra, where coassociativity

$$(\Delta \otimes \mathrm{id}_H)\Delta = (\mathrm{id}_H \otimes \Delta)\Delta$$

does not hold, but holds up to a specified 3-tensor $\Phi \in H^{\otimes 3}$:

$$\Phi \cdot (\Delta \otimes \mathrm{id}_H)\Delta(x) \cdot \Phi^{-1} = (\mathrm{id}_H \otimes \Delta)\Delta(x).$$

There are notions of *quasi-triangular quasi-Hopf algebras* and *ribbon quasi-Hopf algebras*.
These notions are translated into symmetric monoidal categories.

A ribbon quasi-Hopf algebra in $\widehat{\mathbf{A}}$

Theorem (Massuyeau–H)

For each Drinfeld associator $\varphi = \varphi(X, Y) \in \mathbb{Q}\langle\langle X, Y\rangle\rangle$, *there is a ribbon quasi-Hopf algebra structure in* $\widehat{\mathbf{A}}$ *such that the morphisms* $\mu, \eta, \Delta, \epsilon, S$ *are as before, and*

$$\begin{aligned} \Phi &= \varphi_*(c_{12}, c_{23}) \; : 0 \longrightarrow 3, \\ R &= \exp_*(c/2) \; : 0 \longrightarrow 2, \\ \mathbf{r} &= \exp_*(\mu c/2) \; : 0 \longrightarrow 1, \end{aligned}$$

where $*$ *denotes convolution.*

Remark

Let $\mathfrak{g}$ be a Lie algebra and let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be an ad-invariant symmetric tensor. Then, by Drinfeld's work, there is a ribbon quasi-Hopf algebra structure on $U(\mathfrak{g})[[\hbar]]$. The above theorem may be regarded as a diagrammatic version of this fact.