The Goldman-Turaev Lie bialgebra and the Johnson homomorphisms

(joint work with Nariya Kawazumi)

2013.6.6

Yusuke Kuno
(Tsuda College)

§1 Introduction

$\Sigma_{g,1}$: surface of genus g with 1 boundary component

$\mathcal{M}_{g,1}$: the mapping class group of $\Sigma_{g,1}$

$\triangledown$

$\mathcal{M}_{g,1}(k)$: the k-th Johnson subgroup $(\mathcal{M}_{g,1}(1) = \mathcal{I}_{g,1}, \mathcal{M}_{g,1}(2) = \mathcal{K}_{g,1})$

$gr^k(\mathcal{I}_{g,1}) := \mathcal{M}_{g,1}(k) / \mathcal{M}_{g,1}(k+1)$

$\bigoplus_{k=1}^{\infty} gr^k(\mathcal{I}_{g,1})$: graded Lie algebra ([,] : commutator product)

$\boxed{\tau_k : \mathcal{M}_{g,1}(k) \longrightarrow H \otimes \mathcal{L}_{\mathbb{Z}}(k+1)}$ the k-th Johnson homom

$\left(\begin{array}{l} H = H_1(\Sigma_{g,1}; \mathbb{Z}) \\ \mathcal{L}_{\mathbb{Z}}(i) : \text{deg } i \text{ part of the free Lie alg gen by } H \end{array} \right)$

$\mathfrak{h}^{\mathbb{Z}}_{g,1}(k) = \mathrm{Ker}\left([\ ,\ ] : H \otimes \mathcal{L}_{\mathbb{Z}}(k+1) \longrightarrow \mathcal{L}_{\mathbb{Z}}(k+2)\right)$

$\bigoplus_{k=1}^{\infty} \mathfrak{h}^{\mathbb{Z}}_{g,1}(k)$: graded Lie alg (the Lie alg of symplectic derivations of $\mathcal{L}_{\mathbb{Z}}$)

<u>Thm</u> (Morita)

① $\operatorname{Im}\tau_k \subset \mathfrak{h}^{\mathbb{Z}}_{g,1}(k)$, and so $\tau_k : \mathcal{M}_{g,1}(k) \to \mathfrak{h}^{\mathbb{Z}}_{g,1}(k)$

② $\tau = \{\tau_k\}_k : \bigoplus_{k=1}^{\infty} gr^k(\mathcal{I}_{g,1}) \to \bigoplus_{k=1}^{\infty} \mathfrak{h}^{\mathbb{Z}}_{g,1}(k)$ is an inj. graded Lie alg homom

③ τ is not surjective

<u>Problem</u> Determine $\operatorname{Im}\tau$ ($\otimes\mathbb{Q}$)

<u>Goal</u> ① construct an extension of $\{\tau_k\}_k$ to $\mathcal{I}_{g,1}$ in a geometric context (equiv. to Massuyeau's total Johnson map)

② generalization to compact surfaces with <u>non</u>-empty boundary

③ geometric constraint for $\operatorname{Im}\tau$ (Turaev cobracket)

④ tensorial descriptions

Plan

§2 The Goldman Lie algebra and its completion

§3 The Dehn-Nielsen embedding and its infinitesimal analogue

$$\begin{tikzcd}
\mathrm{MCG} \arrow[dr, bend left, "\text{aut}"] & \\
& \boxed{\begin{array}{l}\text{paths on}\\ \text{surface}\end{array}} \\
\mathrm{GLA} \arrow[ur, bend left, "\text{derivation}"'] &
\end{tikzcd}$$

§4 Geometric Johnson homomorphism

§5 Tensorial descriptions

§2 The Goldman Lie algebra and its completion

S : (connected) oriented surface

$\hat{\pi}(S) = [S^1, S]$, $\quad | \ | : \pi_1(S) \to \hat{\pi}(S)$ natural projection

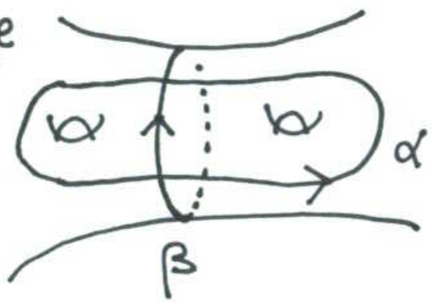
Goldman bracket

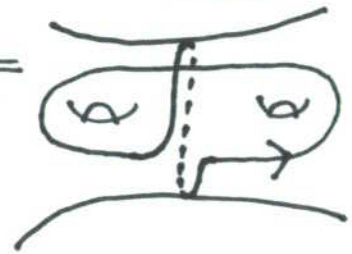
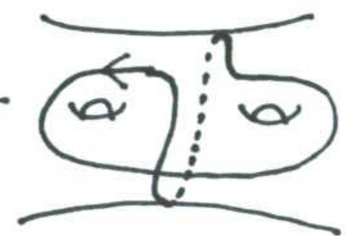
$\alpha, \beta : S^1 \to S$ free loops in general position

For $p \in \alpha \cap \beta$, $\begin{cases} \varepsilon(p : \alpha, \beta) \in \{\pm 1\} & \text{: the local intersection number} \\ \alpha_p \cdot \beta_p \in \pi_1(S) = \pi_1(S, p) & \text{: the conjunction} \end{cases}$

$$\boxed{[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p : \alpha, \beta) |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}(S)}$$

Example



$[\alpha, \beta] =$  $-$ 

Thm (Goldman)

$(\mathbb{Z}\hat{\pi}(S), [\ ,\ ])$ is a Lie algebra

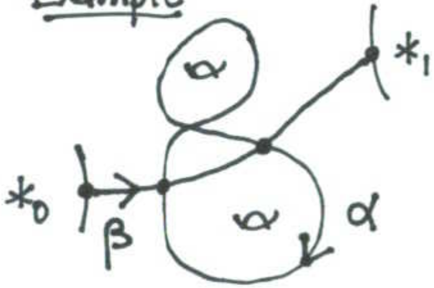
Action on based paths

For $*_0, *_1 \in \partial S$, $\quad \Pi S(*_0, *_1) := \left[\left([0,1], 0, 1\right), \left(S, *_0, *_1\right)\right]$

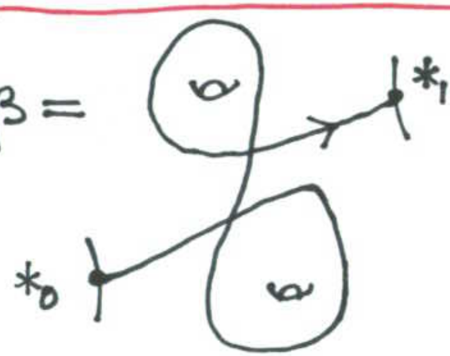
$$\left\{\begin{array}{ll} \alpha : S^1 \to S & \text{free loop} \\ \beta : \left([0,1], 0, 1\right) \to \left(S, *_0, *_1\right) & \end{array}\right. \qquad \text{in general position}$$

$$\boxed{\sigma(\alpha)\beta := \sum_{p \in \alpha \cap \beta} \varepsilon(p ; \alpha, \beta)\, \beta_{*_0 p} \cdot \alpha_p \cdot \beta_{p *_1} \in \mathbb{Z}\Pi S(*_0, *_1)}$$

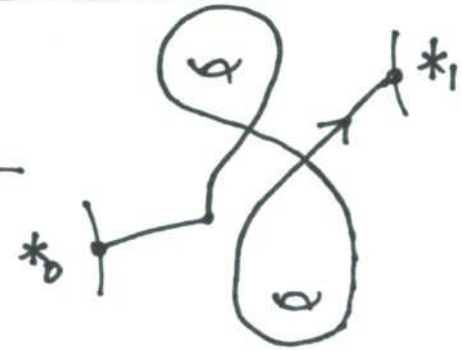
Example



$\sigma(\alpha)\beta =$



$-$



Extending $\mathbb{Z}$-bilinearly, we obtain

$$\sigma = \sigma_{*_0, *_1} : \mathbb{Z}\hat{\pi}(S) \otimes \mathbb{Z}\Pi S(*_0, *_1) \longrightarrow \mathbb{Z}\Pi S(*_0, *_1)$$

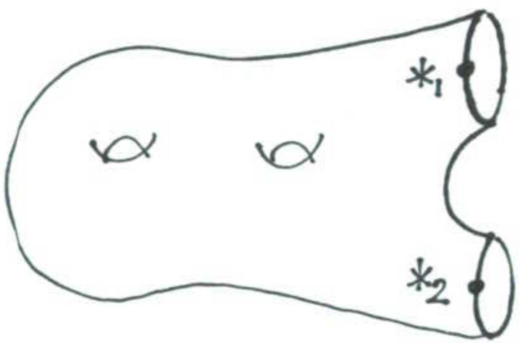
$$\sigma = \sigma_{*_0,*_1} : \mathbb{Z}\hat{\pi}(S) \otimes \mathbb{Z}\Pi S(*_0,*_1) \longrightarrow \mathbb{Z}\Pi S(*_0,*_1)$$
$$\alpha \otimes \beta \longmapsto \sigma(\alpha)\beta$$

<u>Thm</u> (Kawazumi–K)

① $\sigma([\alpha_1,\alpha_2])\beta = \sigma(\alpha_1)(\sigma(\alpha_2)\beta) - \sigma(\alpha_2)(\sigma(\alpha_1)\beta)$, i.e. $\mathbb{Z}\Pi S(*_0,*_1)$ is a $\mathbb{Z}\hat{\pi}(S)$-module.

② For $\beta_1 \in \mathbb{Z}\Pi S(*_0,*_1)$, $\beta_2 \in \mathbb{Z}\Pi S(*_1,*_2)$, $\sigma(\alpha)(\beta_1 \cdot \beta_2) = (\sigma(\alpha)\beta_1)\cdot\beta_2 + \beta_1(\sigma(\alpha)\beta_2)$

Setting S: compact, $\partial S \neq \emptyset$, $E = \{*_i\}_i \subset \partial S$, $E \xrightarrow{\cong} \pi_0(\partial S)$



Small category $\mathbb{Z}\Pi S|_E$

Ob : E

Mor : $\mathbb{Z}\Pi S(*_i,*_j)$

$$\mathrm{Der}(\mathbb{Z}\Pi S|_E) := \{ D = \{D_{ij}\}_{i,j} \mid D_{i,j} \in \mathrm{End}(\mathbb{Z}\Pi S(*_i,*_j)), \text{ "derivation" } \}$$

$\rightsquigarrow$ a Lie alg homom $\sigma : \mathbb{Z}\hat{\pi}(S) \longrightarrow \mathrm{Der}(\mathbb{Z}\Pi S|_E)$

$$\alpha \longmapsto \{\sigma_{*_i,*_j}(\alpha)\}_{i,j}$$

Hereafter we consider $/\mathbb{Q}$ instead of $/\mathbb{Z}$, (e.g. $\mathbb{Z}\hat{\pi}(S) \rightsquigarrow \mathbb{Q}\hat{\pi}(S)$)
Since we relate derivations and automorphisms by log and exp

<u>Completion</u>

$I\pi_1(S) = \mathrm{Ker}(\mathbb{Q}\pi_1(S) \longrightarrow \mathbb{Q},\ \pi_1(S) \ni x \mapsto 1)$: the augmentation ideal

$\mathbb{Q}\pi_1(S)$ is filtered by $(I\pi_1(S))^n$, $n \geq 0$

$|\ | : \mathbb{Q}\pi_1(S) \to \mathbb{Q}\hat{\pi}(S)$ $\mathbb{Q}$-linear extension of $|\ | : \pi_1(S) \to \hat{\pi}(S)$

[<u>Def</u> For $n \geq 0$, $\mathbb{Q}\hat{\pi}(S)(n) := |\mathbb{Q}\mathbb{1} + (I\pi_1(S))^n|$]

($\mathbb{1}$: the class of a constant loop. <u>Rem</u> $\sigma(\mathbb{1})\beta = 0$ $\forall \beta$)

[<u>Lem</u> $[\mathbb{Q}\hat{\pi}(S)(m), \mathbb{Q}\hat{\pi}(S)(n)] \subset \mathbb{Q}\hat{\pi}(S)(m+n-2)$]

$\boxed{\widehat{\mathbb{Q}\hat{\pi}}(S) := \varprojlim_m \mathbb{Q}\hat{\pi}(S) / \mathbb{Q}\hat{\pi}(S)(m)}$: the completed Goldman Lie algebra

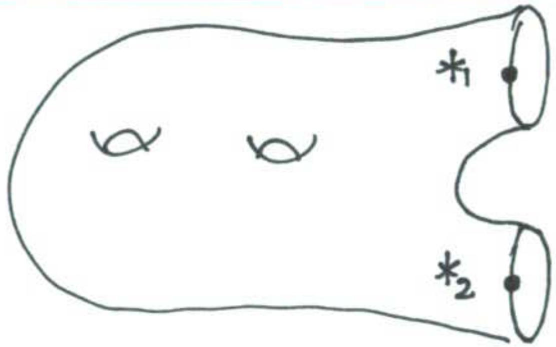
Similarly one can construct a completion $\widehat{\mathbb{Q}\pi S|_E}$, and has a Lie alg homom

$$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \longrightarrow \mathrm{Der}(\widehat{\mathbb{Q}\pi S|_E})$$

§3 The Dehn-Nielsen embedding and its infinitesimal analogue

Setting S : compact surface, $\partial S \neq \emptyset$, $E = \{*_i\}_i \subset \partial S$, $E \xrightarrow{\cong} \pi_0(\partial S)$

$\boxed{\mathcal{M}(S) := \{\varphi : S \to S \text{ diffeo} \mid \varphi|_{\partial S} = \mathrm{id}_{\partial S}\} / \text{isotopy rel } \partial S}$ the mapping class group



Small category $\Pi S|_E$

Ob : E

Mor : $\Pi S(*_i, *_j)$

$\mathcal{M}(S) \curvearrowright \Pi S|_E$

Thm (Dehn-Nielsen embedding)

$DN : \mathcal{M}(S) \longrightarrow \mathrm{Aut}(\Pi S|_E)$ is injective

DN induces $\widehat{DN} : \mathcal{M}(S) \hookrightarrow \mathrm{Aut}(\widehat{\mathbb{Q}\Pi S}|_E)$

Recall $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \longrightarrow \mathrm{Der}(\widehat{\mathbb{Q}\pi S}|_E)$

$$\boxed{\mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi S}|_E) := \{ D \in \mathrm{Der}(\widehat{\mathbb{Q}\pi S}|_E) \mid D({}^{\forall}\text{boundary of } S) = 0 \}}$$

$\mathrm{Im}\,\sigma \subset \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi S}|_E)$

<u>Thm</u> (Kawazumi-K, the infinitesimal Dehn-Nielsen theorem)
$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \longrightarrow \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi S}|_E)$ is an isomorphism

$$\begin{tikzcd}
\mathcal{M}(S) \arrow[r, hook, "\widehat{DN}"] & \mathrm{Aut}(\widehat{\mathbb{Q}\pi S}|_E) \arrow[r, phantom, "\supset"] & {} \arrow[d, shift right, "\log"'] \\
\widehat{\mathbb{Q}\hat{\pi}}(S) \arrow[r, "\sigma", "\cong"'] & \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi S}|_E) \arrow[r, phantom, "\supset"] & {} \arrow[u, shift right, "\exp"']
\end{tikzcd}$$

§4 Geometric Johnson homomorphism

Setting S : compact surface, $\partial S \neq \emptyset$, $E = \{*_i\}_i \subset \partial S$, $E \xrightarrow{\cong} \pi_0(\partial S)$

Torelli group

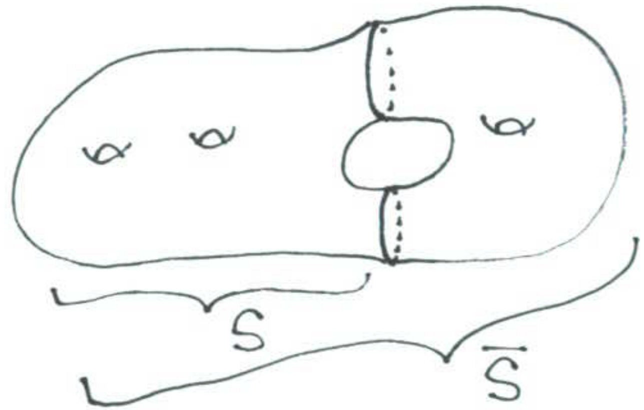
If ∂S is not connected, there are natural choices for the Torelli group (Putman)

① The "smallest" Torelli group

$$\mathcal{I}(S) := \mathrm{Ker}(\mathcal{M}(S) \to \mathrm{Aut}(H_1(S,E)))$$

$$\iota : S \to \overline{S}$$

$$\mathcal{I}(S) = \iota_*^{-1}(\mathcal{I}(\overline{S}))$$



② The "largest" Torelli group

$$\mathcal{I}^L(S) := \mathrm{Ker}\left(\mathcal{M}(S) \to \mathrm{Aut}\left(\frac{H_1(S)}{\mathrm{Im}(H_1(\partial S) \to H_1(S))}\right)\right)$$

$$\iota : S \to \widetilde{S}$$

$$\mathcal{I}^L(S) = \iota_*^{-1}(\mathcal{I}(\widetilde{S}))$$

Our result: Johnson homom for $\mathcal{I}(S)$ and $\mathcal{I}^{+}(S)$

(Question: How to generalize to other kinds of Putman's Torelli groups?)

Construction for $\mathcal{I}(S)$

coproduct of $\Pi S|_E$: For $*_0, *_1 \in E$, define

$$\Delta = \Delta_{*_0,*_1} : \mathbb{Q}\Pi S(*_0,*_1) \longrightarrow \mathbb{Q}\Pi S(*_0,*_1) \otimes \mathbb{Q}\Pi S(*_0,*_1)$$

$$\Pi S(*_0,*_1) \ni x \longmapsto x \otimes x$$

Δ extends naturally to completions

$D \in \mathrm{Der}(\widehat{\mathbb{Q}\Pi S|_E})$ $\quad$ D stabilizes $\Delta \overset{\text{def}}{\Longleftrightarrow} (D \hat{\otimes} 1 + 1 \hat{\otimes} D)\Delta = \Delta \circ D$

Def

$$L^{+}(S,E) := \left\{ u \in \widehat{\mathbb{Q}\hat{\pi}}(S) \;\middle|\; \begin{array}{l} u \in \mathrm{Ker}(\widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(3)) \\ \sigma(u) \text{ stabilizes } \Delta \end{array} \right\}$$

- pro-nilpotent Lie algebra
- It is a group w.r.t. BCH series

$$u \cdot v = \log(e^u \cdot e^v) = u + v + \frac{1}{2}[u,v] + \frac{1}{12}[u-v,[u,v]] + \cdots$$

For $\varphi \in \mathcal{I}(S)$, $\log(\widehat{DN}(\varphi)) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\widehat{DN}(\varphi) - \mathrm{id})^n \in \mathrm{Der}(\widehat{\mathbb{Q}\pi S}|_E)$ exists. 12

Since φ preserves $\begin{cases} {}^\forall \text{boundary of } S \\ \text{the coproduct } \Delta \end{cases}$, we have $\sigma^{-1}(\log(\widehat{DN}(\varphi))) \in L^+(S, E)$
$$\overset{\|}{\tau(\varphi)}$$

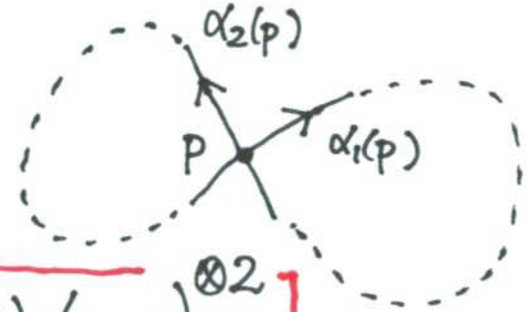
$\boxed{\tau : \mathcal{I}(S) \to L^+(S, E)}$ the geometric Johnson homomorphism

<u>Rem</u> The largest case

- The existence of $\log(\widehat{DN}(\varphi))$ is straightforward
- The target is more complicated

Turaev cobracket

$\alpha : S^1 \to S$ generic immersion



$D(\alpha) \subset S$: the set of double points of α

$$\boxed{\delta(\alpha) := \sum_{p \in D(\alpha)} \alpha_1(p) \otimes \alpha_2(p) - \alpha_2(p) \otimes \alpha_1(p) \in \left(\mathbb{Z}\hat{\pi}(S) / \mathbb{Z}\cdot\mathbb{1} \right)^{\otimes 2}}$$

Example

$$\delta(\alpha) = \alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1$$

$\mathbb{Z}\hat{\pi}(S)/\mathbb{Z}\cdot\mathbb{1}$: the Goldman-Turaev Lie bialgebra

δ extends naturally to completions

[**Thm** (Kawazumi-K)

$$\delta \circ \tau = 0 : \mathcal{I}(S) \xrightarrow{\tau} L^+(S, E) \xrightarrow{\delta} \left(\widehat{\mathbb{Q}\hat{\pi}(S)} \right)^{\hat{\otimes} 2}$$]

(Key fact: Any diffeo preserves the self-intersections of loops on S)

[**Conj** $\quad \overline{\tau(\mathcal{I}(S))} = \mathrm{Ker}\left(\delta|_{L^+(S,E)} \right)$]

§5 Tensorial descriptions

1. "Coordinates" for π

$\pi = \pi_1(\Sigma_{g,1}, *)$ $\qquad \widehat{\mathbb{Q}\pi} := \varprojlim_m \mathbb{Q}\pi / (I\pi)^m$ the completed group ring of π

$H = H_1(\Sigma_{g,1}; \mathbb{Q}) \cong \pi^{abel} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\qquad \widehat{T} := \prod_{k=0}^{\infty} H^{\otimes k}$

the completed tensor algebra generated by H

Def (Massuyeau)

$\theta : \pi \to \widehat{T}$ symplectic expansion

$\Leftrightarrow$ ① $\forall x \in \pi \quad \theta(x) = 1 + [x] + (\text{higher})$

② $\forall x, \forall y \in \pi \quad \theta(xy) = \theta(x)\theta(y)$

③ $\forall x \in \pi \quad \theta(x)$ is group-like

④ $\theta(\text{boundary}) = \exp(\omega)$, where $\omega = \sum_{i=1}^{g} [A_i, B_i] \in H^{\otimes 2}$

Fact Symplectic expansion do exist! (Kawazumi/$\mathbb{R}$, Massuyeau, K., ...)

θ : fix

$$\begin{array}{ccc} \widehat{\mathbb{Q}\pi} & \xrightarrow[\theta]{\cong} & \widehat{T} \\ \cup & & \cup \\ \log(\text{boundary}) & \longmapsto & \omega \end{array}$$

: isomorphism of complete Hopf algebras

2. Goldman bracket

$$\mathfrak{a}_g^- := \{ D : \hat{T} \to \hat{T} \mid \text{(continuous) derivation},\ D(\omega) = 0 \}$$

(An enhancement of) Kontsevich's "associative"

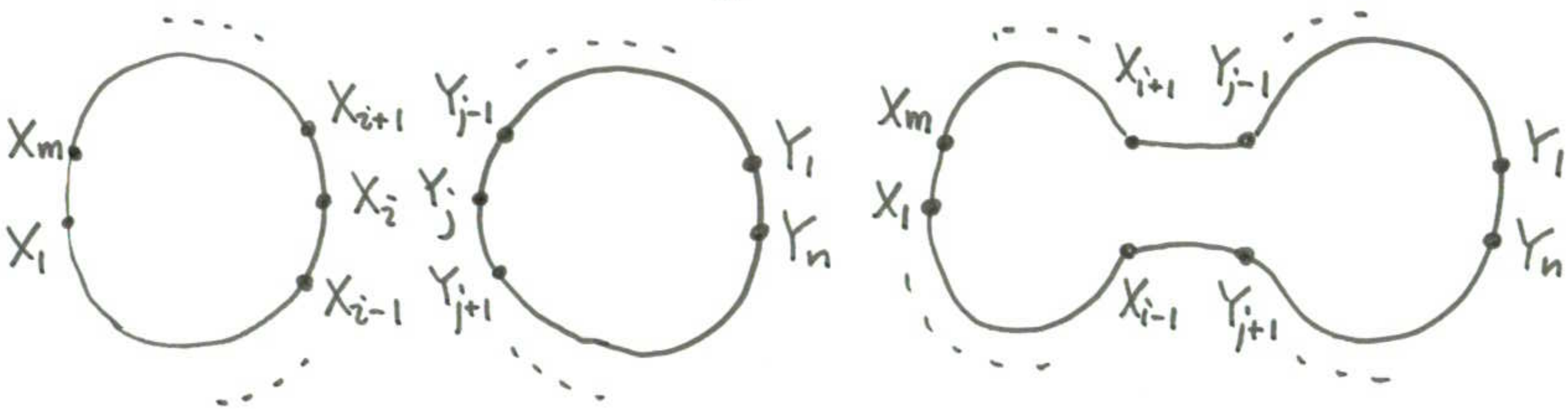
$$\begin{array}{ccccc} \mathfrak{a}_g^- & \subset & \mathrm{Hom}(H, \hat{T}) \cong H \otimes \hat{T} = \hat{T}_{\geq 1} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ D & \mapsto & D|_H \end{array}$$

Then $\mathfrak{a}_g^- \cong N(\hat{T}_{\geq 1})$, where $N : \hat{T}_{\geq 1} \to \hat{T}_{\geq 1}$

$$N(X_1 X_2 \cdots X_m) = \sum_{i=1}^{m} X_i X_{i+1} \cdots X_m X_1 \cdots X_{i-1} \qquad (X_j \in H)$$

Lie bracket

$$[N(X_1 X_2 \cdots X_m), N(Y_1 Y_2 \cdots Y_n)] = -\sum_{i,j} (X_i \cdot Y_j) N(X_{i+1} \cdots X_m X_1 \cdots X_{i-1} Y_{j+1} \cdots Y_n Y_1 \cdots Y_{j-1})$$



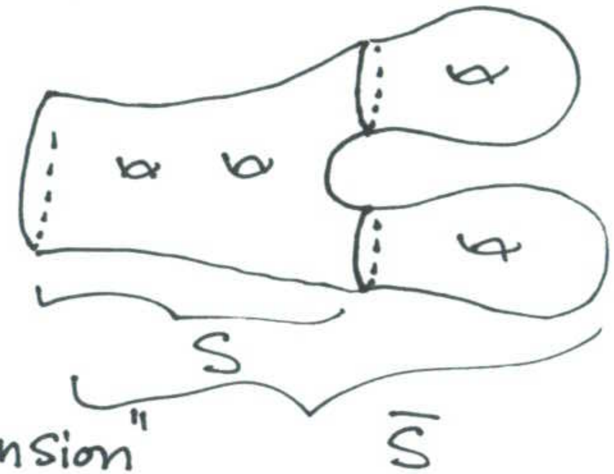
Thm (Kawazumi-K)

Any symplectic expansion θ induces a Lie algebra isom

$$-N\theta : \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}) \xrightarrow{\cong} N(\hat{T}_{\geq 1}) = \mathfrak{a}_g^- , \quad \hat{\pi}(\Sigma_{g,1}) \ni |x| \mapsto -N(\theta(x)-1)$$

Rem General case (Massuyeau-Turaev , Kawazumi-K)

quiver theory



- A choice of a section of $H_1(S) \to H_1(\bar{S})$

  $\rightsquigarrow$ A Lie algebra structure on

  $$N(\hat{T}(H_1(S))_{\geq 1})$$

- Furthermore, a choice of a "symplectic expansion" for the groupoid $\Pi S|_E$

  $\rightsquigarrow$ A Lie algebra isom

  $$\widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} N(\hat{T}(H_1(S))_{\geq 1})$$

3. Johnson homomorphism

$\mathfrak{l}_g^+ := \{ D \in \mathfrak{a}_g^- \mid D(H) \subset \hat{T}_{\geq 2},\ D \text{ stabilizes the coproduct} \}$

$= \prod_{k=1}^{\infty} \mathfrak{h}_{g,1}^{\mathbb{Z}}(k) \otimes \mathbb{Q}$

__Fact__ $\mathfrak{l}_g^+ = N((\hat{\mathcal{L}} \hat{\otimes} \hat{\mathcal{L}})_{\geq 3}) \subset N(\hat{T}_{\geq 1}) = \mathfrak{a}_g^-$, where $\hat{\mathcal{L}} \subset \hat{T}$ is the set of primitive elements

θ : fix

For $\varphi \in \mathcal{I}_{g,1}$, $T^\theta(\varphi) := \theta \circ \varphi \circ \theta^{-1} : \hat{T} \to \hat{T}$ (Kawazumi's total Johnson map)

__Massuyeau__ $\tau^\theta(\varphi) := \log T^\theta(\varphi)$ exists and lies in $\mathfrak{l}_g^+$

$\boxed{\tau^\theta : \mathcal{I}_{g,1} \longrightarrow \mathfrak{l}_g^+}$ Massuyeau's total Johnson map

- $-N\theta : \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}) \xrightarrow{\cong} \mathfrak{a}_g^-$ restricts to an isom $L^+(\Sigma_{g,1}, \{*\}) \xrightarrow{\cong} \mathfrak{l}_g^+$
- $\tau^\theta = (-N\theta) \circ \tau : \mathcal{I}_{g,1} \longrightarrow \mathfrak{l}_g^+$
- $gr(\tau^\theta) = \{\tau_k\}_k$

4. Turaev cobracket

$$\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}) \cong N(\hat{T}_{\mathbb{Z}1}) = \mathfrak{a}_g^-$$

$$\delta \leftrightarrow \delta^\theta = ?$$

<u>Thm</u> (Massuyeau-Turaev, Kawazumi-K)

$\delta^\theta = \delta^{alg} + \delta^\theta_{(1)} + \delta^\theta_{(2)} + \cdots$, where

① δ^{alg} : of degree -2, Schedler's cobracket

$$\delta^{alg}(N(X_1X_2\cdots X_m)) = -\sum_{i<j}(X_i \cdot X_j)\begin{pmatrix} N(X_{i+1}\cdots X_{j-1}) \otimes N(X_{j+1}\cdots X_m X_1 \cdots X_{i-1}) \\ - N(X_{j+1}\cdots X_m X_1 \cdots X_{i-1}) \otimes N(X_{i+1}\cdots X_{j-1}) \end{pmatrix}$$

② $\delta^\theta_{(j)}$: of degree j (Kawazumi-K: it <u>does</u> depend on θ)

Our proof uses the tensorial description of the homotopy intersection form due to Massuyeau-Turaev. We do not know how does this theorem generalize to other compact surfaces.

<u>Cor</u> $\delta^{alg} \circ \tau_k = 0 : \mathfrak{m}_{g,1}(k) \longrightarrow \bigoplus_{p+q=k} N(H^{\otimes p}) \otimes N(H^{\otimes q})$

- The Morita trace $\mathrm{Tr}_k : \mathfrak{h}^{\mathbb{Z}}_{g,1}(k) \otimes \mathbb{Q} \to S^k H$ factors through δ^{alg}.
- The Enomoto-Satoh's "anti-Morita obstruction" does not !

$[1^k]$ ($k \equiv 1 \bmod 4$, $k \geq 5$)