June, 2013

Galois obstructions in Johnson cokernel

Naotake Takao (RIMS)

Johnson homomorphisms

Lecture Hall on the ground floor of the Mathematical Science Building, the University of Tokyo

abstract

The purpose of this survey lecture is to introduce a number-theoretic approach to study obstructions to the surjectivity of the Johnson homomorphism. In particular, the obstruction arising from the outer Galois action on the pro-ℓ fundamental group of the projective line minus three points over the rationals will be focused on. This obstruction has been studied by Mamoru Asada, Makoto Matsumoto, Hiroaki Nakamura, Takayuki Oda,and the lecturer. A recent celebrated theorem on the action of the Tannakian fundamental group of the category of mixed Tate motives over the rational integers on the motivic fundamental group of projective line minus three points by Francis Brown made it possible to determine the "size" of this obstruction.

Acknowledgement

I would like to thank the organisers for giving me the opportunity of my lecture.

Contents

Notations

(g, n) : a pair of non-negative integer such that $2g - 2 + n > 0$

$\Pi_{g,n} :=$ the fundamental group of a (hyperbolic) Riemann surface of type (g, n)

$\mathcal{M}_{g,n}$: the mapping class group of type (g, n)

$\Pi_{g,n}[m]$: the weight filtration of $\Pi_{g,n}$ defined by

$$\Pi_{g,n}[1] = \Pi_{g,n},$$
$$\Pi_{g,n}[2] = [\Pi_{g,n}, \Pi_{g,n}] \cdot \langle \text{ all inertia subgroups } \rangle$$
$$\Pi_{g,n}[k] = \langle [\Pi_{g,n}[k'], \Pi_{g,n}[k'']] | k' + k'' = k \rangle \qquad (k \geq 3)$$

Depending on the situation, g, n is often omitted .

$\mathcal{L}_k := \mathrm{gr}^k \Pi = \Pi[k]/\Pi[k+1]$ $(\Pi = \Pi_{g,n})$

$\mathcal{L} := \mathrm{Gr}\, \Pi = \oplus_{k \geq 1} \mathcal{L}_k$

$\mathfrak{h}_k := \{D \in \mathrm{Der}(\mathcal{L}) | D(\mathcal{L}_d) \subset \mathcal{L}_{d+k}$ for any $d \geq 1$,
D preserves each inertia Lie ideals$\}/\mathrm{Inn}_{\mathcal{L}_k}(\mathcal{L})$

$\mathfrak{h} = \mathrm{Der}^c(\mathcal{L})/\mathrm{Inn}(\mathcal{L}) := \oplus_{k \geq 1} \mathfrak{h}_k$

$\varphi : \mathcal{M} \to \mathrm{Out}^+(\Pi)$: the Dehn-Nielsen map (isomorphism)

$\mathcal{M}[k] := \mathrm{Ker}(\mathcal{M} \to \mathrm{Out}^+(\Pi/\Pi[m+1]))$

$\mathrm{gr}^k \mathcal{M} := \mathcal{M}[k]/\mathcal{M}[k+1]$

$\tau_k : \mathrm{gr}^k \mathcal{M} \to \mathfrak{h}_k$: the k-th integral Johnson homomorphism in this lecture

$\tau_k \otimes_{\mathbb{Z}} \mathbb{Q} : \mathrm{gr}^k \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathfrak{h}_k \otimes_{\mathbb{Z}} \mathbb{Q}$: the k-th Johnson homomorphism in this lecture

$\mathrm{Coker}(\tau_k \otimes_{\mathbb{Z}} \mathbb{Q})$: the Johnson cokernel in this lecture

$\mathbb{M}_{g,n}$: the moduli stack over $\mathbb{Q}$ of proper smooth geometrically connected curves of genus g with disjoint ordered n sections (often refered as (g,n)-curve in this lecture)

$\mathrm{G}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$: the absolute Galois group of rationals

$\pi_1(X)$: the étale fundamental group of a connected scheme/stack X

ℓ : a prime

$\Pi^{pro-\ell}$:

$$\text{the pro-}\ell\text{ completion of } \Pi = \varprojlim_{\substack{\Pi \rhd N, \\ (\Pi:N)=\ell\text{-th power}}} \Pi/N$$

This is often called the pro-ℓ (geometric) fundamental group,
which is isomorphic to the maximal pro-ℓ quotient of the étale fundamental group of a curve X over an algebraically closed field of characteristic 0, whose kernel is a characteristic subgroup of $\pi_1(X)$.

$\Pi^{pro-\ell}[k]$: the topological closure of $\Pi[k]$

Note that $\mathrm{gr}^k(\Pi^{pro-\ell}) \simeq \mathcal{L}_k \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$.

In what follows, depending on the situation, ℓ is omitted.

1 Background

1.1 Quick review of some results on the Johnson cokernel

A list of several (NOT ALL!) results on $\mathrm{Coker}(\tau_k \otimes_{\mathbb{Z}} \mathbb{Q})$ and $\mathrm{Coker}(\tau_k \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$.

0.[Nakamura-Tsunogai] The $\mathrm{Sp}(2g, \mathbb{Q}_\ell) \times \mathfrak{S}_n$-module structure of $\mathfrak{h}_k \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is "computable".

1.[Johnson, Morita] $1 \leq k \leq 2 \Rightarrow \tau_k \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphism.

2.For $1 \leq k \leq 6$, the $\mathrm{Sp}_{2g} \times \mathfrak{S}_n$-module structure of $gr^k \mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \mathbb{Q}$ is as follows:

[Johnson, Asada-Nakamura] $\mathrm{gr}^1 \mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq [1,1,1]_{\mathrm{Sp}} \otimes 1 + [1]_{\mathrm{Sp}} \otimes \begin{bmatrix} n \\ 1 \end{bmatrix}$

[Morita, Asada-Nakamura] $\mathrm{gr}^2 \mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq [2,2]_{\mathrm{Sp}} \otimes 1 + [1,1]_{\mathrm{Sp}} \otimes \begin{bmatrix} n \\ 1 \end{bmatrix} + [0]_{\mathrm{Sp}} \otimes \begin{bmatrix} n \\ 2 \end{bmatrix}$

[Asada-Nakamura] $\mathrm{gr}^3 \mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq [3,1,1]_{\mathrm{Sp}} \otimes 1 + [2,1]_{\mathrm{Sp}} \otimes \begin{bmatrix} n \\ 1 \end{bmatrix} + [1]_{\mathrm{Sp}} \otimes \wedge^2 \begin{bmatrix} n \\ 1 \end{bmatrix}$

[Morita]
$\mathrm{gr}^4 \mathcal{M}_{g,1} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq [4,2]_{\mathrm{Sp}} + [3,1,1,1]_{\mathrm{Sp}} + [2,2,2]_{\mathrm{Sp}} + 2[3,1]_{\mathrm{Sp}} + 2[2,1,1]_{\mathrm{Sp}} + 2[2]_{\mathrm{Sp}}$

[Morita-Sakasai-Suzuki] $\mathrm{gr}^5 \mathcal{M}_{g,1} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \cdots$ (sorry for not making on time) $(g >> k)$

[Morita-Sakasai-Suzuki] $\mathrm{gr}^6 \mathcal{M}_{g,1} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \cdots$ (sorry for not making on time) $(g >> k)$

In particular, together with Nakamura-Tsunogai's results, when $k = 3, 4, 5, 6$, τ_k is not surjective.

3.[Morita] For $g \geq 2$, $k \equiv 1 \pmod 2$,
in $\mathrm{Coker}(\tau_{(g,1)k} \otimes_{\mathbb{Z}} \mathbb{Q})$, $[k]_{\mathrm{Sp}}$ appears with multiplicity 1.

4.[Enomoto-Sato] For $g >> k$, $k \equiv 1 \pmod 4$,
in $\mathrm{Coker}(\tau_{(g,1)k} \otimes_{\mathbb{Z}} \mathbb{Q})$, $[1^k]_{\mathrm{Sp}}$ appears with multiplicity 1.

5.[Conant-Kassabov-Voghtmann]
A generalization of the Morita's trace-
infinite obstructions in the $\mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{Q}/[\mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{Q}, \mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{Q}]$

6.[Oda, Ihara, Matsumoto, Nakamura, Ueno, T + Hain-Matsumoto, Brown] For $2g - 2 + n > 0$, $k \equiv 0 \pmod 2$, $k \neq 2, 4, 8, 12$, in $\mathrm{Coker}(\tau_{(g,n)k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$, $[0]_{\mathrm{Sp}}$ appears with multiplicity $\geq m_k \doteqdot 1.3^{\frac{k}{2}}$ $(k >> 1)$.

Conclusively, (roughly speaking,) when k is odd and $n \geq 1$, then many kinds of obstructions to surjectivity of $\tau_{(g,n)k}$ exist.

but when

k is even or $n = 0$,

the lecturer knows little results. So I think that the results 6. is remarkable in this sense. From now on, I introduce this result in this lecture.

1.2 pro-ℓ universal monodromy representation

Roughly speaking, what is it?

The pro-ℓ universal monodromy representation $\varphi^{arith} : \pi_1(\mathbb{M}) \to \mathrm{Out}^c(\Pi^{pro-\ell})$ can be regarded as an arithmetic version of the Dehn-Nilsen map:

$$\varphi : \mathcal{M} \simeq \mathrm{Out}^+(\Pi),$$
$$\varphi^{arith} : \pi_1(\mathbb{M}) \to \mathrm{Out}^c(\Pi^{pro-\ell}).$$

$\mathrm{Out}^c(\Pi^{pro-\ell}) := \{f \in \mathrm{Aut}(\Pi^{pro-\ell}) | f$ preserves each inertia subgroup $\}/\mathrm{Inn}(\Pi^{pro-\ell})$

The Johnson homomorphism $\tau_k : \mathrm{gr}^k \mathcal{M} \to \mathfrak{h}_k$ is a Lie algebra version of Dehn-Nielsen map $\varphi : \mathcal{M} \to \mathrm{Out}^+(\Pi)$.

$$\begin{array}{ccc} \tau_k \otimes_{\mathbb{Z}} \mathbb{Z}_\ell : \mathrm{gr}^k \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \hookrightarrow & \mathfrak{h}_k \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \\ \downarrow & \circlearrowleft & \downarrow \\ \tau_k^{arith} : \mathrm{gr}^k \pi_1(\mathbb{M}) & \hookrightarrow & \mathfrak{h}_k \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \end{array}$$

(the left vertical arrow is injective and the right vertical arrow is the identity)

Formulation of φ^{arith}

$\mathbb{M}_{g,n+1} \to \mathbb{M}_{g,n}$: the universal family of (g,n)-curve

$\bar{x} \to \mathbb{M}_{g,n}$: a geometric point, $\quad \bar{X} := \mathbb{M}_{g,n+1} \times_{\mathbb{M}_{g,n}} \bar{x}$

$\rightsquigarrow$

$$1 \to \pi_1(\bar{X}) \to \pi_1(\mathbb{M}_{g,n+1}) \to \pi_1(\mathbb{M}_{g,n}) \to 1,$$

$\rightsquigarrow$

$$\begin{array}{ccccccccc}
1 \to & \pi_1(\bar{X}) & \to & \pi_1(\mathbb{M}_{g,n+1}) & \to & \pi_1(\mathbb{M}_{g,n}) & \to 1 \\
 & \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & \\
1 \to & \mathrm{Inn}(\pi_1(\bar{X})) & \to & \mathrm{Aut}(\pi_1(\bar{X})) & \to & \mathrm{Out}(\pi_1(\bar{X})) & \to 1 \\
 & \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow & \\
1 \to & \mathrm{Inn}(\Pi^{pro-\ell}) & \to & \mathrm{Aut}(\Pi^{pro-\ell}) & \to & \mathrm{Out}(\Pi^{pro-\ell}) & \to 1
\end{array}$$

in which all rows are exact. And we have another exact sequence

$$1 \to \pi_1(\mathbb{M}_{g,n} \otimes \overline{\mathbb{Q}}) \to \pi_1(\mathbb{M}_{g,n}) \overset{p_{g,n}}{\to} \mathrm{G}_{\mathbb{Q}} \to 1,$$

so we have

$$
\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & \downarrow & & \\
\mathcal{M}_{g,n} & \to & \widehat{\mathcal{M}_{g,n}} & \simeq & \pi_1(\mathbb{M}_{g,n} \otimes \overline{\mathbb{Q}}) & & \\
 & & & & \downarrow & & \\
 & & & \varphi_{g,n}^{arith}: & \pi_1(\mathbb{M}_{g,n}) & \to & \mathrm{Out}^c(\Pi_{g,n}^{pro-\ell}) \\
 & & & & \downarrow & & \\
 & & & & \mathrm{G}_{\mathbb{Q}} & & \\
 & & & & \downarrow & & \\
 & & & & 1. & &
\end{array}
$$

($\varphi_{g,n}^{arith}$ is called the pro-ℓ universal monodromy representation.)

$$
\begin{aligned}
\pi_1(\mathbb{M}_{g,n})[k] &:= \mathrm{Ker}(\pi_1(\mathbb{M}_{g,n}) \to \mathrm{Out}^C(\Pi_{g,n}^{pro-\ell}/\Pi_{g,n}^{pro-\ell}[k+1]) \\
\mathbb{Q}_{g,n}^{(\ell)}(k) &:= \bar{\mathbb{Q}}^{p_{g,n}(\pi_1(\mathbb{M}_{g,n})[k])} \\
\mathcal{G}_{g,n}^{(\ell)}(k) &:= \mathrm{Gal}(\mathbb{Q}_{g,n}^{(\ell)}(k+1)/\mathbb{Q}_{g,n}^{(\ell)}(k)) \\
\mathcal{G}_{g,n}^{(\ell)} &:= \oplus_{k\geq 1}\mathcal{G}_{g,n}^{(\ell)}(k)
\end{aligned}
$$

$\rightsquigarrow$

so we have

$$
\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & \downarrow & & \\
\tau_k \otimes_{\mathbb{Z}} \mathbb{Z}_\ell : & \operatorname{gr}^k \mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \hookrightarrow & & \operatorname{gr}^k \pi_1(\mathbb{M}_{g,n} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) & & \\
 & & & & \downarrow & & \\
 & & & \tau_k^{arith} : & \operatorname{gr}^k \pi_1(\mathbb{M}_{g,n}) & \hookrightarrow & \mathfrak{h}_k \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \\
 & & & & \downarrow & & \\
 & & & & \mathcal{G}_{g,n}^{(\ell)}(k) & & \\
 & & & & \downarrow & & \\
 & & & & 1 & &
\end{array}
$$

Definition 1.1 (Galois obstruction) For each $k \geq 1$, then we have

$$\operatorname{Coker}(\tau_{(g,n)k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \hookleftarrow \mathcal{G}_{g,n}^{(\ell)}(k) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Each element of $\mathcal{G}_{g,n}^{(\ell)}(k)$ can be regarded as a obstruction to the surjectivity of the Johnson homomorphism, which is called a Galois obstruction in this lecture.

This obstruction appears as

$$[0]_{\mathrm{Sp}} \text{ when } k \in 2\mathbb{Z} \setminus \{2,4,8,12\} \text{ even if } n = 0,$$

which is a central object in this lecture.

2 Main results

2.1 Oda conjecture

Theorem 2.1 (Ihara, Matsumoto, Nakamura, Takao)
(1) $\{\mathbb{Q}_{g,n}^{(\ell)}(k)\}_{k\geq 1}$ is independent of n

and almost independent of g and n in the following sense :

$$\mathbb{Q}_{1,1}^{(\ell)}(k) \supset \mathbb{Q}_{g,n}^{(\ell)}(k) \supset \mathbb{Q}_{0,3}^{(\ell)}(k),$$

$$[\mathbb{Q}_{1,1}^{(\ell)}(k) : \mathbb{Q}_{0,3}^{(\ell)}(k)] < \infty.$$

(2) $\mathbb{Q}_{g,n}^{(\ell)}$ is independent of g and n, where $\mathbb{Q}_{g,n}^{(\ell)} = \cup_{k\geq 1}\mathbb{Q}_{g,n}^{(\ell)}(k)$.

In particular,

$$\mathcal{G}_{g,n}^{(\ell)} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \mathcal{G}_{0,3}^{(\ell)} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Corollary 2.2 (Galois obstruction) (If $2g-2+n>0$, then for $k \geq 1$)

$$\mathrm{Coker}(\tau_{(g,n)\,k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \hookrightarrow \mathcal{G}^{(\ell)}_{0,3}(k) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

In particular, writing $r_k := \dim_{\mathbb{Q}_\ell} \mathcal{G}^{(\ell)}_{0,3}(k) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$,

$$\dim_{\mathbb{Q}_\ell} \mathrm{Coker}(\tau_{(g,n)\,k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \geq r_k.$$

2.2 Deligne-Ihara Conjecture

Remark 2.3 (As $\mathcal{M}_{0,3} = \{0\}$,) $\mathcal{G}_{0,3}^{(\ell)}(k)$ is a free $\mathbb{Z}_\ell$-module of finite rank r_k, so $\mathfrak{h}_{0,3} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookleftarrow \mathcal{G}_{0,3}^{(\ell)}$.

And $\mathcal{G}_{0,3}^{(\ell)}(k) \simeq \mathbb{Z}_\ell(\frac{k}{2})^{\oplus r_k}$ as a $\mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_\ell^\times$-module when k is even.
On the contrary $\mathcal{G}_{0,3}^{(\ell)}(k) = \{0\}$, as $\mathfrak{h}_{0,3}^{(\ell)} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \{0\}$ when k is odd.

Theorem 2.4 (Soulé-Ihara) $k \equiv 2 \pmod 4$ and $k \geq 6$

$\Rightarrow$

$\exists$ natural nontrivial character

$$\kappa_k : \mathcal{G}_{0,3}^{(\ell)}(k) \to \mathbb{Z}_\ell(\frac{k}{2})$$

Take a $\sigma_k \in \mathcal{G}_{0,3}^{(\ell)}(k)$ such that $\mathcal{G}_{0,3}^{(\ell)}(k) = \mathrm{Ker}(\kappa_k) + \mathbb{Z}_\ell \sigma_k$ ($k \equiv 2 \pmod 4$ and $k \geq 6$)

Theorem 2.5 (Hain-Matsumoto, Brown)
$\mathcal{G}_{0,3}^{(\ell)} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is freely generated by σ_k ($k \equiv 2 \pmod 4$ and $k \geq 6$) as a graded Lie algebra/$\mathbb{Q}_\ell$.

Especially,

Corollary 2.6

$$r_{2k'} = \frac{1}{k'} \sum_{d|k'} \mu\left(\frac{k'}{d}\right) (\sum_{i=1}^{3} (\alpha_i^d - 1 - (-1)^d)),$$

where α_i ($1 \leq i \leq 3$) are the roots of $x^3 - x - 1$.

In particular, if $k \neq 2, 4, 8, 12$ and k is even,

then $\tau_{(g,n)k} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is not surjective.

the calculated value of r_k for small k;

k	1	2	3	4	5	6	7	8	9	10
r_k	0	0	0	0	0	1	0	0	0	1

k	11	12	13	14	15	16	17	18	19	20
r_k	0	0	0	1	0	1	0	1	0	1

k	21	22	23	24	25	26	27	28	29	30
r_k	0	2	0	2	0	3	0	3	0	4

k	31	32	33	34	35	36	37	38	39	40
r_k	0	5	0	7	0	7	0	10	0	12

....

we have $r_{2k} \doteqdot 1.3^k$ for k is enough large

2.3 A sketch of the proof

Oda conjecture

The proof consists of four parts:

Step1 [Nakamura-Takao-Ueno] $\mathbb{Q}_{g,n}(k)$ is independent of n ($n > 0$).

Srep2 [Nakamura] $\mathbb{Q}_{1,1}(k) \supset \mathbb{Q}_{g,n}(k) \supset \mathbb{Q}_{0,3}(k)$ ($n > 0$).

Step3 [Ihara-Nakamura] $\mathbb{Q}_{g,n} = \mathbb{Q}_{0,3}$ ($n > 0$), $[\mathbb{Q}_{1,1}(k) : \mathbb{Q}_{0,3}(k)] < \infty$.

Step4 [Takao] $\mathbb{Q}_{g,1}(k) = \mathbb{Q}_{g,0}(k)$.

Step1 [Nakamura-Ueno-Takao] $\mathbb{Q}_{g,n}(k)$ is independent of n ($n > 0$).

A key idea is to consider "a braid version" of pro-ℓ universal monodromy representation, that is, the pro-ℓ monodromy representation associated to the universal family of the pure configuration space of curves.
We write $\mathfrak{h}^{(r)}, \mathbb{Q}_{g,n,r}(k), \cdots$ for the corresponding objects.
Then the proof can be reduced to the following three lemmas.

Lemma 2.7 $\mathbb{Q}_{g,n,r}(k) \subset \mathbb{Q}_{g,n+1,r}(k)$.

Lemma 2.8 $\mathbb{Q}_{g,n,r}(k) \subset \mathbb{Q}_{g,n-1,r+1}(k)$.

Lemma 2.9 $\mathbb{Q}_{g,n,r}(k) \subset \mathbb{Q}_{g,n,r-1}(k)$.

The third lemma is shown by (tough) combinatorial Lie algebraic calculations in "braid Lie algebra". A key theorem to show the last lemma is

Theorem 2.10 The $\mathbb{Z}_\ell$-Lie algebra homomorphism

$$\mathfrak{h}^{(r+1)} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathfrak{h}^{(r)} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

is injective.

Key principle of the proof is as follows:

Let D is the (canonical) lift of any element of the kernel of the above map.

Find $A, B \in \mathcal{L}_{k+1} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ such that $[A, B] = 0$ and $D(B) = 0$.

$\Rightarrow D(A) = 0$.

$\rightsquigarrow$ (omitted tough calculation) $\rightsquigarrow$

we can show that $D = 0$.

Step2 [Nakamura] $\mathbb{Q}_{1,1}(k) \supset \mathbb{Q}_{g,n}(k) \supset \mathbb{Q}_{0,3}(k)$ $(n > 0)$
Key ingredients in this step are

(1)Knudsen's Clutching morphisms to relate (compactified) moduli stacks of different type

(2)Grothendieck-Murre theory to describe the fundamental group of the formal tubular neighborhood of the locus of a type of stable curves arising from the clutching, and to combine the fundamental groups of moduli stacks of different type

(3)Nakamura's invention to couple universal monodromy representations of different types, using Grothendieck-Murre sequences

Rough (and incorrect) sketch is as follows: Let $\mathbb{M}^*_{g,n}$ be the Deligne-Mumford compactification of $\mathbb{M}_{g,n}$. Taking, a boundary irreducible divisor $D \simeq \mathbb{M}^*_{g_1,n_1} \times \mathbb{M}^*_{g_2,n_2}$ in $\mathbb{M}^*_{g,n}$ $(g_1 + g_2 = g,\ n_1 + n_2 = n + 2)$, which is, roughly speaking, the image of the clutching morphism. Then , by the theory of Grothendieck-Murre, we have

$$\begin{array}{ccccccccc} 1 \to & \hat{\mathbb{Z}}(1) & \to & \pi_1(]D[\backslash D) & \to & \pi_1(D) & \to & 1 \\ & & & \downarrow & & \downarrow & & \\ & & & \pi_1(\mathbb{M}_{g,n}) & & \pi_1(\mathbb{M}_{g_1,n_1} \times \mathbb{M}_{g_2,n_2}). & & \end{array}$$

A Key theorem is

Theorem 2.11 (Coupling Theorem). There exists a closed subgroup N of $\pi_1(]D[\backslash D)$ which is (almost) surjectively mapped onto $\pi_1(D)$, making the following diagram commutes:

$$\begin{array}{ccccc} & & \pi_1(\mathbb{M}_{g_1,n_1} \times \mathbb{M}_{g_2,n_2}) & \to & \mathrm{Out}^c(\Pi^{pro-\ell}_{g_1,n_1}) \times \mathrm{Out}^c(\Pi^{pro-\ell}_{g_2,n_2}) \\ & \nearrow & & & \\ N & & & & \downarrow \\ & \searrow & & & \\ & & \pi_1(\mathbb{M}_{g,n}) & \to & \mathrm{Out}^c(\Pi^{pro-\ell}_{g,n}). \end{array}$$

A key idea of the proof is Nakamura's invention to couple the universal monodromy representation over $\mathbb{M}_{g,n}$ with the universal monodromy representation over $\mathbb{M}_{g_1,n_1} \times \mathbb{M}_{g_2,n_2}$. More precisely, we consider two another clutching morphisms:

$$c_1 : \mathbb{M}^*_{g_1,n_1+1} \times \mathbb{M}^*_{g_2,n_2} \longrightarrow \mathbb{M}^*_{g,n+1},$$
$$c_2 : \mathbb{M}^*_{g_1,n_1} \times \mathbb{M}^*_{g_2,n_2+1} \longrightarrow \mathbb{M}^*_{g,n+1}.$$

Denoting the boundary component corresponding to c_1 (resp.c_2) by D_1 (resp. D_2), we have

$$
\begin{array}{ccccccccc}
1 \to & \hat{\mathbb{Z}}(1) & \to & \pi_1(]D_1[\backslash D_1) & \to & \pi_1(D_1) & \to & 1 \\
 & & & \downarrow & & \downarrow & & \\
 & & & \pi_1(\mathbb{M}_{g,n+1}) & & \pi_1(\mathbb{M}_{g_1,n_1+1} \times \mathbb{M}_{g_2,n_2}), & &
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
1 \to & \hat{\mathbb{Z}}(1) & \to & \pi_1(]D_2[\backslash D_2) & \to & \pi_1(D_2) & \to & 1 \\
 & & & \downarrow & & \downarrow & & \\
 & & & \pi_1(\mathbb{M}_{g,n+1}) & & \pi_1(\mathbb{M}_{g_1,n_1} \times \mathbb{M}_{g_2,n_2+1}), & &
\end{array}
$$

by Grothendieck-Murre theory.

By considering the formal tubular neighborhood $]D_1 \cup D_2[\backslash D_1 \cup D_2$, we get conclusion after some group theoretical treatments.

Corollary 2.12 $\mathbb{Q}_{g_1,n_1}(k)\mathbb{Q}_{g_2,n_2}(k) \supset \mathbb{Q}_{g,n}(k)$

Corollary 2.13 $\mathbb{Q}_{1,1}(k) \supset \mathbb{Q}_{g,n}(k) \supset \mathbb{Q}_{0,3}(k)$ $(n > 0)$

Remark 2.14 Matsumoto also related $g = 0$ and $g > 0$ by another method.

Step3 [Ihara-Nakamura] - very rough sketch

$\mathbb{Q}_{g,n} = \mathbb{Q}_{0,3}$ $(n > 0)$

$[\mathbb{Q}_{1,1}(k) : \mathbb{Q}_{0,3}(k)] < \infty$ $(n > 0)$

Key ingredients in this step are

(1)Tangential structure to construct a canonical universal deformation family of maximally degenerate stable marked curves

(2)Formal patching (Artin, Deligne, Harbarter) and Grothendieck-Murre theory to describe the Galois action on the fundamental group of the formal tubular neighborhood of maximally degenerate stable marked curves in the canonical deformation space

(3)A certain weight argument to prove the latter statement

A key theorem is

Theorem 2.15 If $\sigma \in \mathrm{G}_{\mathbb{Q}}$ acts trivially on the pro-ℓ geometric fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with respect to the set of Deligne's tangential base points, then σ has an extension $\tilde{\sigma} \in G_{\mathbb{Q}((q))}$ that acts trivially on the pro-ℓ fundamental groupoid of the generic geometric fiber of the family constructed in (1) with respect to the set of "tangential base points" which are associated with (ordinary double) singular points in the maximal degenerated stable marked curve. In particular, the outer action of $\tilde{\sigma}$ on the pro-ℓ fundamental group of the generic geometric fiber of the family constructed in (1) is trivial.

Very roughly speaking, the proof is done by "gluing Grothendieck-Murre sequence" using formal patching. It can be shown that the pro-ℓ fundamental group of the generic geometric fiber of the family constructed in (1) is "almost" isomorphic to the free product of all the pro-ℓ geometric fundamental groups of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

A key idea of the proof of finite index part is to consider the nilpotency class. Then we have $\mathbb{Q}_{1,1}(k) \subset \mathbb{Q}_{0,3}(2k+4)$. Using a certain weight argument, we get the conclusion.

Step4 [Takao] $\mathbb{Q}_{g,1}(k) = \mathbb{Q}_{g,0}(k)$ $(g \geq 2)$ - very rough sketch

Enough to prove

Lemma 2.16 $\mathbb{Q}_{g,0,2}(k) \subset \mathbb{Q}_{g,0,1}(k)$

A presentation of $\mathcal{L}_{g,0,2}$ is as folllows:

$$\text{generators} \qquad X_i^{(j)}, Z \qquad (1 \leq i \leq 2g, 1 \leq j \leq 2),$$

$$\text{relations} \qquad \sum_{i=1}^{g} [X_i^{(j)}, X_{i+g}^{(j)}] + Z = 0 \qquad (1 \leq j \leq 2),$$

$$[X_i^{(j)}, X_{i'}^{(j')}] = \begin{cases} 0 & (j \neq j',\ i \leq i' \text{ and } i' \neq i+g), \\ Z & (j \neq j',\ i \leq i' \text{ and } i' = i+g). \end{cases}$$

In particular, we have

$$[X_i^{(1)} + X_i^{(2)}, Z] = 0 \qquad (1 \leq i \leq 2g).$$

Now the outline is the same as in Step1.
But there is NOT! any generator A such that $[A, Z] = 0$.
We use $X_i^{(1)} + X_i^{(2)}$ instead.
$\mathrm{C}(X_i^{(1)} + X_i^{(2)})$ is not clear, but we can see

Lemma 2.17 For $1 \leq i \leq 2g$,

$$\mathrm{C}(X_i^{(1)} + X_i^{(2)}) \cap \mathrm{Gr}\, N^{(2)} = \langle X_i^{(2)}, Z \rangle_{Lie}.$$

Here, $N^{(2)}$ is the fiber subgroup obtained by forgetting the second component (i.e. the kernel of the natural projection of braid groups).

Key ingredients of the proof are

· Elimination theorem for free Lie algebras

· Degree discussion on free Lie algebras

$\rightsquigarrow$ (omitted tough calculation) $\rightsquigarrow$

we can show that $D = 0$.

Remark 2.18 Hoshi and Mochizuki proved the non-filtered part of this step and step1 in a quite different way from a quite different motivation.

Deligne-Ihara conjecture - very very rough sketch

$\cdot$ The Lie algebra of the pro-unipotent part of the motivic Galois group (=the Tannakian fundamental group of the category of mixed Tate motives over $\mathrm{Spec}(\mathbb{Z})$) is freely generated by Soulè cyclotomic elements of the K-theory.

$\cdot$ The motivic Galois group acts on the motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

$\cdot$ Brown proved that this action is faithful.

$\cdot$ $\mathcal{G}_{0,3}^{(\ell)}$ is the the ℓ-adic realization of the image of this action.

$\rightsquigarrow$

Theorem 2.5(Deligne-Ihara conjecture)

By what kind of method did Brown prove the faithfulness?

In fact, Brown also proved that Hoffman conjecture on multiple zeta values

$$\zeta(m_1, m_2, \cdots, m_r) = \Sigma_{0<n_1\cdots<n_r} \frac{1}{n_1^{m_1}\cdots n_r^{m_r}} \quad (1 \leq m_1, \cdots, m_r, 2 \leq m_r).$$

Theorem 2.19 (Hoffman conjecture) The k-th graded piece of $\mathbb{Q}$-graded algebra of multiple zeta values is generated by $\{\zeta(m_1, m_2, \cdots, m_r) | m_i = 2, 3,\ m_1 + \cdots + m_r = k\}$

The proof was done by proving the motivic version of Hoffman conjecture.
The motivic version of Hoffman conjecture states that
$\{\zeta^M(m_1, m_2, \cdots, m_r) | m_i = 2, 3,\ m_1 + \cdots + m_r = k\}$ is a basis of the k-th graded piece of the algebra of motivic multiple zeta values, which leads to faithfulness of the action.

3 Further interest

3.1 On the image of pro-ℓ universal monodromy representaion

$\mathrm{Im}(\tau_k \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \cap \mathcal{G}_k \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \{0\}.$

But $\varphi^{arith}(\mathcal{M}) \subset \phi_X(G_k)$ if X/k is ℓ-monodromically full defined by Hoshi. Here $\phi_X : \mathrm{G}_k \to \mathrm{Out}(\Pi^{pro-\ell})$ is the pro-ℓ outer Galois representation associated to X/k. Matsumoto and Tamagawa proved that many such curves exist.

So the image of any mapping classes can be described as the image of some element of Galois group. For example, How can Dehn twist be described ?

To the contrary, it seems interesting to characterize the image of mapping class group in the image of Galois group. For example, Can some Forbenius element be described by Mapping class?

3.2 Towards to the integral Oda conjecture

Finally speaking, the lecturer is now interested in the "integral Oda problem" and "integral Deigne-Ihara problem", which is, roughly speaking, the problem to investigate the structure of the associated graded Lie algebra $\mathcal{G}_{g,n}^{(\ell)}$ over $\mathbb{Z}_\ell$. That is because this structure may has some arithmetic information. In fact, for example, this structure depends on ℓ.

Conclusively, to solve the integral Oda problem, we must analyse "the integral Johnson homomorphism". For example, if $\mathrm{Coker}(\tau_{(g,n)k}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is torsion-free, then $\mathcal{G}_{g,n}^{(\ell)}(k)$ is torsion-free. So the lecturer is also interested in Integral Johnson homomorphism.

The lecture is end. Thank you for listening.