Two Applications of Analytic Functors

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Abstract
We apply the theory of analytic functors to two topics related to theoretical computer science. The one is a mathematical foundation of certain syntactic well-quasi-orders and well-orders appearing in graph theory, the theory of term rewrting systems, and proof theory. The other is a new verification of the Lagrange-Good inversion formula using several ideas appearing in semantics of lambda calculi, especially the relation between categorical traces and fixpoint operators.

Keyword: analytic functor, recursive path ordering, Lagrange-Good inversion formula.

Combinatorial species are introduced by Joyal [37] as a categorical framework of the theory of enumerative combinatorics. Analytic functors are equivalent to the combinatorial species [38]. A principal machinery in enumerative combinatorics is the generating functions, which are formal power series where the coefficients of x^n equal the numbers of those objects on an n point set which one wants to enumerate. An analytic functor is the functor on the category of sets having a form similar to a formal power series. Analytic functors can manipulate the structures themselves directly while generating functions the numbers of the structures.

In this paper, we give two applications of the analytic functor to completely different directions related to theoretical computer science. The first is the theory of term rewriting systems. A main theme of the theory is to develop the tools to prove the termination of computation. One of the most useful methods is Dershowitz' recursive path ordering [14]. This method is superior in respect that we can define a well-order directly on the set of terms, rather than assign to terms, say, natural numbers. This property, which is on the one hand a great advantage, may be a disadvantage on the other. In an attempt to extend the original definition to an inenumerable variation, we are lack of the principle to rely on if we insist on the syntactic structures. One of such syntactic monsters appears in proof theory. Takeuti's ordinal diagram is a powerful well-ordering

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to prove the consistency of logical systems. But it requires several pages for just definition and dozens of pages for the proof of well-orderedness [61, 63]. We want to demonstrate that the ordering including the recursive path ordering and the ordinal diagrams can be handled in a single framework using the analytic functor, thus giving a mathematical foundation to a class of well-orderings. The recursive path ordering is based on the theory of well-quasi-ordering, including the celebrated theorem by Kruskal [41]. The analytic functor is useful also to deal with a class of well-quasi-orderings in a unified framework. These points are discussed in section 2.

The second application is to the semantics of the lambda calculi. The lambda calculus is widely used as the foundation of functional programming from its beginning. However, the calculus has many concepts that challenge if we try to construct its model. For instance, we must deal with a function and a data in a single regime, and we must find the interpretation of the fixpoint combinator that produces a fixpoint for any function. This challenge stimulated the development of the domain theory after the work by Scott [57, 26]. This theory shows that, using posets, we can construct models where these characteristic phenomena of the lambda calculus are interpreted consistently. These models are extensively used to develop the theory of the lambda calculus, but they are still difficult to analyze. Later Girard gave a model of lambda calculi where the terms of the calculi are interpreted by functors in a special class of analytic functors [23]. Since analytic functors generalize formal power series, we may hope that we can directly manipulate the model of the lambda calculus, using a handful weapons of mathematics developed in centuries of its history. Indeed, we proved that the interpretations of the pure lambda terms are regarded as systems of formal power series in integer coefficients, and we gave an explicit formula for the interpretation of the fixpoint combinator in [30]. In section 3, we give a formal power series interpretation of system PCF. The system contains the fixpoint combinator, thus its interpretation by an analytic functor is expected to give a formula computing fixpoints of formal power series. In analysis, this kind of formula has been known for long as the Lagrange inversion formula [21]. We give a new proof of the generalization of the inversion formula to several variables, known as the Lagrange-Good inversion formula [24]. To this end, we use the recent result [27, 28] establishing fixpoints and categorical traces [40].

This paper is an extension of the talk presented in the workshop "Theories of Types and Proofs", and written as a lecture note surveying the author's recent works. Several results have already been published elsewhere (see the reference). Part of section 2 and the most of section 3 are new. Also the results by others are included to facilitate the reader's understanding of this ongoing field. In the first section, we give an elementary theory of analytic functors needed to read the following sections.

1 Analytic Functors

Preliminaries

We begin with preliminaries of category theory needed in this paper.

A *weak pullback* of a family of morphisms $A_i \xrightarrow{f_i} B$ is an object P with a family of morphisms $P \xrightarrow{p_i} A_i$ satisfying that all $f_i \circ p_i$ are equal, and that, for each family of morphisms $D \xrightarrow{g_i} A_i$ subject to the condition that all $f_i \circ g_i$ are equal, there is a morphism h from D to P such that $g_i = p_i \circ h$ for every i. The difference with the usual pullback lies in that we do not require the uniqueness of the intermediate morphism h. In this paper, whenever we refer pullbacks or weak pullback, we intend those for arbitrarily many legs.

Set is the category of all small sets and all functions. For a small category $\mathbf{C}$, we denote by $\mathbf{Set}^{\mathbf{C}}$ the category of all presheaves over $\mathbf{C}$ and all natural transformations. Here a presheaf over $\mathbf{C}$ is simply a functor from $\mathbf{C}$ to **Set**.

A *finitely presentable* object in a category $\mathbf{C}$ is the object X satisfying that the representable functor $\mathbf{C}(X, \text{-})$ preserves filtered colimits [2]. Finitely presentable objects in **Set** are simply finite sets. If A is a discrete category, i.e., a set, a finitely presentable object of $\mathbf{Set}^A$ is equivalent to a function f carrying $a \in A$ to a natural number, where a natural number n is identified with the set of all natural numbers satisfying $x < n$, this f subject to the condition that $f(a)$ is 0 except for finitely many a. We can regard such a function as a finite multiset γ where a is in γ with multiplicity m iff $f(a) = m$.

For a category $\mathbf{C}$ and its object B, the *slice category* $\mathbf{C}/B$ is defined as follows: Its objects are the pairs of an object A and a morphism $A \xrightarrow{f} B$ of $\mathbf{C}$. A morphism g from (A, f) to (A', f') in the slice category is a morphism $A \xrightarrow{g} A'$ of $\mathbf{C}$ rendering the triangle diagram

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commutative. More in general, for a functor $F : \mathbf{C} \to \mathbf{D}$ and an object B of $\mathbf{D}$, the slice category F/B is defined as follows: The objects are pairs of an object A of $\mathbf{C}$ and a morphism $FA \xrightarrow{f} B$ in $\mathbf{D}$. A morphism g from (A, f) to (A', f') is a morphism $A \xrightarrow{g} A'$ in $\mathbf{C}$ rendering the triangle diagram

$$\begin{tikzcd} FA \arrow[rr, "Fg"] \arrow[dr, "f"'] & & FA' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

commutative. There is an obvious forgetful functor $F/B \to \mathbf{C}$ taking the first component.

1.1 Definition
Let $F : \mathbf{C} \to \mathbf{Set}$ be a functor from some category $\mathbf{C}$.

The *category of elements* $\mathrm{el}(F)$ is defined as follows: The objects are the pairs (B, a) where B is an object of $\mathbf{C}$ and a is a member of FB. A morphism $f : (B, a) \to (B', a')$ in $\mathrm{el}(F)$ is a morphism $B \xrightarrow{f} B'$ satisfying $Ff(a) = a'$.

There is an obvious forgetful functor from $\mathrm{el}(F)$ to $\mathbf{C}$ taking the first component. If the functor F preserves pullbacks, the forgetful functor $\mathrm{el}(F) \to \mathbf{C}$ creates pullbacks. Namely, for a family of morphisms $(A_i, a_i) \xrightarrow{f_i} (B, b)$ in $\mathrm{el}(F)$, if P is a pullback of $A_i \xrightarrow{f_i} B$ in category $\mathbf{C}$, there is an element $c \in FP$ such that (P, c) is a pullback of f_i in the category $\mathrm{el}(F)$ of elements. Likewise, if F preserves weak pullbacks, for a family of morphisms f_i in $\mathrm{el}(F)$ and a weak pullback P of f_i in $\mathbf{C}$, there is a weak pullback (P, c). Similar facts hold for other types of limits as well.

To each functor $F : \mathbf{C} \to \mathbf{D}$ between small categories, we associate functors $F^* : \mathbf{Set}^{\mathbf{D}} \to \mathbf{Set}^{\mathbf{C}}$ and $F_! : \mathbf{Set}^{\mathbf{C}} \to \mathbf{Set}^{\mathbf{D}}$ such that $F_! \dashv F^*$. The inverse image functor F^* is easily defined. For each presheaf G over $\mathbf{D}$, the composition $G \circ F$ yields a presheaf over $\mathbf{C}$. The composition $(\cdot) \circ F$ is exactly the inverse image F^*.

The functor $F_! : \mathbf{Set}^{\mathbf{C}} \to \mathbf{Set}^{\mathbf{D}}$ is defined by left Kan extension. We put $F_!(T) \triangleq \mathrm{Lan}_F(T)$ for each presheaf T over $\mathbf{C}$ where $\mathrm{Lan}_F(T)$ denotes the left Kan extension of T along F. Namely, $F_!(T) = \mathrm{Lan}_F(T)$ is the presheaf over $\mathbf{D}$ satisfying the isomorphism

$$\mathbf{Set}^{\mathbf{D}}(\mathrm{Lan}_F(T), G) \;\cong\; \mathbf{Set}^{\mathbf{C}}(T, F^*(G))$$

natural in $G \in \mathbf{Set}^{\mathbf{D}}$. We have the following pointwise construction of $F_!(T) = \mathrm{Lan}_F(T)$. For each object B of $\mathbf{D}$, we have a diagram $F/B \to \mathbf{C} \xrightarrow{T} \mathbf{Set}$ where the left arrow is the forgetful functor from the slice category mentioned above. We denote this diagram by $T(F/B)$. Then the value of the left Kan extension at B is given by the colimit $F_!(T)(B) \triangleq \varinjlim T(F/B)$ in the cocomplete category $\mathbf{Set}$. This mapping $F_!$ gives rise to a functor and the isomorphism above turns out to be natural also in T. Hence $F_!$ is a left adjoint of F^*.

By the dual construction, we can define a right adjoint F_* of the functor F^*, although we do not use this.

A group G is regarded as a category with a single object $*$ and with morphisms that are the elements of G. Then a presheaf T over G determines a left G-set A as the image $T(*)$ of the single object, and vice versa. We recall that a left

G-set is a set A endowed with a left action of the group G, that is, with a group homomorphism $G \to \mathrm{Aut}_{\mathbf{Set}}(A)$. We consider the functor $F_! : \mathbf{Set}^G \to \mathbf{Set}^1$ where F is a functor from the group G to the trivial category 1 having a single object and a single morphism. For each G-set A determined by a presheaf T, the diagram $T(F/*)$ determined by the unique object $*$ of 1 is simply the G-set A itself endowed with the morphisms $p \cdot (\text{-})$ given by the left action for $p \in G$. Hence the colimit $F_!(T)$ in $\mathbf{Set}^1 \cong \mathbf{Set}$ is exactly the quotient set A/G, that is, the set of all orbits by the left action of G.

We denote by $\mathrm{Aut}_{\mathbf{C}}(A)$ the group of all automorphisms of A (i.e., invertible morphisms from A to A), for a category $\mathbf{C}$ and its object A.

Analytic Functors

We give definition of analytic functors by Joyal [38]. Let X be a finite set and G a subgroup of the symmetric group on X. Namely G is a group of bijections from X to itself, regarded to act on X from left by $p \cdot x = p(x)$ for $p \in G$ and $x \in X$. Then, for each set A, the group G acts on the homset $\mathbf{Set}(X, A)$ by composition of inverse, that is, $p \cdot f := f \circ p^{-1}$ for $p \in G$ and $f : X \to A$. In other words, G acts on $\mathbf{Set}(X, A)$ from right. Let $\mathbf{Set}(X, A)/G$ denote the set of all orbits by this action. Then $\mathbf{Set}(X, \text{-})/G$ gives rise to a functor from $\mathbf{Set}$ to $\mathbf{Set}$. An analytic functor is by definition a sum of these functors:

1.2 Definition
An *analytic functor* from $\mathbf{Set}$ to $\mathbf{Set}$ is a (not finite, in general) coproduct $\sum_{i \in I} \mathbf{Set}(X_i, \text{-})/G_i$ where X_i is a finite set and G_i is a subgroup of the symmetric group $\mathrm{Aut}_{\mathbf{Set}}(X_i)$.

A leading example frequently appearing in this paper is the functor $\exp X$. For a set A, we define $\exp A$ as the set of all finite multisets of members of A. This operation turns out to be an analytic functor. Namely, if we write $X^n = \mathbf{Set}(n, X)$,

$$\exp X \;=\; 1 + X/S_1 + X^2/S_2 + X^3/S_3 + \cdots + X^n/S_n + \cdots$$

where S_n is the symmetric group over n letters. The power X^n is the set of all lists $\langle a_1, a_2, \ldots, a_n \rangle$ of members of X of length n. The symmetric group S_n acts on the set of lists by permuting the components. The orbits by this action are the lists where the order of components are ignored. These are exactly multisets $\{a_1, a_2, \ldots, a_n\}$.

The reason this functor is named $\exp X$ lies in its similarity to the Taylor series of the exponential function $\exp x = 1 + x/1! + x^2/2! + \cdots + x^n/n! + \cdots$. If we identify the symmetric group S_n with its order $n!$, the functor $\exp X$ has the same form as the Taylor series of the exponential function. So one may regard analytic functors as generalization of formal power series. In general, transferring from $\mathbf{Set}(n, X)/G$ to $x^n/|G|$ where $|G|$ is the order of the group G, we can associate a formal power series to each analytic functor.

One of the subjects of mathematics where the formal power series are principal vehicles is enumerative combinatorics. Generating functions are a machinery to attack the problems of enumeration. The coefficient of x^n in a generating function is the number of structures satisfying the condition of the problem. The analytic functors are introduced to give a foundation to the theory of generating functions in enumerative combinatorics. We refer the reader to [37, 7]. The relation between analytic functors and enumerative combinatorics is employed to answer a question of the lambda calculus addressed by Girard [30].

We have two other equivalent conditions for analytic functors. To this end, we need several notions.

A *transitive object* is an object X satisfying the following two conditions for every object A: (i) $\mathrm{Hom}(X, A)$ is non-empty; (ii) The right action of $\mathrm{Aut}(X)$ on $\mathrm{Hom}(X, A)$ by composition is transitive. Namely, for all $f, g : X \rightrightarrows A$, there is an automorphism $p : X \xrightarrow{\sim} X$ such that $f = gp$. Obviously any object isomorphic to a transitive object is transitive.

1.3 Lemma

(i) *If X is a transitive object, every endomorphism $f : X \to X$ is invertible. Namely* $\mathrm{Hom}(X, X) = \mathrm{Aut}(X)$.

(ii) *Any two transitive objects are isomorphic.*

(*Proof*) (i) It suffices to show that each $X \xrightarrow{f} X$ has right inverse. Since the action of $\mathrm{Aut}(X)$ is transitive, there is $p \in \mathrm{Aut}(X)$ such that $f \circ p = id_X$, giving the right inverse of f. (ii) Suppose that X and Y are transitive objects. By the non-emptiness condition, there are morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$. If we let the inverses of gf and fg be p and q, then pg and gq give the left and the right inverse of f, respectively. Thus they must be equal and are the inverse of f. $\square$

Remark: An initial object is a special case of a transitive object X, where the group $\mathrm{Aut}(X)$ is a unit group.

1.4 Definition

Let $\mathbf{C}$ be a category and A its object.

A *weak normal form* of A is a transitive object $X \to A$ in the slice category $\mathbf{C}/A$.

What interests us later on are weak normal forms in the category $\mathrm{el}(F)$ of elements. We give an example for the case that F is the analytic functor $FX = \exp X$ of finite multisets. Let us consider an object $(\mathbb{N}, \{5,5,4\})$ of $\mathrm{el}(F)$. A weak normal form of this object has the shape $(3, \{0,1,2\}) \xrightarrow{f} (\mathbb{N}, \{5,5,4\})$ where the function f is not unique. For example we may set f to be given by $0 \mapsto 5$, $1 \mapsto 5$ and $2 \mapsto 4$. But an arbitrary function $3 \xrightarrow{f'} \mathbb{N}$ carrying any two elements of the domain to 5 and the remaining single element to 4 can take place of f, giving a weak normal form as well.

By the preceding lemma, a weak normal form is determined uniquely up to isomorphism, if it exists. The *weak normal form property* is satisfied by the category $\mathbf{C}$ iff each object in the slice category $\mathbf{C}/A$ has a weak normal form for every object A. Furthermore, we call X a weak normal form if an identity $X \xrightarrow{1} X$ is a weak normal form.

1.5 Lemma
Let $A \xrightarrow{g} B$ be a morphism in a category $\mathbf{C}$.

(i) *If $X \xrightarrow{h} B$ is a weak normal form, then there is a weak normal form $X \xrightarrow{f} A$ such that $h = gf$.*
(ii) *Suppose that $\mathbf{C}$ has the weak normal form property. If $X \xrightarrow{f} A$ is a weak normal form, then $X \xrightarrow{gf} B$ is a weak normal form.*

(*Proof*) (i) We repeatedly use the hypothesis that h is a weak normal form, so we refer this as WNF. By WNF, there is $X \xrightarrow{f} A$ such that $h = gf$. We show that f is a weak normal form in $\mathbf{C}/A$. For this, let $C \xrightarrow{e} A$ be any object of $\mathbf{C}/A$. By WNF, there is $X \xrightarrow{d} C$ such that $h = ged$, i.e., $h \xrightarrow{ed} g$ over B. By WNF again, there is $p : X \xrightarrow{\sim} X$ over B such that $edp = f$, i.e., $f \xrightarrow{dp} e$ over A. Since $h \xrightarrow{dp} ge$ over B, dp is unique up to $\mathrm{Aut}_{\mathbf{C}/B}(h)$ by WNF. We must show that dp is unique up to $\mathrm{Aut}_{\mathbf{C}/A}(f)$. So let us assume that $f \xrightarrow{d'} e$ over A. By WNF, there is $q : X \xrightarrow{\sim} X$ over B such that $d'q = dp$. Then $f = edp = ed'q = fq$, showing q is in $\mathrm{Aut}_{\mathbf{C}/A}(f)$.

(ii) Let $Y \xrightarrow{h} B$ be a weak normal form. Then there is $Y \xrightarrow{k} A$ such that $h = gk$. By (i), k is a weak normal form of A. So there is an isomorphism $f \cong k$, inducing $gf \cong gk (= h)$. Hence gf is a weak normal form of B. $\square$

As an easy corollary of this lemma, we have the following: Suppose that a category $\mathbf{C}$ satisfies the weak normal form property. Then $X \xrightarrow{f} A$ is a weak normal form iff X is a weak normal form. The direction $(\Leftarrow)$ is an immediate consequence of (ii) of the preceding lemma. For $(\Rightarrow)$, an application of (i) of the preceding lemma implies the existence of a weak normal form $X \xrightarrow{p} X$, this morphism p being invertible as a morphism between transitive objects f. Hence p is isomorphic to an identity 1_X in the slice category $\mathbf{C}/X$, concluding that also the identity is a weak normal form.

Now we can state two other equivalent conditions for analytic functor. The one of them is in terms of universal conditions, and the other is by the weak normal form property.

1.6 Theorem

For a functor $F : \mathbf{Set} \to \mathbf{Set}$, the following three conditions are equivalent:

(i) *F is isomorphic to an analytic functor.*

(ii) *F preserves all filtered colimits and all weak pullbacks (not only binary ones).*

(iii) *The category $\mathrm{el}(F)$ of elements has the weak normal form property. Moreover, X is a finite set for every weak normal form (X, x).*

Remark: To extend the theorem to more general categories, we should read the finite set X in the condition (iii) as a finitely presentable object.

The proof of (iii)⇒(i) is easy. Let (X_i, x_i) $(i \in I)$ be a set of representatives of all isomorphism classes of weak normal forms in $\mathrm{el}(F)$. For each element $a \in FA$, there is unique (X_i, x_i) such that a morphism $(X_i, x_i) \xrightarrow{f} (A, a)$ is a weak normal form in $\mathrm{el}(F)$. Hence, for every $(X_i, x_i) \xrightarrow{f'} (A, a)$, we have an invertible morphism p such that $f' = f \circ p$, since f' is a weak normal form as well by the corollary mentioned after Lemma 1.5 and a morphism between weak normal forms is invertible. If we put G_i to be $\mathrm{Aut}_{\mathrm{el}(F)}(X_i, x_i)$, namely the stabilizer of x_i with respect to the action of $\mathrm{Aut}_{\mathbf{Set}}(X_i)$ on FX_i, this morphism p is an element of G_i. Hence FA has a one-to-one correspondence to the sum of $\mathbf{Set}(X_i, A)/G_i$. Furthermore, if $(X_i, x_i) \xrightarrow{f} (A, a)$ is a weak normal form and $g : A \to A'$, then $(X_i, x_i) \xrightarrow{gf} (A', Fg(a))$ is a weak normal form by Lemma 1.5 (ii). This shows that Fg equals the sum of $\mathbf{Set}(X_i, g)/G_i$. Hence F is isomorphic to an analytic functor $\sum_i \mathbf{Set}(X_i, \text{-})/G_i$. This concludes the proof of (iii)⇒(i). Next we prove (i)⇒(ii).

1.7 Lemma

Each analytic functor preserves filtered colimits.

(*Proof*) A representable functor $\mathbf{Set}(X, \text{-})$ for finite X preserves filtered colimits as an endofunctor on $\mathbf{Set}$, since finite sets are exactly the finitely presentable objects of $\mathbf{Set}$. We immediately see that, if G is a subgroup of the symmetry group $\mathrm{Aut}_{\mathbf{Set}}(X)$, the functor $\mathbf{Set}(X, \text{-})$ indeed sends filtered colimits in $\mathbf{Set}$ to filtered colimits in G-$\mathbf{Set}$. Moreover, for unique $F : G^{\mathrm{op}} \to 1$, the functor $F_! : \mathbf{Set}^{G^{\mathrm{op}}} \to \mathbf{Set}$ taking quotients preserves all colimits, since $F_! = \mathrm{Lan}_F$ is the left adjoint of the inverse image functor F^*. Hence a functor $\mathbf{Set}(X, \text{-})/G$ preserves filtered colimits. Finally coproduct preserves all colimits. Therefore an analytic functor preserves filtered colimits. □

1.8 Lemma
Each analytic functor preserves weak pullbacks (including infinite ones).

(*Proof*) Coproduct (possibly infinite) on **Set** preserves pullbacks, thus preserving weak pullbacks [45]. So it suffices to show that the functor $\mathbf{Set}(X,\text{-})/G$ preserves weak pullbacks for a finite set X and a subgroup G of $\mathrm{Aut}_{\mathbf{Set}}(X)$.

Let $W \xrightarrow{c_i} A_i$ be a weak pullback of a family of morphisms $A_i \xrightarrow{h_i} B$. We can conclude that FW is a weak pullback of Fh_i if there is a function ϕ from a pullback $(\prod_{FB} FA_i)$ to FW satisfying $F(c_i) \circ \phi = \pi_i$ where π_i is the i-th projection from the pullback. To obtain ϕ, let us take $\langle [f_i]_G; i\rangle$ in $\prod_{FB} FA_i$. Namely all $h_i f_i$ are equal modulo the action of G. Thence we fix i_0, put $f_0 := h_{i_0} f_{i_0}$, and we take $p_i \in G$ such that $f_0 = h_i f_i p_i$. Since W is a weak pullback, there is a morphism $X \xrightarrow{k} W$ such that $f_i p_i = c_i k$ for all i. We define the function ϕ by $\phi(\langle [f_i]_G; i\rangle) = [k]_G$. (Remark: $[k]_G$ depends on the choice of representatives f_i). We have $F(c_i)([k]_G) = [c_i k]_G = [f_i p_i]_G = [f_i]_G$, this verifying FW to be a weak pullback. $\square$

From the last two lemmas, we have the proof of (i)$\Rightarrow$ (ii). Finally we verify (ii)$\Rightarrow$(iii).

In place of manipulating $\mathrm{el}(F)$ directly, we provide conditions for a category $\mathbf{C}$ to have a weak normal form property. First we introduce a definition. A *minimal object* in a category $\mathbf{C}$ is an object S to which every morphism $A \to S$ is an epimorphism. The conditions on $\mathbf{C}$ are the following two:

(1) For each family of morphisms $B_i \xrightarrow{f_i} A$ with a common target, there is a family of morphisms $C \xrightarrow{g_i} B_i$ such that $f_i g_i$ are equal. We refer this situation as "each fan is spanned."
(2) For each object A, there is a morphism $S \xrightarrow{f} A$ from a minimal object S which has only finitely many quotient objects.

First we show that the existence of minimal objects is related to the weak normal form property of a category $\mathbf{C}$.

1.9 Lemma
Let $\mathbf{C}$ *be a category fulfilling the conditions (1) and (2) above, and let* X *be an object of* $\mathbf{C}$.

Every $X \to A$ *is a transitive object in the slice category* $\mathbf{C}/A$ *for all object* A *if and only if all morphism* $S \to X$ *from each minimal object* S *is isomorphic.*

(*Proof*) ($\Rightarrow$) Suppose $S \xrightarrow{f} X$ is given where S is minimal. In $\mathbf{C}/X$, there is a morphism g from id_X to f. Hence $fg = id_X$. By minimality, g is epi. So f is an isomorphism. ($\Leftarrow$) Suppose that

$$\begin{tikzcd} & C \arrow[d, "h"] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

is given. We show the following two: (a) there is a morphism $X \to C$ over A. (b) if $g, g' : X \rightrightarrows C$ over A, then there is an isomorphism $p : X \xrightarrow{\sim} X$ over A such that $gp = g'$. For (a), we take a square diagram

$$\begin{tikzcd} D \arrow[r, "q'"] \arrow[d, "q"'] & C \arrow[d, "h"] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

and take $S \xrightarrow{m} D$ where S is minimal. Here the conditions (1) and (2) are used. Then $S \xrightarrow{qm} X$ is iso by the hypothesis. Hence $X \xrightarrow{(qm)^{-1}} S \xrightarrow{q'm} C$ is a morphism from f to h in $\mathbf{C}/A$. For (b), apply the same argument for (g, g') in place of (f, h). $\square$

1.10 Lemma
A category $\mathbf{C}$ *has the weak normal form property if* $\mathbf{C}$ *fulfills the conditions (1) and (2) above.*

(*Proof*) By the preceding lemma, it suffices to show that, for every minimal object X, there is a minimal object X' with an (epi)morphism $X \leftarrow X'$ such that every morphism $X' \leftarrow S$ from each minimal object S is isomorphic.

For contradiction, let us assume that there is a sequence of epimorphisms $X_0 \xleftarrow{e_0} X_1 \xleftarrow{e_1} X_2 \xleftarrow{e_2} \cdots$ where all X_i are minimal objects and none of the morphisms are isomorphisms. Let us put $e_{ij} = e_i \circ e_{i+1} \circ \cdots \circ e_{j-1}$ for $i < j$. If $i = j$, we put e_{ij} to be an identity. We consider a fan of ω morphisms $X_0 \xleftarrow{e_{0i}} X_i$, which is spanned by a family of morphisms $X_i \xleftarrow{d_i} X_\omega$. Take $X_\omega \xleftarrow{c} S$ with minimal S which has only finitely many (say, n) quotient objects. We consider $n+1$ morphisms $e_{i\,n+1} d_{n+1} c$ from S to X_i. By the pigeonhole principle, there are $i < j$ such that $e_{i\,n+1} d_{n+1} c$ is equal to $e_{j\,n+1} d_{n+1} c$ as quotient objects. Hence there is an invertible morphism $X_j \xrightarrow{p} X_i$ such that $e_{i\,n+1} d_{n+1} c = p e_{j\,n+1} d_{n+1} c$. Since $e_{j\,n+1} d_{n+1} c$ is epi, we have $e_{ij} = p$. So e_i, e_{i+1} etc. must be invertible. Contradiction. $\square$

1.11 Lemma
The category $\mathrm{el}(F)$ *of elements has the weak normal form property if* $F : \mathbf{Set} \to \mathbf{Set}$ *preserves filtered colimits and weak pullbacks (including infinite ones).*

(*Proof*) We verify that $\mathrm{el}(F)$ fulfills the conditions (1) and (2) above. The condition (1) immediately follows from the assumption that F preserves weak pullbacks (see the paragraph after Def 1.1). We prove (2). To each object (A,a) of $\mathrm{el}(F)$ there is a morphism $(X,x) \to (A,a)$ with finite set X, since F preserves filtered colimits. Each finite set X has only finitely many subobjects. Hence we can find $(X_0,x_0) \to (X,x)$ with monic $X_0 \to X$ satisfying that there is no $(Y,y) \to (X_0,x_0)$ with monic $Y \to X$. Therefore (X_0,x_0) is a minimal object over (A,a), observing that **Set** has epi-mono factorization and that $(Y,y) \xrightarrow{f} (X_0,x_0)$ is epi in $\mathrm{el}(F)$ iff f is epi in **Set**. Finally each finite set X_0 has only finitely many quotient objects, thus so does (X_0,x_0). $\square$

Remark: In the argument of this section, we used only countable fans. Accordingly, in order to prove the weak normal form property of $\mathrm{el}(F)$, it suffices that F preserves weak pullbacks of countably many morphisms.

The last lemma completes the proof of (ii)⇒(iii). So we have proved the equivalence of the three conditions in Thm. 1.6.

Normal Functors

Girard introduced normal functors for the purpose of giving models of various systems of lambda calculi. It turns out that normal functors are a special case of analytic functors. Namely the flat species [42] correspond to the normal functors. So they obtain the same concept from entirely different motivations. In [23], analytic functor is used as an alias of normal functor. To avoid confusion, we reserve the name of analytic functor for more general ones by Joyal and call normal functors for these specialized functors.

1.12 Definition
A *normal functor* from **Set** to **Set** is a coproduct $\sum_{i \in I} \mathbf{Set}(X_i, \text{-})$ of representable functors where all X_i are finite sets (finitely presentable objects of **Set**).

Therefore normal functors are simply analytic functors where the involving groups are all a unit group.

A typical example of normal functors is the functor $\mathrm{List}(X)$. For a set A, we define $\mathrm{List}(A)$ as the set of all finite lists $\langle a_1, a_2, \ldots, a_n \rangle$ of members of A. As in the case of analytic functor $\exp X$, we have the following representation as a formal power series

$$\mathrm{List}(X) \;=\; 1 + X + X^2 + X^3 + \cdots + X^n + \cdots$$

where X^n is equal to the collection of all lists of exactly n components. Another example of a normal functor is $\mathrm{BinTr}\, X$ that is the collection of all ordered binary trees with leaves labeled by members of X. Here an ordered tree means a tree with root such that the set of immediate successors of each node is linearly ordered. We have the representation

$$\mathrm{BinTr}\, X \;=\; X + X^2 + 2X^3 + 5X^4 + \cdots + C_n X^n + \cdots$$

where the coefficient C_n is the Catalan number $(1/n)\binom{2n-2}{n-1}$. Here we identify finite sets with their cardinalities, which are non-negative integers. So $C_n X^n$ is the cartesian product of two sets C_n and X^n in the category $\mathbf{Set}$.

Corresponding to Thm. 1.6, we have two equivalent characterizations of normal functors. First we define a normal form, that is a special case of a weak normal form.

1.13 Definition

A *normal form* of an object A in a category $\mathbf{C}$ is an initial object $X \to A$ in the slice category $\mathbf{C}/A$. If every object has a normal form, the category $\mathbf{C}$ fulfills the *normal form property.*

Normal forms are developed by Girard in the theory of ordinal notations called dilators in mathematical logic [22].

1.14 Theorem

For a functor $F : \mathbf{Set} \to \mathbf{Set}$, the following three conditions are equivalent:

(i) *F is isomorphic to a normal functor.*
(ii) *F preserves all filtered colimits and all pullbacks (including infinite ones).*
(iii) *The category* $\mathrm{el}(F)$ *of elements enjoys the normal form property. Moreover, X is a finite set for every normal form (X, x).*

In [23], it is assumed, by definition, a normal functor preserves equalizers as well. This condition, however, follows from the preservation of filtered colimits and pullbacks. In fact, by the preceding theorem, the preservation of these ensures that the functor is isomorphic to the formal power series $\sum_{i \in I} \mathbf{Set}(X_i, \text{-})$. Since the representable functors preserve any limits and the coproduct preserves equalizers, the functor must preserve equalizers. Indeed the functor preserves all connected limits, i.e., the limits the underlying diagram of which is connected graphically.

The proof of Thm. 1.14 is a modification of the proof of Thm. 1.6. In the proof of (iii)$\Rightarrow$(i) therein, the stabilizer G_i turns out to be a unit group in the current case, since a normal form is defined as an initial object, on which the automorphism group is a unit group. The direction (i)$\Rightarrow$(ii) is obvious, as mentioned in the preceding paragraph. We prove (ii)$\Rightarrow$(iii).

1.15 Lemma

Let A be an object of a category $\mathbf{C}$.

(i) *If an identity $A \xrightarrow{1_A} A$ is a weak normal form in a category $\mathbf{C}$, it must be a normal form.*

(ii) *Suppose that $\mathbf{C}$ has binary pullbacks. If $X \xrightarrow{f} A$ is a normal form, then $X \xrightarrow{gf} B$ is a normal form for every $A \xrightarrow{g} B$.*

(*Proof*) (i) The automorphism group $\mathrm{Aut}_{\mathbf{C}/A}(1_A)$ on the identity 1_A is always a unit group. (ii) Let $C \xrightarrow{h} B$ is an object of the slice category over B. We take a pullback $A \times_B C$ in category $\mathbf{C}$. Since $X \xrightarrow{f} A$ is a normal form, there is a unique morphism from f to the projection $A \times_B C \to A$. Hence we have a morphism $X \to A \times_B C \to C$, which turns out to be a morphism from gf to h in the slice category over B. Furthermore, every morphism from gf to h must factor through $A \times_B C$. So, by the universal property of the pullback, the morphism must be unique. $\square$

We start the proof of (ii)$\Rightarrow$(iii) of Thm. 1.14. By hypothesis (ii), F is an analytic functor, so it satisfies the weak normal form property. We prove that every weak normal form $(X,z) \xrightarrow{f} (A,a)$ is actually a normal form. By Lemma 1.5, it is implied that an identity $(X,z) \xrightarrow{1_X} (X,z)$ is a weak normal form. But the lemma above shows that it must be a normal form. Since F preserves pullbacks by hypothesis, the category $\mathrm{el}(F)$ of elements has pullbacks (see the argument after Def. 1.1). So, by (ii) of the preceding lemma, $(X,z) \xrightarrow{f} (A,a)$ must be a normal form. This completes the proof.

Composition

Composition of analytic functors is an analytic functor. This is obvious by the universal conditions satisfied with analytic functors: If both G and H preserve filtered colimits as well as weak pullbacks, then so does $H \circ G$. Later, however, we need the weak normal forms of composition of analytic functors. The following theorem shows that the weak normal forms of $H \circ G$ are obtained by taking the weak normal forms of weak normal forms

1.16 Theorem

Let G and H be analytic functors and let c be an element of $H(G(A))$.

The weak normal form of (A,c) in $\mathrm{el}(H \circ G)$ has the shape $\overline{f} : (X_1 + X_2 + \cdots + X_n, \overline{w}) \to (A,c)$ which is given as follows: Let $(Y,w) \xrightarrow{g} (GA,c)$ be a *weak normal form in* el(H). *We put the finite set* Y *to be* $\{1,2\ldots,n\}$, *and we take a family of weak normal forms* $(X_i,z_i) \xrightarrow{f_i} (A,g(i))$ *in* el(G) *for each* $i=1,2,\ldots,n$. *Function* $\overline{f}$ *is the unique map from the coproduct induced by* f_i*'s. Furthermore, we put* $\overline{w}=He(w)$ *where* $Y \xrightarrow{e} G(X_1+X_2+\cdots+X_n)$ *carries* i *to* $\overline{z}_i$ *which is the image of* $z_i \in G(X_i)$ *by the injection.*

(*Proof*) Let $(B,d) \xrightarrow{k} (A,c)$ be a morphism of el(HG). First we verify the existence of a morphism from $(\sum X_i,\overline{w})$ to (B,d) over (A,c). The morphism k induces a morphism $(GB,d) \xrightarrow{Gk} (GA,c)$ in el(H). Hence there is $(Y,w) \xrightarrow{h} (GB,d)$ over (GA,c). In turn, this induces a morphism $(B,h_i) \xrightarrow{k} (A,g_i)$ in el(G). Hence there is $(X_i,z_i) \xrightarrow{l_i} (B,h_i)$ over (A,g_i). Let us put $\overline{l}=[l_1,l_2,\ldots,l_n]$, the unique map from the coproduct $X_1+X_2+\cdots+X_n$ induced by l_i's. The mapping $G\overline{l}\circ e$ carries each $i\in Y$ to $G\overline{l}(\overline{z}_i)=h_i$. So we have $h=G\overline{l}\circ e$, and thus $d=Hh(w)=HG\overline{l}(\overline{w})$, concluding the existence of $(\sum X_i,\overline{w}) \xrightarrow{\overline{l}} (B,d)$ over (A,c).

Next we verify the morphism $\overline{l}$ is unique up to isomorphisms. So we suppose to have another morphism $\overline{l}'$ from $(\sum X_i,\overline{w})$ to (B,d) in el(HG). They induce morphisms $G\overline{l}\circ e$ and $G\overline{l}'\circ e$ from (Y,w) to (GB,d) in el(H). Hence there is an isomorphism $q:(Y,w) \xrightarrow{\sim} (Y,w)$ in el(H) from $G\overline{l}\circ e$ to $G\overline{l}'\circ e$ over (GB,d). Namely, the equation $G\overline{l}(\overline{z}_i)=G\overline{l}'(\overline{z}_{q_i})$ holds for each $i\in Y$. These equations induce morphisms $(X_i,q_i)\to(B,h_i)$ and $(X_{q_i},z_{q_i})\to(B,h_i)$ in el(G) where $h_i=G\overline{l}(\overline{z}_i)$. Thus we have a family of isomorphisms $p_i:(X_i,z_i) \xrightarrow{\sim} (X_{q_i},z_{q_i})$ in el(G) over (B,h_i). So there is a bijection r determined by $(q,\sum p_i)$ on $\sum X_i$ and it holds that r is an automorphism on $(\sum X_i,\overline{w})$ in el(HG), giving a morphism from $\overline{l}$ to $\overline{l}'$ over (B,d). $\square$

As is easily seen, the automorphism $r=(q,\sum p_i)$ in this proof can be described using the wreath product of groups. Let us recall definition of wreath product [54].

1.17 Definition

Let G and H be permutation groups acting on sets X and Y respectively.

The *wreath product* $G \operatorname{Wr} H$ is the semidirect product $G^Y \rtimes H$ where G^Y is the cartesian product of card$\,Y$ copies of G with an obvious right action of H.

More concretely, the wreath product is the group having the cartesian product of $G^Y\times H$ as its underlying set. For $\gamma,\delta\in G^Y$ and $p,q\in H$, the multiplication $(\gamma,p)\cdot(\delta,q)$ is defined by $(\gamma q\cdot\delta,\,qp)$, and the inverse $(\gamma,p)^{-1}$ is defined by $(\gamma^{-1}p^{-1},\,p^{-1})$. Here $\gamma q\cdot\delta$ is an element of G^Y carrying $y\in Y$ to $\gamma(qy)\cdot\delta(y)$, and $\gamma^{-1}p^{-1}$ carries $y\in Y$ to $(\gamma(p^{-1}y))^{-1}$.
The wreath product $G \operatorname{Wr} H$ acts on the cartesian product $X\times Y$ from left by $(\gamma,p)\cdot(x,y)=(\gamma(y)\cdot x,\,py)$. We note that the wreath product turns out to be

a permutation group by this action, namely, that two elements of the wreath product give different actions.

We let G_S denote the setwise stabilizer where G is a permutation group acting on a set X and S is a subset of X. Namely, G_S is the collection of $p \in G$ satisfying $px \in S$ for every $x \in S$. Moreover we denote by G_S^S the permutation group acting on the subset S induced by the setwise stabilizer G_S. Composition of analytic functors is described using the wreath product as follows [64].

1.18 Corollary
Let $G = \sum_{i \in I} \mathbf{Set}(X_i, \text{-})/G_i$ and $H = \sum_{j \in J} \mathbf{Set}(Y_j, \text{-})/H_j$ be analytic functors on **Set***.*

The composition $H \circ G$ is an analytic functor presented as follows:

$$H(G(A)) \;\cong\; \sum_{[e] \in HI} \mathbf{Set}\big(\textstyle\sum_{y \in Y_j} X_{e(y)},\ A\big)/K_e$$

where the group K_e for $e : Y_j \to I$ is given by $\prod_{i \in I} G_i \,\mathrm{Wr}\, (H_j)^{e^{-1}\{i\}}_{e^{-1}\{i\}}$ (cartesian product of wreath products) where the i-th wreath product acts on $X_i \times e^{-1}(i)$.

We need several comments for the meaning of the formula. First $X_i \times e^{-1}(i)$ is the sum of $e^{-1}(i)$ copies of X_i regarded as a subset of $\sum_{y \in Y_j} X_{e(y)}$. We recall the cartesian product of permutation groups acts on the disjoint sum of sets. Second, if $[e] = [e']$ holds (i.e., $e = e'p$ for some $p \in H_j$), there is a group isomorphism $(H_j)^{e^{-1}\{i\}}_{e^{-1}\{i\}}$ onto $(H_j)^{e'^{-1}\{i\}}_{e'^{-1}\{i\}}$ by conjugate $p^{-1}(\cdot)p$ for each $i \in I$. Thus a group isomorphism $K_e \cong K_{e'}$ is induced, although it is not canonical. If we identify K_e and $K_{e'}$ through a chosen isomorphism, the formula in the right hand side makes sense irrelevant of the choice of the representative e.

(*Proof*) We consider Thm. 1.16. We recall that the index set I in the formal power series representation of G equals the set of isomorphism classes of weak normal forms in $\mathrm{el}(G)$. Let (Y_j, w) be a weak normal form in $\mathrm{el}(H)$ corresponding to the index $j \in J$. Then $He(w) = He'(w)$ iff $e, e' : Y_j \rightrightarrows I$ are in the same orbit for the action of H_j, that is, $[e] = [e']$. So the isomorphism classes of weak normal forms in $\mathrm{el}(HG)$ correspond bijectively to the set of $[e] \in HI$. Furthermore, if we transfer the function e in the preceding theorem into the map (denoted by the same symbol e) carrying $i \in Y_j$ to the index $e(i) \in I$ which corresponds to the weak normal form (X_i, z_i) in the preceding theorem, then the stabilizer of $\overline{w}$ is isomorphic to the group K_e here. $\square$

Remark: If X_i is an empty set for some i, the wreath product $G_i \,\mathrm{Wr}\, (H_j)^{e^{-1}\{i\}}_{e^{-1}\{i\}}$ has no points to act on, although this wreath product may still be non-trivial. If this is the case, K_e is not a permutation group. So it is more precise to say that K_e is the permutation group induced by $\prod_{i \in I} G_i \,\mathrm{Wr}\, (H_j)^{e^{-1}\{i\}}_{e^{-1}\{i\}}$.

Combinatorial Species

Combinatorial species are introduced in [37] to give a foundation to the theory of enumerative combinatorics. They are shown to be equivalent to analytic functors. To prove this, first we form the category of analytic functors.

A *weak cartesian* natural transformation is a natural transformation $\nu : F \to G$ subject to the condition that the square diagram

$$\begin{tikzcd} FC \arrow[r, "\nu_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FD \arrow[r, "\nu_D"'] & GD \end{tikzcd}$$

is a weak pullback for every morphism $C \xrightarrow{f} D$. If F and G are analytic functors, we can show that ν preserves and reflects weak normal forms as follows:

1.19 Lemma

Let F and G be analytic functors and let $F \xrightarrow{\nu} G$ be a weak cartesian natural transformation.

(X, x) is a weak normal form in $\mathrm{el}(F)$ *if and only if $(X, \nu_X(x))$ is a weak normal form in* $\mathrm{el}(G)$.

(*Proof*) We verify the direction ($\Rightarrow$) only, leaving the other to the reader. Let $(Y, y') \xrightarrow{h} (X, \nu_X(x))$ be a weak normal form in $\mathrm{el}(G)$. It suffices to prove that h is invertible. Since the square diagram

$$\begin{tikzcd} FY \arrow[r, "\nu_Y"] \arrow[d, "Fh"'] & GY \arrow[d, "Gh"] \\ FX \arrow[r, "\nu_X"'] & GX \end{tikzcd}$$

is a weak pullback, there is a member $y \in FY$ such that $y' = \nu_Y(y)$ and $x = Fh(y)$. The latter equality means that $(Y, y) \xrightarrow{h} (X, x)$ is a morphism in $\mathrm{el}(F)$. Since (X, x) is assumed to be a weak normal form, there is a morphism $(X, x) \xrightarrow{m} (Y, y)$ in $\mathrm{el}(F)$ satisfying $hm = 1_X$. We note that $(X, \nu_X(x)) \xrightarrow{m} (Y, y')$ is a morphism of $\mathrm{el}(G)$. This follows from $Gm(\nu_X(x)) = \nu_Y(Fm(x)) = \nu_Y(y) = y'$. Since (Y, y') is a weak normal form, the endomorphism mh on (Y, y') must be invertible. Now, both hm and mh being invertible, h is invertible. $\square$

We note that the proof uses the global property of analytic functors, that is, the existence of weak normal forms for all objects in the category of elements.

1.20 Definition (of $[\mathbf{Set},\mathbf{Set}]_{\mathbf{AF}}$)
$[\mathbf{Set},\mathbf{Set}]_{\mathbf{AF}}$ is the category of all analytic functors on **Set** and all weak cartesian natural transformations between them.

Let $\mathbf{B}$ be the category defined as follows: The objects are all natural numbers n, and the morphisms from n to n are all permutations over n letters. There are no morphisms between m and n if $m \neq n$. Hence $\mathbf{B}$ is the groupoid of the disjoint sum of symmetric groups S_n where n ranges over $\mathbb{N}$. We note that $\mathbf{B}$ is equivalent to the (large) category of all finite sets and all bijections.

A *combinatorial species* is a functor from $\mathbf{B}$ to **Set**. A morphism between combinatorial species is simply a natural transformation. Hence the category of combinatorial species is nothing but the category $\mathbf{Set}^{\mathbf{B}}$ of presheaves. We denote a combinatorial species by $F[\cdot]$ using square brackets while an analytic functor by $F(\cdot)$ using braces. We warn that $F[n]$ differs from $F(n)$ even if n is a natural number that may be regarded as a finite set of n elements.

Our purpose is to verify that there is a categorical equivalence between the category $[\mathbf{Set},\mathbf{Set}]_{\mathbf{AF}}$ of analytic functors and the category $\mathbf{Set}^{\mathbf{B}}$ of combinatorial species. First we define a functor from $\mathbf{Set}^{\mathbf{B}}$ to $[\mathbf{Set},\mathbf{Set}]_{\mathbf{AF}}$. The object map of the functor is given as follows. To each presheaf $F[\cdot]$, we associate a functor

$$F(A) \triangleq \sum_{n\in\mathbb{N}} \mathbf{Set}(n,A) \times_{S_n} F[n]$$

where A ranges over **Set**. Here $\mathbf{Set}(n,A)\times_{S_n} F[n]$ is the quotient of the cartesian product by equating (f,x) to $(f\circ p^{-1},\, p\cdot x)$ for all $p \in S_n$, where $n \xrightarrow{f} A$ and x is a member of $F[n]$.

This functor $F(A)$ is actually an analytic functor, since we have an isomorphism $F(A) \cong \sum_i \mathbf{Set}(X_i,A)/G_i$ where the summation ranges over all pairs of natural numbers X_i and orbits o_i in $F[X_i]$, and G_i is the stabilizer subgroup of an arbitrary element in the orbit o_i. Note that the group G_i is determined up to conjugation.

For the morphism map of the functor, a natural transformation $\nu[\cdot] : F[\cdot] \to G[\cdot]$ in $\mathbf{Set}^{\mathbf{B}}$ yields a natural transformation $\nu(\cdot) : F(\cdot) \to G(\cdot)$. For a set A, the function $\nu(A)$ by definition sends (f,x) to $(f,\, \nu[n](x))$ if $n \xrightarrow{f} A$ and $x \in F[n]$.

We must show that $\nu(\cdot)$ is weak cartesian. Let $A \xrightarrow{k} B$ be a function. We prove that the following square diagram is a weak pullback.

$$\begin{tikzcd}
F(A) \arrow[r, "\nu(A)"] \arrow[d, "F(k)"'] & G(A) \arrow[d, "G(k)"] \\
F(B) \arrow[r, "\nu(B)"'] & G(B)
\end{tikzcd}$$

Let (g, x) be a member of $F(B)$ and (f, y) a member of $G(A)$, and suppose that these members are sent to the same element of $G(B)$. Hence there is a permutation p such that $g = kfp^{-1}$ and $\nu[n](x) = p \cdot y$. Then $(f,\ p^{-1} \cdot x)$ is an element of $F(A)$, this element sent to (g, x) and (f, y) by $F(k)$ and $\nu(A)$ respectively. This verifies that $F(A)$ is a weak pullback.

1.21 Theorem
There is a categorical equivalence between the category $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{AF}}$ *of analytic functors and the category* $\mathbf{Set}^{\mathbf{B}}$ *of combinatorial species.*

(*Proof*) We leave to the reader to show that the functor from $\mathbf{Set}^{\mathbf{B}}$ to $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{AF}}$ defined above is full and faithful. It remains to show that every analytic functor $F(\cdot)$ induces a combinatorial species $F[\cdot]$ satisfying a natural isomorphism $FA \cong \sum_{n\in\mathbb{N}} \mathbf{Set}(n, A) \times_{S_n} F[n]$.

For each natural number n, we define the set $F[n]$ as the sum of the set $S_n/\mathrm{Aut}_{\mathrm{el}(F)}(n, c)$ of cosets where the summation is over the equivalence classes of all weak normal forms (n, c) in $\mathrm{el}(F)$. The symmetric group S_n acts on $F[n]$ by the canonical left actions on cosets.

Note that $\sum_{n\in\mathbb{N}} \mathbf{Set}(n, A) \times_{S_n} F[n]$ is naturally isomorphic to $\sum \mathbf{Set}(X_i, A)/G_i$ where i ranges over the equivalence classes of weak normal forms and, if (n, c) is a representative, X_i equals n and G_i is $\mathrm{Aut}_{\mathrm{el}(F)}(n, c)$. Furthermore a natural isomorphism $\sum \mathbf{Set}(X_i, A)/G_i \cong F(A)$ is determined by the function carrying $X \xrightarrow{f} A$ to a member a of $F(A)$ given by the weak normal form $(X, c) \xrightarrow{f} (A, a)$. $\square$

Many Variables

A theorem similar to 1.21 holds for normal functors. In this case, however, we have a better result, that the normal functors may have additional parameters. From this result, we can form a cartesian closed category having normal functors as morphisms. We demonstrate this in the following.

We extend normal functors to finitely or infinitely many variables. Let A be a set. The category $\mathbf{Set}^A$ of presheaves is regarded as the cartesian product of $\mathrm{card}\, A$ copies of $\mathbf{Set}$. A *normal functor* from $\mathbf{Set}^A$ to $\mathbf{Set}$ is by definition a coproduct $\sum_{i \in I} \mathbf{Set}^A(X_i, \text{-})$ of representable functors where all X_i are finitely presentable objects of $\mathbf{Set}^A$. By tupling these functors, we can define also a normal functor from $\mathbf{Set}^A$ to $\mathbf{Set}^B$ for sets A and B. Namely a normal functor F from $\mathbf{Set}^A$ to $\mathbf{Set}^B$ is a family of normal functors $F_b : \mathbf{Set}^A \to \mathbf{Set}$ where b ranges over the set B.

Let us recall that a finitely presentable object of $\mathbf{Set}^A$ is regarded as a finite multiset of members of A. If A is a finite set n, a presheaf Z in $\mathbf{Set}^n$ is regarded as a tuple $(Z_0, Z_1, \ldots, Z_{n-1})$ of sets. If a finitely presentable object of $\mathbf{Set}^n$ is given as a multiset γ containing $0, 1, \ldots, n-1$ with multiplicity $m_0, m_1, \ldots, m_{n-1}$ respectively, then the value $\mathbf{Set}^n(\gamma, Z)$ is exactly a monomial ${Z_0}^{m_0} {Z_1}^{m_1} \cdots {Z_{n-1}}^{m_{n-1}}$. For a general set A, in the same way, we obtain monomials of $\operatorname{card} A$ variables. So, as a sum of these monomials, a normal functor from $\mathbf{Set}^A$ to $\mathbf{Set}$ is a formal power series in $\operatorname{card} A$ variables.

We can consider a category where a morphism is a normal functor from $\mathbf{Set}^A$ to $\mathbf{Set}^B$. First we characterize the objects. We recall the Lindenbaum-Tarski duality asserting that the category of all complete atomic Boolean algebra and all homomorphisms preserving sups and infs is equivalent to the opposite category $\mathbf{Set}^{\mathrm{op}}$ [36]. We define categories that behave similarly to complete atomic Boolean algebras.

A coproduct $A = \sum_{i \in I} X_i$ is *disjoint* iff every injection $X_i \to A$ is a monomorphism and, in addition, the pullback $X_i \times_A X_j$ is an initial object for all $i \neq j$. Moreover, a coproduct $A = \sum_{i \in I} X_i$ is *universal* iff, for every $B \xrightarrow{g} A$, the object B is a colimit of the pullbacks $B \times_A X_i$ [37].

A *complete atomic accessible category* is a category $\mathbf{C}$ with finite limits and small coproducts that are disjoint and universal, this category $\mathbf{C}$ subject to the condition that (i) the lattice $\mathrm{Sub}(1)$ of the subobjects of a terminal object 1 is small, and (ii) every object is a coproduct of atomic elements in the lattice $\mathrm{Sub}(1)$. It follows that the lattice $\mathrm{Sub}(1)$ is a complete atomic Boolean algebra. We state the following theorem analogous to the Lindenbaum-Tarski duality.

1.22 Theorem

Let $\mathbf{CAAcc}_{\mathrm{CC}}$ be the category of all complete atomic accessible categories and all functors that are both continuous and cocontinuous.

There is a categorical equivalence $\mathbf{CAAcc}_{\mathrm{CC}} \cong \mathbf{Set}^{\mathrm{op}}$.

(*Proof*) Sketch. For each set A, the category $\mathbf{Set}^A$ is a complete atomic accessible category. Conversely, each complete atomic accessible category $\mathbf{C}$ is equivalent to the category $\mathbf{Set}^{\mathrm{Atm}(\mathbf{C})}$ where $\mathrm{Atm}(\mathbf{C})$ is the set of all atomic elements in the lattice $\mathrm{Sub}(1)$. $\square$

By this theorem, we may identify a complete atomic accessible category with a category of presheaves over a set (with an abuse: a complete atomic accessible category is only equivalent to a category $\mathbf{Set}^A$ of presheaves, not identical).

We define $(\mathbf{Set}^A, \mathbf{Set}^B)_{\mathrm{NF}}$ as the set of all isomorphism classes of normal functors from $\mathbf{Set}^A$ into $\mathbf{Set}^B$. Hence, in this set, we do not distinguish two normal functors that are isomorphic. Taking this as a hom-set, the following category is defined.

1.23 Definition (of $\mathbf{CAAcc}_{\mathrm{NF}}$)
The (large) category $\mathbf{CAAcc}_{\mathrm{NF}}$ has all complete atomic accessible categories $\mathbf{Set}^A$ as objects and all isomorphism classes of normal functors as morphisms.

We may lift the category just defined to a 2-category by introducing the following notion specializing weak cartesian natural transformations defined earlier. A *cartesian* natural transformation is a natural transformation $\nu : F \to G$ subject to the condition that the square diagram

$$\begin{tikzcd} FC \arrow[r, "\nu_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FD \arrow[r, "\nu_D"'] & GD \end{tikzcd}$$

is a pullback for every morphism $C \xrightarrow{f} D$.

However the 2-category obtained in this way is not so well-behaved. The trouble occurs when we prove the cartesian closedness as we do in Thm. 1.24 below. In order to circumvent the problem, we have to identify certain cartesian natural transformations. We regard two cartesian natural transformations $\nu, \nu' : F \rightrightarrows G$ to be equal if and only if $(X, \nu_X(a)) \cong (X, \nu'_X(a))$ holds in $\mathrm{el}(G)$ for every normal form (X, a) in $\mathrm{el}(F)$. The last condition signifies that, for every normal form (X, a) in $\mathrm{el}(F)$, two elements $\nu_X(a)$ and $\nu'_X(a)$ in $G(X)$ are in the same orbit for the action of $\mathrm{Aut}_{\mathbf{C}}(X)$. The policy behind this identification is that we should not distinguish the difference of choice of a representative from each orbit.

We define $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{NF}}$ as the category having all normal functors from $\mathbf{Set}^A$ to $\mathbf{Set}^B$ as objects and all cartesian natural transformations where we identify certain cartesian natural transformations as above. For the later application to model lambda calculi, we use sets $(\mathbf{Set}^A, \mathbf{Set}^B)_{\mathrm{NF}}$ identifying isomorphic normal functors, though we sometimes employ categories $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{NF}}$ to verify certain properties.

We apply Thm. 1.21 to a normal functor $F : \mathbf{Set} \to \mathbf{Set}$. For each set X, we have $F(X) \cong \sum_{n \in \mathbb{N}} \mathbf{Set}(n, X) \times_{S_n} F[n]$. Since F is normal, the action of the symmetric group S_n on $F[n]$ is free, i.e., the stabilizer of every element in $F[n]$ is the trivial group. Thus, if we put $\tilde{F}[n] = F[n]/S_n$, the set $\mathbf{Set}(n, X) \times_{S_n} F[n]$ has a one-to-one correspondence to $\mathbf{Set}(n, X) \cdot \tilde{F}[n]$ where (-)·(-) is the shorthand for cartesian product. Here n is regarded to range over the finitely presentable objects of $\mathbf{Set}$, and S_n is the automorphism group $\mathrm{Aut}_{\mathbf{Set}}(n)$. Likewise, if we have a normal functor $F : \mathbf{Set}^A \to \mathbf{Set}$, there is a natural isomorphism $F(X) \cong \sum_{\gamma \in \exp A} \mathbf{Set}^A(\gamma, X) \times_{\mathrm{Aut}(\gamma)} F[\gamma]$. We recall that finitely presentable objects of $\mathbf{Set}^A$ are regarded as finite multisets in $\exp A$. Since F is normal, if we put $\tilde{F}[\gamma] = F[\gamma]/\mathrm{Aut}_{\mathbf{Set}^A}(\gamma)$, we have the formula

$$F(X) \;\cong\; \sum_{\gamma \in \exp A} \mathbf{Set}^A(\gamma, X) \cdot \tilde{F}[\gamma].$$

So we might regard each $\tilde{F}[\gamma]$ as the coefficient of the γ-th power in the formal power series given by the normal functor. From the observation above, we have the following theorem:

1.24 Theorem
Bijection $(\mathbf{Set}^{A+B}, \mathbf{Set}^C)_{\mathrm{NF}} \cong (\mathbf{Set}^A, \mathbf{Set}^{\exp B \times C})_{\mathrm{NF}}$ *holds for all sets* A*,* B *and* C*.*

(*Proof*) We prove a stronger assertion: categorical equivalence $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{NF}} \cong \mathbf{Set}^{\exp A \times B}$ holds for all sets A and B. From this, categorical equivalence $[\mathbf{Set}^{A+B}, \mathbf{Set}^C]_{\mathrm{NF}} \cong [\mathbf{Set}^A, \mathbf{Set}^{\exp B \times C}]_{\mathrm{NF}}$ follows if we note that $\exp(A+B) \cong \exp A \times \exp B$. Thence the identification of isomorphic normal functors yield the theorem.

A functor from $\mathbf{Set}^{\exp A \times B}$ into $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{NF}}$ is given by associating normal functor $F_b : \mathbf{Set}^X \to \mathbf{Set}$ to each $\tilde{F} \in \mathbf{Set}^{\exp A \times B}$ by defining $F_b(X) = \sum \mathbf{Set}^A(\gamma, X)\tilde{F}[\gamma, b]$ for each $b \in B$. This association gives rise to the morphism map carrying $\tilde{\nu} : \tilde{F} \to \tilde{G}$ to the cartesian natural transformation $\nu : F \to G$ defined by $\nu_{b,X} : (\gamma, k, x) \mapsto (\gamma, k, \tilde{\nu}_{\gamma,b}(x))$ where $\gamma \xrightarrow{k} X$ and $x \in \tilde{F}[\gamma, b]$. The functor is essentially onto as proven in the paragraph immediately before this theorem.

We prove that the functor is full. Let F and G be the normal functors induced from $\tilde{F}$ and $\tilde{G}$. Let us take an arbitrary cartesian natural transformation $F \xrightarrow{\nu} G$. For every $x \in \tilde{F}[\gamma, b]$, applying Lemma 1.19 to the normal form $(\gamma, (\gamma, id_\gamma, x))$ in $\mathrm{el}(F_b)$, we infer that $(\gamma, \nu_{b,\gamma}(\gamma, id_\gamma, x))$ is a normal form in $\mathrm{el}(G_b)$ (the lemma is proved only to the case that the domain is the category $\mathbf{Set}$, but we notice that the proof works for every category). Hence there is a unique $y \in \tilde{G}[\gamma, b]$ such that $(\gamma, \nu_{b,\gamma}(\gamma, id_\gamma, x)) \cong (\gamma, (\gamma, id_\gamma, y))$, since every normal form in $\mathrm{el}(G_b)$ is isomorphic to this shape for a unique y. We put this y to be $\tilde{\mu}_{\gamma,b}(x)$, deriving $\tilde{\mu} : \tilde{F} \to \tilde{G}$, which in turn induces a cartesian natural transformation $F \xrightarrow{\mu} G$. Then the isomorphism immediately above means $(\gamma, \nu_{b,\gamma}(\gamma, id_\gamma, x)) \cong (\gamma, \mu_{b,\gamma}(\gamma, id_\gamma, x))$. This implies also $(Z, \nu_{b,Z}(c)) \cong (Z, \mu_{b,Z}(c))$ for every normal form (Z, c) in $\mathrm{el}(F_b)$ by taking a normal form $(\gamma, (\gamma, id_\gamma, x))$ isomorphic to (Z, c). Hence two cartesian natural transformations ν and μ are identified in the category $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{NF}}$, concluding that the involving functor is full. Finally it is easy to see that the functor is faithful. $\square$

The equivalence between $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{NF}}$ and $\mathbf{Set}^{\exp A \times B}$ would fail, if we did not identify certain cartesian natural transformations. A counter-example is given in [62]. For the normal functor $F : \mathbf{Set} \to \mathbf{Set}$ given by $F(X) = X^2$, there are two cartesian natural transformations from F to F, the one being an identity and the other switching the components, but the corresponding presheafs in $\mathbf{Set}^\omega$ are the same.

The category $\mathbf{Set}^{A+B}$ is the cartesian product of $\mathbf{Set}^A$ and $\mathbf{Set}^B$. Moreover it is easy to show that the bijection of the theorem of above is natural in $\mathbf{Set}^A$ and $\mathbf{Set}^C$. Therefore we conclude the following:

1.25 Corollary
Category $\mathbf{CAAcc}_{\mathrm{NF}}$ *is cartesian closed.*

The extension of analytic functors to many variables is defined likewise. An analytic functor from $\mathbf{Set}^A$ to $\mathbf{Set}$ may be defined as $F(X) = \sum_i \mathbf{Set}^A(\gamma_i, X)/G_i$ where γ_i is an element of $\exp A$ and G_i is a subgroup of $\mathrm{Aut}_{\mathbf{Set}^A}(\gamma_i)$.

There is a subtle point, however, for this extension. For example, we consider the analytic functor $F : \mathbf{Set}^2 \to \mathbf{Set}$ defined by $F(X, Y) = (X^2 Y^2)/G$ for the subgroup G of $S_2 \times S_2$ generated by (σ, σ) where σ is the generator of S_2. Then, the functor $G_X(\text{-}) \triangleq F(X, \text{-})$ for fixed X is an analytic functor, since it preserves filtered colimits and weak pullbacks. So there is a bijection $F(X, Y) \cong G_X(Y)$, but it is not a natural isomorphism in X, since the group G is not decomposed into a cartesian product of subgroups of S_2. Hence we fail to obtain cartesian closedness in this way.

Initial Algebra

Let $F : \mathbf{C} \to \mathbf{C}$ be an endofunctor. An F-*algebra* is a pair of an object A and a morphism $FA \xrightarrow{f} A$ of $\mathbf{C}$. A morphism g between F-algebras (A, f) and (A', f') is a morphism $A \xrightarrow{g} A'$ of $\mathbf{C}$ rendering the following square diagram commutative

$$\begin{tikzcd} FA \arrow[r, "Fg"] \arrow[d, "f"'] & FA' \arrow[d, "f'"] \\ A \arrow[r, "g"'] & A' \end{tikzcd} .$$

An *initial F-algebra* μF is an initial object of the category of F-algebras. From the universal condition of an initial object, it is derived that for an initial algebra $F(\mu F) \xrightarrow{f} \mu F$ the morphism f is invertible. So an initial algebra is a categorical counterpart of a least fixpoint.

Let us consider an analytic functor F. Then F preserves filtered colimits. Hence an initial F-algebra is given as an inductive colimit of the ω-chain $A_0 \xrightarrow{e_0} A_1 \xrightarrow{e_1} A_2 \xrightarrow{e_2} \cdots$ in category **Set**. Here A_0 is an empty set and e_0 is an empty map. Moreover, recursively, the set A_{n+1} is defined by FA_n and e_{n+1} by $F(e_n)$.

An analytic functor preserves monomorphisms. This is obvious from the form of the analytic functor as a sum of quotients of representable functors. Thus all e_n is monomorphic, since e_0 is monomorphic as an empty map. So the initial algebras μF as an inductive colimit of A_n may be regarded as an increasing union $\bigcup_{n=0}^{\infty} A_n$. Hence, for each element $a \in \mu F$, there is the least index n such that a is a member of A_n.

Notation: For each member $a \in \mu F$, we let $p(a)$ denote the least index n such that $a \in A_n$.

We consider an initial algebra of an analytic functor in several variables. For example, if $F(X,Y)$ is an analytic functor on $\mathbf{Set}^2$, we may regard an initial algebra $\mu Y.\, F(X,Y)$ with respect to the second argument. The initial algebra $\mu F(X) = \mu Y.\, F(X,Y)$ is a functor in X, since initial algebras are given as colimits of ω-chains of bifunctors. We want to show that $\mu F(X)$ is actually an analytic functor.

1.26 Lemma
The category $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{AF}}$ is cocomplete, and colimits are pointwise.

(*Proof*) The first statement is immediate from the equivalence $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{AF}} \cong \mathbf{Set}^{\mathbf{B}}$, since the latter is cocomplete. A colimit $\varinjlim F_i$ in $[\mathbf{Set}, \mathbf{Set}]_{\mathrm{AF}}$ is defined to correspond to the colimit $\varinjlim F_i[\cdot]$ in the category $\mathbf{Set}^{\mathbf{B}}$ of presheaves. To show that the colimit is pointwise, we note that

$$\begin{aligned} (\varinjlim F_i)(A) &\cong \textstyle\sum_n \mathbf{Set}(n, A) \times_{S_n} \varinjlim F_i[n] \\ \varinjlim (F_i(A)) &\cong \varinjlim \textstyle\sum_n \mathbf{Set}(n, A) \times_{S_n} F_i[n]. \end{aligned}$$

But sum $\sum_n$, product $\mathbf{Set}(n, A) \times (\cdot)$, and the quotient of S_n-sets all preserve colimits, since they are left adjoint. Hence we must have $(\varinjlim F_i)(A) \cong \varinjlim(F_i(A))$. $\square$

Remark: The category $[\mathbf{Set}, \mathbf{Set}]_{\mathbf{AF}}$ is also complete, since so is $\mathbf{Set}^{\mathbf{B}}$. But limits are not pointwise.

1.27 Proposition
Let $F : \mathbf{Set}^A \times \mathbf{Set} \to \mathbf{Set}$ be a binary analytic functor.

The initial algebra μF turns out to be an analytic functor from $\mathbf{Set}^A$ to $\mathbf{Set}$. Furthermore, if F is normal then μF is normal

(*Proof*) For each object Y of $\mathbf{Set}^A$, the initial algebra $\mu F(Y)$ is defined as the colimit of $A_n(Y)$ where $A_0(Y) = \emptyset$ and $A_{n+1}(Y) = F(Y, A_n(Y))$. We let $(\gamma_n)_Y : A_n(Y) \to \mu F(Y)$ denote the colimiting cone. By Lemma 1.26, γ_n is a weak cartesian natural transformation. We must verify that each object (B, b) of $\mathrm{el}(\mu F)$ has a weak normal form. There is an index n and $b_0 \in A_n(B)$ such that $b = (\gamma_n)_B(b_0)$. Hence, if we take a weak normal form $(Z, c_0) \xrightarrow{f} (B, b_0)$ in $\mathrm{el}(A_n)$, then, for $c = (\gamma_n)_Z(c_0)$, the morphism $(Z, c) \xrightarrow{f} (B, b)$ is a weak normal form in $\mathrm{el}(\mu F)$ by Lemma. 1.19 since γ_n is weakly cartesian. $\square$

2 Divisibility Orderings and Recursive Path Orderings

In this section, we show that analytic functors provide a foundation to the theory of well-partial-orderings including tree-embeddings, and the theory of recursive path orderings. The former developed in graph theory, and the latter in the theory of term rewriting system in theoretical computer science.

We define the divisibility ordering which is a generalization of the notion of the same name introduced by Higman [32], and the recursive path ordering which is a generalization of the one introduced by Dershowitz [14]. We note that the topological or syntactical parts in these notions are taken place by the weak normal forms developed in the last section.

Well-Partial-Order

Well-partial-order is a generalization of well-order, which is a linear order with no infinite strictly decreasing sequences. A well-partial-order is by definition a poset that has neither infinite strictly decreasing sequences nor infinite antichains. We recall that an *antichain* is a set where every pair of its members is incomparable. A further generalization is a well-quasi-order, which is simply a quasi-order that turns out to be a well-partial-order if we collapse it by the equivalence relation induced by the quasi-ordering.

The notion of well-partial-order frequently appeared in the literature. It started being known after the work by Higman [32] who showed that a family of the sets of finite trees with bounded number of immediate successors are endowed with well-partial-orders, and by Kruskal [41] who showed that the set of all finite trees is a well-partial-order with respect to the topological embedding, settling Vazsonyi's conjecture. This result, now known as Kruskal's theorem, is given a short proof by Nash-Williams [48]. An important result in graph theory is a series of results including an extension of Kuratowski's theorem to general surfaces by Robertson and Seymour [53]. In this theorem, it is proved that certain classes of graphs form well-partial-orders with respect to the graph-minor relation.

The well-quasi-order has a simple definition, if we introduce the notion of bad sequences. A *bad sequence* in a quasi-ordered set is a finite or infinite sequence $\langle a_0, a_1, \ldots \rangle$ of its members subject to the condition $\forall\, i < j.\, a_i \not\trianglelefteq a_j$.

2.1 Definition
A *well-quasi-order* is a quasi-ordered set that has no infinite bad sequences.

In the following proposition, a strictly decreasing sequence in a quasi-ordered set is defined to be a sequence $\langle a_0, a_1, \ldots \rangle$ satisfying $a_i \trianglerighteq a_{i+1}$ but $a_i \not\trianglelefteq a_{i+1}$ for all i.

2.2 Proposition
A quasi-ordered set A is a well-quasi-order iff A has neither infinite antichains nor infinite strictly decreasing sequences.

(*Proof*) ($\Rightarrow$) is obvious. For ($\Leftarrow$), we assume that A has an infinite bad sequence $\langle a_0, a_1, \ldots \rangle$. We color the set $[\mathbb{N}]^2$ of all subsets of natural numbers of cardinality two by blue and red. For each $\{i, j\}$ with $i < j$, we color it by blue if $a_i \trianglerighteq a_j$, and by red if a_i and a_j are incomparable. Then an infinite Ramsey theorem [25] implies the existence of an infinite subset $S \subseteq \mathbb{N}$ such that $[S]^2$ is monochromatic. Let us consider the subsequence $\langle a_{k(0)}, a_{k(1)}, \ldots \rangle$ where $k(n)$ enumerates the set S in the increasing order. According to which the color, blue or red, the set $[S]^2$ has, the subsequence is either strictly decreasing or an antichain. $\square$

The following lemma is immediate by definition of well-quasi-orders.

2.3 Lemma
Let $\trianglelefteq$ and $\trianglelefteq'$ be two quasi-orderings on the same set A, these quasi-orderings satisfying that $a \trianglelefteq b$ implies $a \trianglelefteq' b$.

If $(A, \trianglelefteq)$ is a well-quasi-order, then $(A, \trianglelefteq')$ is a well-quasi-order.

The *linearization* of a poset $(A, \trianglelefteq)$ is a linear order $<$ on the set A satisfying that $a \trianglelefteq b$ implies $a \leq b$, where $\leq$ is the reflexive closure of the strict linear order $<$. Since a linear order that is a well-partial-order at the same time is exactly a well-order, the next corollary follows.

2.4 Corollary
A linearization of a well-partial-order is a well-order.

Let $Bad(A)$ be the tree of all finite bad sequences in a quasi-order A. The tree $Bad(A)$ is well-founded if and only if A is a well-quasi-order. We can assign an ordinal to a well-founded tree by the following procedure. Let T be a well-founded tree. For each node s, we assign the ordinal $|s|$ that is the supremum of $|t| + 1$ where the nodes t range over all immediate successors of s. In particular, every leaf has the ordinal 0. The ordinal assigned to the tree T itself is defined as the ordinal assigned to the root of T. We denote this ordinal by $|T|$.

In particular, if A is a well-order, the ordinal assigned to A as above is exactly the same as the order type of well-ordered sets in the usual sense, that is, the ordinal isomorphic to A. If $\trianglelefteq$ and $\trianglelefteq'$ are well-quasi-orders on the same set A satisfying $a \trianglelefteq b \Rightarrow a \trianglelefteq' b$, then the ordinal assigned to $Bad(A, \trianglelefteq')$ is less than or equal to the ordinal assigned to $Bad(A, \trianglelefteq)$. In particular, the order type of a linearization of a well-partial-order A is less than or equal to the ordinal of $Bad(A)$. De Jongh and Parikh [13] proved that there is a linearization achieving this upper bound.

2.5 Theorem
Let A be a well-partial-order.

A has a linearization of the order type equal to $|Bad(A)|$.

Divisibility Ordering

We let **QO** denote the category of all quasi-ordered sets and all order-preserving functions, that is, those functions $A \xrightarrow{f} B$ satisfying that $a \trianglelefteq a'$ implies $f(a) \trianglelefteq f(a')$.

Some of analytic functors behave as functors on $\mathbf{QO}$. For example, let us consider the functor $\exp X$. If A is a quasi-order, we can endow $\exp A$ with the quasi-order defined as follows: Let $\alpha = \{s_1, s_2, \ldots, s_m\}$ and $\beta = \{t_1, t_2, \ldots, t_n\}$ be multisets in $\exp A$. Then $\alpha \trianglelefteq \beta$ is defined to hold iff there is a one-to-one function $[m] \xrightarrow{k} [n]$ such that $s_i \trianglelefteq t_{k(i)}$ holds in the quasi-order A for each $i = 1, 2, \ldots, m$. If we regard A as the set of variables $x_1, x_2, \ldots$, and if we identify a multiset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$ with a monomial $x^\alpha = x_{i_1} x_{i_2} \cdots x_{i_n}$, then $\alpha \trianglelefteq \beta$ holds if and only if the monomial x^α divides the monomial x^β. So we call this quasi-ordering on $\exp A$ the divisibility relation.

A *lifting* of an analytic functor F_0 to the category $\mathbf{QO}$ of quasi-orders is a functor $F : \mathbf{QO}^n \to \mathbf{QO}$ rendering the square diagram

$$\begin{tikzcd}
\mathbf{QO}^n \arrow[r, "F"] \arrow[d] & \mathbf{QO} \arrow[d] \\
\mathbf{Set}^n \arrow[r, "F_0"'] & \mathbf{Set}
\end{tikzcd}$$

commutative, where the vertical arrows are the functors forgetting the structure of quasi-orders. We note that F_0 is uniquely determined by F. For simplicity, we may sometimes let the same symbol F denote the functors on both $\mathbf{QO}$ and $\mathbf{Set}$. An *inclusion* in the category $\mathbf{QO}$ is an order-preserving map $A \xrightarrow{f} B$ between quasi-orders subject to the condition that $f(a) \trianglelefteq f(a') \Rightarrow a \trianglelefteq a'$ for all a, a' in A.

Two simple examples of lifting are disjoint sum $+$ and direct product $\times$. The disjoint sum of two quasi-orders is simply the quasi-ordered set with the copies of quasi-orderings A and B , and with no ordering between the members of A and the members of B. The direct product $A \times B$ of quasi-orders is obviously defined. The functor $\exp X$ is another example of lifting, as mentioned above. Moreover, we note that these functors preserve inclusions.

2.6 Lemma
Let $F : \mathbf{QO}^n \to \mathbf{QO}$ be a lifting of an analytic functor on $\mathbf{Set}$.

If F preserves inclusions, then F preserves filtered colimits in $\mathbf{QO}^n$.

(*Proof*) For simplicity, we deal with the case when F is an unary functor. Let A_n be a filtered diagram in $\mathbf{QO}$. There is a commutative diagram in $\mathbf{QO}$

$$\begin{tikzcd}
\varinjlim FA_n \arrow[r, "g"] & F(\varinjlim A_n) \\
FA_n \arrow[u, "\overline{c}_n"] \arrow[ur, "Fc_n"'] &
\end{tikzcd}$$

where c_n and $\overline{c}_n$ are colimiting cones. We note that the forgetful functor $\mathbf{QO} \to \mathbf{Set}$ creates filtered colimits. So g is a bijection as a function between sets, since F is a lifting of an analytic functor preserving filtered colimits by definition. Hence it suffices to verify that the inverse g^{-1} is a homomorphism between quasi-orders.

To this end, let us take two elements a and a' in $F(\varinjlim A_n)$ satisfying $a \trianglelefteq a'$. Since g is a bijection, there are b and b' in some FA_n such that $a = Fc_n(b)$ and $a' = Fc_n(b')$. Let $(X, z) \xrightarrow{f} (A_n, b)$ and $(X', z') \xrightarrow{f'} (A_n, b')$ be weak normal forms in $\mathrm{el}(F)$, and we set the quasi-order C to be the restriction of A_n to he union of images $\mathrm{Im}(f) \cup \mathrm{Im}(f')$. Since the underlying set of C is finite, there is $A_n \xrightarrow{e} A_p$ such that the restriction of $\varinjlim A_n$ to the image of $C \hookrightarrow A_n \xrightarrow{c_n} \varinjlim A_n$ is order-isomorphic to the restriction of A_p to the image of $C \hookrightarrow A_n \xrightarrow{e} A_p$. Namely, if we put $\overline{C}$ to be the restriction of A_p to the image of C under e, then $\overline{C} \hookrightarrow A_p \xrightarrow{c_p} \varinjlim A_n$ is an inclusion. Since F preserves inclusions, we have $Fe(b) \trianglelefteq Fe(b')$ in $F\overline{C}$. Noting $\overline{c}_n = \overline{c}_p \circ e$, this implies $\overline{c}_n(b) \trianglelefteq \overline{c}_n(b')$, that is $g^{-1}(b) \trianglelefteq g^{-1}(b')$. $\square$

Higman introduced divisibility orderings in [32] on a family of sets of finite trees with bounded number of immediate successors. We provide an abstract definition of divisibility orderings using analytic functors, so that our definition covers Higman's definition as well as the topological embedding on trees.

2.7 Definition
Let $F : \mathbf{QO} \to \mathbf{QO}$ be a lifting of an analytic functor. Moreover, we suppose that F preserves inclusions.

The *divisibility ordering* on the initial algebra μF is an ω-inductive limit in the category $\mathbf{QO}$ of the diagram $A_0 \xrightarrow{e_0} A_1 \xrightarrow{e_1} \cdots$ where A_0 is the empty quasi-order and e_0 is an empty map. Moreover we define A_{n+1} and e_{n+1} as follows:

(i) The underlying set of A_{n+1} equals the underlying set of $F(A_n)$. The quasi-ordering on A_{n+1} is the transitive closure of the union of the following two relations: The one is the quasi-order $F(A_n)$, and the other is the collection of all $a^\circ \trianglelefteq a$ for each weak normal form $(X, z) \xrightarrow{f} (A_n, a)$ in $\mathrm{el}(F)$ and each a° in the image of $X \xrightarrow{f} A_n \xrightarrow{e_n} A_{n+1}$.

(ii) $e_{n+1} : A_{n+1} \to A_{n+2}$ is simply $F(e_n)$.

Remark: Since the empty map e_0 is an inclusion, every e_n is an inclusion as well. As an ω-inductive limit of inclusions, each morphism $A_n \xrightarrow{c_n} \mu F$ of the colimiting cone is an inclusion.

One may regard a° occurring in this definition as a subterm or a subtree of a. See the example after Lemma 2.9 below.

2.8 Lemma
The functions $e_n : A_n \to A_{n+1}$ in Def. 2.7 are homomorphisms between quasi-orders.

(*Proof*) Induction on n. If $n = 0$, an empty map e_0 is a homomorphism between quasi-orders. So we verify that $e_{n+1} : A_{n+1} \to A_{n+2}$ is order-preserving, provided that e_n is so. Let us take $a, a' \in FA_n$, and put $\overline{a} = e_{n+1}(a)$ and $\overline{a}' = e_{n+1}(a')$. If $a \trianglelefteq a'$ in FA_n, then $\overline{a} \trianglelefteq \overline{a}'$ holds since $Fe_n : FA_n \to FA_{n+1}$ preserves order. If $a^\circ \trianglelefteq a$ where a° is in the image of $X \xrightarrow{f} A_n \xrightarrow{e_n} A_{n+1}$ for a weak normal form $(X, z) \xrightarrow{f} (A_n, a)$, Lemma 1.5 implies that $(X, z) \xrightarrow{e_n f} (A_{n+1}, \overline{a})$ is a weak normal form. So if we put $\overline{a}^\circ = e_{n+1}(a^\circ)$, we have $\overline{a}^\circ \trianglelefteq \overline{a}$, since $\overline{a}^\circ$ is in the image of $X \xrightarrow{e_n f} A_{n+1} \xrightarrow{e_{n+1}} A_{n+2}$ for the weak normal form $e_n f$. $\square$

We note that the divisibility ordering on μF differs from the initial F-algebra in the category **QO**, since we include the relation $a^\circ \trianglelefteq a$ in the definition above. For example, if $F(X)$ is defined by $1 + X$, the initial algebra is an antichain of countable elements whereas the divisibility ordering is isomorphic to the ordinary order of natural numbers.

2.9 Lemma
Let F be a lifting of an analytic functor, this F preserving inclusions. Moreover, we suppose the initial algebra μF is endowed with the divisibility ordering.

Let $\gamma : F(\mu F) \xrightarrow{\sim} \mu F$ be the canonical bijection in **Set**. *Then γ is order-preserving, i.e., a morphism of* **QO**.

(*Proof*) Since F preserves filtered colimits of quasi-orders, if $a \trianglelefteq a'$ holds in $F(\mu F)$, then there is A_n such that $a \trianglelefteq a'$ in $F(A_n)$. Hence $a \trianglelefteq a'$ holds in A_{n+1}, implying $a \trianglelefteq a'$ in μF.

Remark: We note that the inverse γ^{-1} of the canonical bijection is not order-preserving, in general.

The divisibility ordering on $\mu X. \exp X$ is exactly the tree-embedding between finite unordered trees with root. Here an unordered tree is such that the order of immediate subtrees of each node is irrelevant. We denote the tree t with immediate subtrees $t_1, t_2, \ldots, t_n$ of the root by $\mathsf{span}(t_1, t_2, \ldots, t_n)$. The tree t is a member of some $A_{k+1} = \exp A_k$ with weak normal form

$$(X,\ \{z_1, z_2, \ldots, z_n\}) \xrightarrow{g} (A_k,\ \{t_1, t_2, \ldots, t_n\})$$

where X is the set of pairwise distinct elements $z_1, z_2, \ldots, z_n$ and the function g carries each z_i to t_i. Hence the subterms t_i of t correspond to the elements in the image of the function g occurring in the weak normal form. Therefore, for a tree $u = \mathsf{span}(u_1, u_2, \ldots, u_p)$, we have $t \trianglelefteq u$ by the divisibility ordering iff either the divisibility relation $\{t_1, t_2, \ldots, t_n\} \trianglelefteq \{u_1, u_2, \ldots, u_p\}$ holds between multisets, or some t_i embeds into u. This is exactly the tree-embedding between finite trees.

Another example $\mu X.\, 1 + X^2$ is the divisibility ordering which gives a tree-embedding on binary trees respecting the order of immediate subtrees. Moreover, the set $\mathrm{List}(A) \stackrel{\triangle}{=} \mu X.\, 1 + A \times X$ is regarded as the set of all finite lists of members of A. The divisibility ordering on $\mathrm{List}(A)$ is the Higman ordering [32, 60, 47].

2.10 Theorem
Let $F : \mathbf{QO} \to \mathbf{QO}$ be a lifting of an analytic functor, this F preserving inclusions.

If F sends well-quasi-orders to well-quasi-orders, then the divisibility ordering on μF is a well-quasi-order.

(*Proof*) The standard minimal bad sequence argument. We recall that $p(a)$ denote the least index n such that $a \in A_n$ for each $a \in \mu F$. Towards contradiction, we assume that there is an infinite bad sequence in μF. Then there is a bad sequence $\langle a_0, a_1, \ldots \rangle$ minimal in the following sense: If $\langle b_0, b_1, \ldots \rangle$ is an infinite bad sequence, and $a_0 = b_0$, $a_1 = b_1$, …, $a_{i-1} = b_{i-1}$ hold, then $p(a_i) \leq p(b_i)$ holds. Induction on i proves the existence of such a sequence $\langle a_0, a_1, \ldots \rangle$ where, at the i-th induction step, we choose a_i as the one with the smallest possible value $p(a_i)$.

For each i, let $(X_i, z_i) \xrightarrow{f_i} (A_{p(a_i)-1}, a_i)$ be a weak normal form in $\mathrm{el}(F)$. We put the quasi-order S to be the restriction of μF to the union of images $\bigcup \mathrm{Im}(f_i)$. We verify that S is a well-quasi-order. Let us assume that S has an infinite bad sequence $\langle b_0, b_1, \ldots \rangle$, for contradiction. Suppose that b_0 is in $\mathrm{Im}(f_{i_0})$. Since the union $\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1) \cup \cdots \cup \mathrm{Im}(f_{i_0-1})$ is finite, without loss of generality we may assume that no b_i belongs to this union, by eliminating finitely many b_i's if necessary. Let us consider an infinite sequence $\langle a_0, a_1, \ldots, a_{i_0-1}, b_0, b_1, \ldots \rangle$. This sequence must be good in μF since $p(b_0)$ must be strictly smaller than $p(a_{i_0})$. Hence there are indices $j < i_0$ and k such that $a_j \trianglelefteq b_k$ in μF. We have $b_k \in \mathrm{Im}(f_l)$ for some l where $i_0 \leq l$ by the assumption above. So both $a_j \trianglelefteq a_l$

and $j < l$ hold, contradicting the assumption that $\langle a_0, a_1, \ldots\rangle$ is bad. Therefore S is a well-quasi-order.

Now $F(S)$ is a well-quasi-order, since F preserves well-quasi-orders. So the sequence $\langle a_0, a_1, \ldots\rangle$ must be good in $F(S)$, thus in $F(\mu F)$. Since the canonical bijection $F(\mu F) \xrightarrow{\sim} \mu F$ preserves orders, the sequence is good also in μF, contradicting that $\langle a_0, a_1, \ldots\rangle$ is bad. $\square$

Remark: In the usual proof of Kruskal's theorem, the minimal bad sequence is defined to be minimal by the lexicographic comparison of the indices $p(a_i)$. This does not work in our case, however. In fact, the sets A_n may be infinite, thus König's lemma to find an infinite path cannot be applied.

We may apply the construction of divisibility orderings iteratively. For example, we discuss the divisibility ordering on $\mu X_1 \mu X_2 \cdots \mu X_n.\ \exp(X_1 + X_2 + \cdots + X_n)$ later. For this iteration to behave well, we must prove the following lemma.

2.11 Lemma

Let $F(X_1, X_2, \ldots, X_n, Y)$ be a lifting of an analytic functor. We suppose F preserves inclusions.

The divisibility ordering on $\mu Y. F(X_1, X_2, \ldots, X_n, Y)$ yields a functor on $\mathbf{QO}^n$ that is a lifting of an analytic functor and preserves inclusions.

(*Proof*) The proof is the same for all $n \geq 1$, so we prove the case when $n = 1$. By Prop. 1.27, the underlying functor of $G(X) = \mu Y. F(X, Y)$ on **Set** is an analytic functor. The initial algebra $G(X)$ is obtained as an inductive limit of $G_{n+1}(X) = F(X, G_n(X))$. We show that, for each $A \xrightarrow{k} B$ in $\mathbf{QO}$, the function $G_{n+1}(k) = F(k, G_n(k))$ is order-preserving. Once we prove this, the assertion of lemma follows easily, since $G(X)$ is an inductive limit and the colimiting cone consists of inclusions.

If $a \trianglelefteq a'$ in $F(A, G_n(A))$, then obviously $\overline{a} \trianglelefteq \overline{a}'$ in $F(B, G_n(B))$ where $\overline{a} = G_{n+1}(k)(a)$ and $\overline{a}' = G_{n+1}(k)(a')$. Next let us suppose $a^\circ \trianglelefteq a$ where a° is in the image of $Z \xrightarrow{f} G_n(A) \longrightarrow G_{n+1}(A)$ where $(Z, z) \xrightarrow{f} (G_n(A), a)$ is a weak normal form in $\mathrm{el}(F(A, \text{-}))$. We note that $F(k, Z)$ is a weakly cartesian natural transformation between the functors $F(A, Z)$ and $F(B, Z)$ in Z, since F preserves weak pullbacks. Hence $(Z, z') \xrightarrow{f} (G_n(A), a')$ is a weak normal from in $\mathrm{el}(F(B, \text{-}))$ by Lemma 1.19 where $z' = F(k, Z)(z)$ and $a' = F(k, G_n(A))(a)$. Since $(G_n(A), a') \xrightarrow{G_n(k)} (G_n(B), \overline{a})$ in $\mathrm{el}(F(B, \text{-}))$, we have a weak normal form $(Z, z') \xrightarrow{G_n(k) f} (G_n(B), \overline{a})$ by Lemma 1.5, (ii). Hence $\overline{a}^\circ \trianglelefteq \overline{a}$ holds in $G_{n+1}(B)$ where $\overline{a}^\circ = G_{n+1}(k)(a^\circ)$. $\square$

Algebras

Employing the observation so far, we have a class of well-partial-orders generated by simple operations.

2.12 Definition
An *algebra* A is generated by the following Backus-Naur form:

$$A \quad ::= \quad \emptyset \ \mid\ 1 \ \mid\ X \ \mid\ A+A \ \mid\ A \times A \ \mid\ \mu X.\, A$$

where X is a variable taken from a given countable set.

We handle algebras as syntactic objects. But we can interpret these constructs as operations of posets in an obvious way: $\emptyset$ is an empty poset, 1 is a singleton, $A+B$ is a disjoint sum, $A \times B$ a direct product, and $\mu X.\, A$ a divisibility ordering.

The *terms* in algebras are generated by the following rules:

$$\frac{}{* \in 1}$$

$$\frac{a \in A}{\iota a \in A+B} \qquad \frac{b \in B}{\iota' b \in A+B}$$

$$\frac{a \in A \qquad b \in B}{\langle a, b\rangle \in A \times B} \qquad \frac{a \in [(\mu X.\, A)/X]A}{\gamma a \in \mu X.\, A}$$

Here $[(\mu X.\, A)/X]$ is the substitution of $\mu X.\, A$ for X. We write the substitution as a prefix operator. Moreover, we often write AB in place of $A \times B$.

To be precise, we should annotate the constructors of terms such as ι and γ by involving algebras so that we can determine the algebras where they belong. For example, the notation $\iota_{A,B}$ may be better to keep track of the information of algebras. But, to avoid cumbersome notations, we omit the annotations if confusions are unlikely. Furthermore, we often omit ι and ι' if they are clear from the context.

The substitutions occurring in the initial algebra construction is canonically determined by the form of the initial algebras. Namely, if $a \in A[D_1, D_2, \ldots, D_n]$ occurs in the derivation tree of terms for an algebra $A[X_1, X_2, \ldots, X_n]$ and its ground substitution, the substituted algebras D_i are determined by the form of A. So we associate the *canonical substitution* η_A to each algebra A (precisely speaking, to each subalgebra A occurring in a fixed closed algebra) as follows: For a given algebra A, we start with the empty substitution $\eta_A = \emptyset$. If $A = \mu X_n.\, B$, we put $\eta_B = \eta_A[A/X_n]$. For the other constructors, the canonical substitutions do not change. For example, if $A = B + C$, both η_B and η_C

are set to equal η_A. Here the concatenation of substitutions has an obvious meaning. For example, if $\eta = [A/X][B[X]/Y]$ then $\eta\, C[X,Y] = C[A, B[A]]$.

Example: Let us consider the algebra $T = \mu X\, \mu Y.1{+}XY$, where we put $F[X] = \mu Y.1 + XY$. We start with $\eta_T = \emptyset$. After stripping the outermost μX, we have $\eta_{F[X]} = [T/X]$. Finally, inside the scope of the operator μY, the canonical substitution becomes $[T/X][F[X]/Y]$. Hence, if we put the last substitution to be η, we have $\eta(1 + XY) = 1 + TF[T]$.

So a canonical substitution should have the form $[D_1/X_1][D_2/X_2]\cdots[D_n/X_n]$ where D_1 is a closed algebra, D_2 may have a free variable X_1, and D_n is in $X_1, X_2, \ldots, X_{n-1}$. Later we use also partial applications of the canonical substitution, $\eta_A^{(k)} \triangleq [D_{k+1}/X_{k+1}]\cdots[D_n/X_n]$. For simplicity, we often omit the suffix recording the algebra, and let simply η denote the involved canonical substitution.

Let A be an algebra where all free variables are given suffixes as $X_1, X_2, \ldots$. We define the *cardinality* $\operatorname{card} A$ by Ω_n with the highest n such that X_n is free in A. If A is closed, we put $\operatorname{card} A = \Omega_0$.

Remark: There is no straightforward relation of this cardinality to the set-theoretic one. But the variables X_n behave somehow as cardinals Ω_n used in the traditional ordinal notations [9, 51]. In particular, there seems to be a close analogy to the Bachmann-Isles' hierarchy [20, 34].

We replace every bound variable by one of X_n by the following procedure: We start from a given algebra where its cardinality is well-defined, and decompose it until initial algebras are encountered. If $\mu X.\, A[X]$ is of cardinality Ω_n, then we set $\mu X_{n+1}.\, A[X_{n+1}]$ and the procedure goes to the next turn where $A[X_{n+1}]$ is manipulated. For example, we transfer

$$\begin{aligned}&\mu X \mu Y.\, (\mu Z.\, 1 + XZ)(\mu Z.\, 1 + YZ)\\ &\qquad \rightsquigarrow\ \mu X_1 \mu X_2.\, (\mu X_2.\, 1 + X_1 X_2)(\mu X_3.\, 1 + X_2 X_3).\end{aligned}$$

We note that the left μZ is replaced by μX_2 while the right by μX_3. The basic idea is to use the smallest number that has not yet been used in the scope of the μ-operator.

We often add a suffix to the constructor γ associated to initial algebras as follows: We write $\gamma_n a$ for members of $\eta(\mu X_n.\, A)$ for an initial algebra $\mu X_n.\, A$ of cardinality Ω_{n-1}.

2.13 Definition
Let a be a term in $A = \eta X_k$ for some variable X_k and its canonical substitution

η. Moreover, let b be a term in an algebra ηB.

The *subterm* relation, written $a \subseteq b$ in symbols, is generated by the following rules:

$$\frac{}{a \subseteq a} \qquad \frac{a \subseteq b}{a \subseteq \iota b} \qquad \frac{a \subseteq c}{a \subseteq \iota' c}$$

$$\frac{a \subseteq b}{a \subseteq \langle b, c\rangle} \qquad \frac{a \subseteq c}{a \subseteq \langle b, c\rangle}$$

$$\frac{a \subseteq b}{a \subseteq \gamma_n b} \quad \text{if } k < n$$

Moreover, for two terms a and $a' = \gamma_k(b')$ in the algebra $A = \eta X_k$, we say that a is a *subterm* of a' and write $a \subset a'$ if and only if $a \subseteq b'$.

Remark: The assertion that a is a subterm of a' makes sense, either if the cardinality of the algebra where a' belongs is greater than that of the algebra where a belongs, or if a and a' are members of the same initial algebras. Which one we intend will be clear from the context.

The sole important point in definition of subterms is the side condition $k < n$ in the last rule. For example, let us consider the algebra $T = \mu X_1 \mu X_2 . 1 + X_1 X_2$. The associated canonical substitution η equals $[T/X_1][F[X_1]/X_2]$ where $F[X_1] = \mu X_2 . 1 + X_1 X_2$. We consider a term $a = \gamma_2\langle \gamma_1 a', a''\rangle$ where a, a' and a'' are members of $\eta X_2 = F[T]$. Then both a' and a'' occur as subexpressions of a. However, we have $a'' \subset a$ whereas $a' \not\subset a$ since $a' \not\subseteq \gamma_1 a'$ by the side condition.

The following proposition asserts the connection between normal forms and subterms. Let us recall that, if a canonical substitution η is given by $[D_1/X_1]$ $[D_2/X_2] \cdots [D_n/X_n]$, then the partial substitution $\eta^{(k)}$ denotes $[D_{k+1}/X_{k+1}] \cdots$ $[D_n/X_n]$. Hence $\eta^{(k)} A$ is a normal functor in $X_1, X_2, \ldots, X_k$ for algebra $A[X_1, X_2, \ldots, X_k, \ldots, X_n]$.

2.14 Proposition

For an algebra $A[X_1, X_2, \ldots, X_n]$ and its canonical substitution η, let $F(X_k)$ be the functor $\eta^{(k)} A$ in variable X_k (as well as $X_1, X_2, \ldots, X_{k-1}$, which are fixed here, however). Moreover let $(Y, z) \xrightarrow{f} (\eta X_k, b)$ be a normal form in $\mathrm{el}(F)$.

A term $a \in \eta X_k$ is in the image of f if and only if a is a subterm of b.

(*Proof*) ($\Leftarrow$) By induction on the derivation of $a \subseteq b$. We prove the case deriving $a \subseteq \gamma_n b$ from $a \subseteq b$ where $\gamma_n b$ is a member of ηA for $A = \mu X_n . B$. For a partial substitution $\eta^{(k)} = [D_{k+1}/X_{k+1}] \cdots [D_{n-1}/X_{n-1}][A/X_n]$, let us put η' to be the substitution without the last component $[A/X_n]$. Then, provided $k < n$, we have $\eta^{(k)}B = \eta'[A/X_n]B$, which turns out to equal $[(\mu X_n.\eta' B)/X_n](\eta' B)$. This is bijective to $\mu X_n.\eta' B$, i.e., $\eta^{(k)}(\mu X_n.B)$ by definition of initial algebra. Moreover, this bijection γ_n is natural in $X_1, X_2, \ldots, X_k$. Regarding $\eta^{(k)}B$ to be a functor $G(X_k)$, let us take a normal form $(Y, z) \xrightarrow{f} (\eta X_k, b)$ in el(G). By induction hypothesis, a is in the image of f. Since the natural isomorphism $\gamma_n : \eta^{(k)}B \xrightarrow{\sim} \eta^{(k)}(\mu X_n.B)$ must be cartesian, also $(Y, \gamma_n z) \xrightarrow{f} (\eta X_k, \gamma_n b)$ is a normal form. Hence the assertion of proposition holds also for $a \subseteq \gamma_n b$, namely, a is in the image of f giving the normal form of $\gamma_n b$.

$(\Rightarrow)$ Induction on construction of B. If $A = \mu X_n.B$ with $k < n$, we have $\eta^{(k)}A = \eta^{(k)}B$. So the image of the function f of normal forms does not change. If $A = B \times C$, a normal form of $\langle b, c\rangle$ has the form $(Y + Y', \langle \overline{z}, \overline{z}'\rangle)$ where (Y, z) and (Y', z') are normal forms of b and c, and $\overline{z}$ is the image of z under the function determined by the injection $Y \to Y + Y'$, and likewise for $\overline{z}'$. Hence a must be in either the image of the normal form of b or the image of the normal form of c. The case $A = B + C$ is easy. If $A = X_k$, the element a must equal b. □

2.15 Lemma

Let us consider three algebras A, $B[X]$ and $C[Y]$ and three terms $a \in A$, $b \in B[A]$ and $c \in C[B[A]]$.

If $a \subseteq b$ and $b \subseteq c$, then $a \subseteq c$.

(*Proof*) Obvious by the preceding proposition and Thm. 1.16. □

We define the embeddability relation $a \trianglelefteq_A a'$ for a, a' in A where A is the ground instance of some algebra by its canonical substitution.

$$\frac{}{* \trianglelefteq_1 *}$$

$$\frac{a \trianglelefteq_A a'}{\iota a \trianglelefteq_{A+B} \iota a'} \qquad \frac{b \trianglelefteq_A b'}{\iota' b \trianglelefteq_{A+B} \iota' b'}$$

$$\frac{a \trianglelefteq_A a' \qquad b \trianglelefteq_B b'}{\langle a, b\rangle \trianglelefteq_{A\times B} \langle a', b'\rangle}$$

$$\frac{b \trianglelefteq_{[A/X_n]B} b'}{\gamma_n b \trianglelefteq_A \gamma_n b'} \qquad \frac{a \trianglelefteq_A a^\circ}{a \trianglelefteq_A a'} \quad \text{if } a^\circ \subset a'.$$

In the last two rules, we assume that A is the initial algebra $\mu X_n.B$. These rules are the same as those in [29], except that, in the last rule, the subterm relation $a^\circ \subset a'$ does not simply mean that a° is a subexpression of a'.

2.16 Lemma

Let A and $B[X]$ be algebras, and let a be a term of A and b, b' terms of $B[A]$.

If $a \subseteq b$ and $b \trianglelefteq_B b'$, there is a term $a' \in A$ such that $a' \subseteq b'$ and $a \trianglelefteq_A a'$.

(*Proof*) Induction on construction of the algebra $B[X]$. The sole non-trivial case is when $B[X]$ is an initial algebra $\mu Y.\, C[X, Y]$. The divisibility ordering on $\mu Y.C[A, Y]$ is the transitive closure of two patterns. The one is the condition that $b = \gamma c$ and $b' = \gamma c'$ with $c \trianglelefteq_C c'$. In this case, we have $a \subseteq c$. By hypothesis on C, there is a term a' such that $a \trianglelefteq_A a'$ and $a' \subseteq c'$, the latter implying $a' \subseteq b'$. The other pattern is $b \subset b'$. In this case, $a \subseteq b \subseteq c'$ if we write $b' = \gamma c'$. So, by Lemma 2.15, we have $a \subseteq c'$, thus $a \subseteq b'$. By taking the transitive closure of these, we prove the assertion of lemma for initial algebras. $\square$

Let us recall that the constructors of algebras have natural meaning as operators on posets. For simplicity of notation, we abuse the same symbols for both syntactic objects and their interpretations. The next theorem shows that the embeddability relation is equal to the intended partial order.

2.17 Theorem

Let a and a' be terms of A that is the ground instance of an algebra by its canonical substitution.

The embeddability relation $a \trianglelefteq_A a'$ holds iff $a \trianglelefteq a'$ in the poset interpreting A.

(*Proof*) The inference rules of the embeddability relation are nearly word-by-word translation of the corresponding definition of the operations on posets, except that the inference by the subterm relation $a^\circ \subset a'$ is restricted to the right hand side of $\trianglelefteq_A$ while the corresponding divisibility ordering, which is defined as the transitive closure of two patterns, allows this anywhere in the sequence of these patterns. So we must verify that the use of $a^\circ \subseteq a'$ can be postponed to the rightmost of the sequence. To this end, we assume that $a^\circ \subset a$ holds, and that $a \trianglelefteq_A a'$ is derived from $b \trianglelefteq_{[A/X]B} b'$ where $a = \gamma b$ and $a' = \gamma b'$. Then we have $a^\circ \subseteq b$. So, by Lemma 2.16, there is $a^\circ \trianglelefteq_A {a'}^\circ$ such that ${a'}^\circ \subseteq b'$, from which ${a'}^\circ \subset a'$ follows. Hence one can collect all applications of subterm relation to the rightmost of the sequence. $\square$

2.18 Corollary

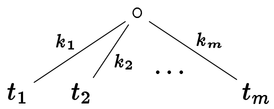
Let A be the ground instance of an algebra by its canonical substitution.

The embeddability relation $\trianglelefteq_A$ on the set of terms in A is a well-partial-order.

The divisibility ordering on an iterated initial algebra gives a version of the embedding between labeled trees with the gap condition [59, 19]. As an example, let us consider the algebra $\mu X_1 \mu X_2 \cdots \mu X_n.\, \mathrm{List}(X_1 + X_2 + \cdots + X_n)$ where $\mathrm{List}(A)$ is defined by $\mu Y.\, 1 + AY$. We put

$$\begin{aligned}
D_1 &= \mu X_1.\, D_2[X_1] \\
D_2[X_1] &= \mu X_2.\, D_3[X_1, X_2] \\
&\vdots \\
D_n[X_1, X_2, \ldots, X_{n-1}] &= \mu X_n.\, D_\infty[X_1, X_2, \ldots, X_{n-1}, X_n] \\
D_\infty[X_1, X_2, \ldots, X_{n-1}, X_n] &= \mathrm{List}(X_1 + X_2 + \cdots + X_{n-1} + X_n)\ .
\end{aligned}$$

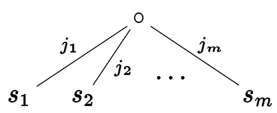
With the shorthand $\overline{\gamma}_i t = \gamma_i \gamma_{i+1} \cdots \gamma_n t$, the members of ηD_∞ are written as lists $\langle \overline{\gamma}_{k_1} t_1, \overline{\gamma}_{k_2} t_2, \ldots, \overline{\gamma}_{k_m} t_m \rangle$. It is helpful to regard the list as the tree

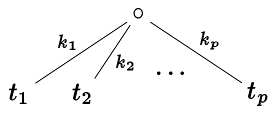


with edges labeled by natural numbers in $1, 2, \ldots, n$, where the order of immediate subtrees $t_1, t_2, \ldots, t_m$ must be the same as the order occurring in the list.

For each pair of s and t in ηD_∞, it holds that $\overline{\gamma}_i s \trianglelefteq \overline{\gamma}_i t$ by the divisibility ordering on ηD_i iff either *(1)* there is a proper subtree t° of t where the edge to the root of t° is labeled by i and the edges occurring on the path from the root of t to the root of t° are labeled by some $j > i$ except the last one, and where t° satisfies $\overline{\gamma}_i s \trianglelefteq \overline{\gamma}_i t^\circ$ in ηD_i, or *(2)* it holds that $\overline{\gamma}_{i+1} s \trianglelefteq \overline{\gamma}_{i+1} t$ in ηD_{i+1}, where if $i = n$ we put $i + 1 = \infty$. Unfolding the part $\overline{\gamma}_i s \trianglelefteq \overline{\gamma}_i t^\circ$ in (*1*) iteratively by these two patterns (increasing the index i if the pattern (*2*) is applied), we conclude $\overline{\gamma}_i s \trianglelefteq \overline{\gamma}_i t$ iff t has a subtree t° (possibly equal to t) such that the edges occurring on the path from the root of t to the root of t° have labels $j \geq i$, this t° satisfying $s \trianglelefteq t^\circ$ in ηD_∞, namely, by the Higman embedding between lists.

Hence, for $s = \langle \overline{\gamma}_{j_1} s_1, \overline{\gamma}_{j_2} s_2, \ldots, \overline{\gamma}_{j_m} s_m \rangle$ and $t = \langle \overline{\gamma}_{k_1} t_1, \overline{\gamma}_{k_2} t_2, \ldots, \overline{\gamma}_{k_p} t_p \rangle$ in ηD_∞, that is, for two trees





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it holds that $s \trianglelefteq t$ iff there is a strictly order-preserving function $[m] \xrightarrow{e} [n]$ such that $j_i = k_{e(i)}$ for all $i = 1, 2, \ldots, m$, and, for each i, we have $\overline{\gamma}_{j_i} s_i \trianglelefteq \overline{\gamma}_{k_{e(i)}} t_{e(i)}$, that is, there is a subtree t° of $t_{e(i)}$ with all edges to the root of t° having labels equal to j_i or more, and furthermore t° satisfies $s_i \trianglelefteq t^\circ$ in ηD_∞.

In this example, we regard the trees where the order of immediate subtrees is respected. Replacing List by exp as

$$\mu X_1 \mu X_2 \cdots \mu X_n.\ \exp(X_1 + X_2 + \cdots + X_n),$$

we can define the edge-labeled trees where the order of immediate subtrees is irrelevant. This is a modification of the tree-embedding with the gap condition [59] to the trees where the edges are labeled. We note that the type of embedding used in [52] is akin to the embedding of edge-labeled trees discussed here.

Recursive Path Ordering

Let **LO** be the category of all linearly ordered sets and all strictly order-preserving maps. Similarly to the case of quasi-orders, some of analytic functors are regarded as functors on **LO**.

Given a linearly ordered set A, the *multiset ordering* [15] is the linear ordering on $\exp A$ defined as follows: For multisets $\alpha, \beta \in \exp A$, and $s_1, s_2, \ldots, s_n, t \in A$ with $n \geq 0$, we define

$$\{s_1, s_2, \ldots, s_n\} \cup \alpha \ < \ \{t\} \cup \beta$$

if $\alpha \leq \beta$ and $s_i < t$ in A for every i. As an alternative characterization, it holds that $\{s_1, s_2, \ldots, s_m\} \leq \{t_1, t_2, \ldots, t_n\}$ in the multiset ordering iff there is a function $k : [m] \to [n]$ such that $s_i \leq t_{k(i)}$ for every i and, in addition, if the equality $s_i = t_{k(i)}$ holds then $k(i) = k(i')$ implies $i = i'$. It is easy to see that, endowed with the multiset ordering, $\exp X$ turns out to be functorial, and that it satisfies the commutative square diagram

$$\begin{tikzcd} \mathbf{LO} \arrow[r, "\exp"] \arrow[d] & \mathbf{LO} \arrow[d] \\ \mathbf{Set} \arrow[r, "\exp"'] & \mathbf{Set} \end{tikzcd}.$$

Furthermore, the multiset ordering on $\exp A$ is a linearization of the divisibility relation on it. This fact is proved easily if one uses the alternative characterization of the multiset ordering. Therefore, as a linearization of a well-partial-order, the multiset ordering on $\exp A$ is a well-order, provided that A is a well-ordered set.

The multiset ordering is not a unique way to lift the functor $\exp X$ to the category **LO**. We may compare two multisets α and β first by the number of members with multiplicity taken into account, and second by the multiset ordering if the numbers are equal. If we regard the members of A as variables

$x_1, x_2, \ldots$ equipped with a linear order on them, and if we identify a multiset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\}$ with a monomial $x_{i_1}x_{i_2}\cdots x_{i_n}$, then this ordering compares the total degrees of monomials first and after arranging the variable in the monomials in the descending order, compare two monomials by the lexicographic ordering. This ordering is called the total degree-lexicographic ordering in [5]. By this means, we have a functor lifing $\exp X$ to **LO**. Also this is a linearization of the divisibility relation, thus the total order-lexicographic ordering is a well-order, provided that A is a well-order.

The recursive path ordering is introduced by Dershowitz [14] as a machinery to prove termination of term rewriting systems. The recursive path ordering is a well-ordering on the set of trees. Since terms are regarded as trees with nodes labeled by function or constant symbols, this method has an advantage that the termination is proved directly.

To define the recursive path ordering, we assume to have a well-ordering $\prec$, called a precedence order, on the set of function and constant symbols. The *multiset path ordering* $s < t$ for terms $s = f(s_1, s_2, \ldots, s_m)$ and $t = g(t_1, t_2, \ldots, t_n)$ is defined by either (i) $s \leq t_j$ for some subterm t_j, or (ii) both $s_i < t$ for all subterms s_i and either $f \prec g$ for the top function symbols or $f = g$ and $\{s_1, s_2, \ldots, s_m\} <^\circ \{t_1, t_2, \ldots, t_n\}$ by the multiset ordering $<^\circ$ induced by the multiset path ordering $<$ itself. We can deal with function symbols of variable arity.

A typical application is the proof of termination of the following term rewriting system to compute the disjunctive normal form [14]:

$$\begin{array}{lcl}
\neg\neg\, x & \to & x \\
\neg(x \,\&\, y) & \to & \neg\, x \vee \neg\, y \\
\neg(x \vee y) & \to & \neg x \,\&\, \neg\, y \\
x \,\&\, (y \vee z) & \to & (x \,\&\, y) \vee (x \,\&\, z) \\
(x \vee y) \,\&\, z & \to & (x \,\&\, z) \vee (y \,\&\, z).
\end{array}$$

An appropriate precedence order to prove termination is

$$\vee \;\prec\; \& \;\prec\; \neg.$$

Then, for each rewriting rule $l \to r$ and each ground substitution σ, one can see in seconds that $\sigma l > \sigma r$ by the multiset path ordering. For example, to see $\neg\, s \vee \neg\, t < \neg(s \,\&\, t)$, first we compare the topmost symbols $\vee$ and $\neg$. Since the latter is greater in the precedence order, we turn to check $\neg\, s < \neg(s \,\&\, t)$ and $\neg\, t < \neg(s \,\&\, t)$, decomposing the left hand side. Since the topmost symbols are the same $\neg$, we should check $s < s \,\&\, t$ for the former and $t < s \,\&\, t$ for the latter. But these are true since s and t are subterms of $s \,\&\, t$. Since the multiset path ordering is a well-order, the relations $\sigma l > \sigma r$ imply that this term rewriting system enjoys the strong normalization property. Namely every rewriting sequence terminates in finite steps.

There are many variations of the multiset path ordering. The *lexicographic path ordering* is obtained by modifying the phrase (ii) in definition of the multiset path ordering , replacing the multiset ordering $\{s_1, s_2, \ldots, s_m\} <^\circ \{t_1, t_2, \ldots, t_n\}$ by $\langle s_1, s_2, \ldots, s_m\rangle <_{\text{lex}} \langle t_1, t_2, \ldots, t_n\rangle$ where $<_{\text{lex}}$ is the lexicographic ordering induced by the ordering $<$ we are defining. In this case, we assume that each function symbol has a fixed arity. More generally, we can mix up the multiset and the lexicographic path orderings to the so-called the recursive path ordering with status [44].

One of frequent users of well-orders is the proof-theoretician. After a celebrated work by Gentzen proving the consistency of Peano arithmetic by ordinal ε_0, well-orders are employed as a measure of the power of proof systems. The proof-theoretical ordinal of a proof system is defined as the least ordinal the well-orderedness of which cannot be proved in the system. For consistency proofs, we must form a term notation for a fragment of the class of ordinals, rather than set-theoretic abstract ordinals as transitive sets. Hence, it is not surprising that proof-theoreticians are forerunners of computer scientists in use of well-orders. For example, the ordering by Ackermann [1] is nothing other than a special case of the recursive path ordering with status. A more elaborate ordinal notation for a larger fragment is Takeuti's ordinal diagrams [61]. We discuss this topic extensively later.

A functor $F : \mathbf{LO}^n \to \mathbf{LO}$ is a lifting of an analytic functor F_0 iff the square diagram

$$\begin{tikzcd} \mathbf{LO}^n \arrow[r, "F"] \arrow[d] & \mathbf{LO} \arrow[d] \\ \mathbf{Set}^n \arrow[r, "F_0"'] & \mathbf{Set} \end{tikzcd}$$

commutes, where the downward arrows are the functor forgetting the structures of linear orders.

Example: (i) The disjoint sum $(\cdot) + (\cdot)$ has a lifting to linear orders. The linear order $A + B$ is given by the copies of linear orders A and B as well as the relations asserting that every member of B is greater than all members of A. We note that the operator $+$ on linear orders is non-commutative.

(ii) The direct product $(\cdot) \times (\cdot)$ has a lifting to linear orders. The linear order $A \times B$ is the inverse lexicographic ordering. Namely, $\langle a, b\rangle < \langle a', b'\rangle$ holds iff either $b < b'$ or both $b = b'$ and $a < a'$. One may prefer the ordinary lexicographic ordering. This is a matter of choice, but traditional ordinal notations adopt

the inverse lexicographic path ordering. The operator $\times$ on linear orders is non-commutative.
(iii) As mentioned above, the functor $\exp X$ has a lifting to linear orders by the multiset ordering.

We introduce an abstract version of the recursive path ordering as a linear order on the initial algebra $\mu X.\, F(X)$, provided that F is a lifting of an analytic functor to linear orders.

2.19 Definition
Let a functor $F : \mathbf{LO} \to \mathbf{LO}$ be a lifting of an analytic functor.

The *recursive path ordering* is a linear order on the initial algebra $\mu X.\, F(X)$, this linear order defined as the inductive limit of the ω-chain $A_0 \xrightarrow{e_0} A_1 \xrightarrow{e_1} \cdots$ in the category $\mathbf{LO}$, where A_0 is an empty linear order with an empty map e_0, and A_{n+1} and e_{n+1} are defined by induction as follows:

(i) The underlying set of A_{n+1} equals the underlying set of $F(A_n)$. The linear order on A_{n+1} is defined by the following: For members a and b in A_{n+1}, let their weak normal forms in $\mathrm{el}(F)$ be $(X, z) \xrightarrow{f} (A_n, a)$ and $(Y, w) \xrightarrow{g} (A_n, b)$. Then the ordering $a < b$ holds iff either
(1) It holds that $a \le b^\circ$ in A_{n+1} for some b° in the image of the composite $Y \xrightarrow{g} A_n \xrightarrow{e_n} A_{n+1}$, or
(2) It holds that $a < b$ in the linear order $F(A_n)$ and, in addition, $a^\circ < b$ holds in A_{n+1} for every a° in the image of the composite $X \xrightarrow{f} A_n \xrightarrow{e_n} A_{n+1}$.

(ii) $e_{n+1} : A_{n+1} \to A_{n+2}$ is defined simply by $F(e_n)$.

Remark: We note that, in the condition *(2)* above, $a < b$ is a comparison by the linear order $F(A_n)$ while $a^\circ < b$ by the linear order A_{n+1} we are defining. Likewise $a < b^\circ$ in the condition *(1)* is by the linear order A_{n+1}. So definition of the recursive path ordering uses an inner induction on $p(a) + p(b)$. We recall that $p(a)$ is the least index n satisfying $a \in A_n$ if we regard $\mu X.\, F(X)$ as the increasing union of A_n.

2.20 Lemma
(i) $e_{n+1} : A_{n+1} \to A_{n+2}$ *in Def. 2.19 is strictly order-preserving.*
(ii) *The recursive path ordering on the initial algebra μF is a linear ordering.*

(*Proof*) To prove these two assertions, we have to prove that each A_n is a linear order and $e_n : A_n \to A_{n+1}$ is strictly order-preserving. Then the recursive path ordering as the inductive limit in category $\mathbf{LO}$ turns out to be a linear order.

First we show that A_n is a linear order. We must show $(a < b) \vee (a = b) \vee (a > b)$ and the disjunctions are exclusive. The proof is left to the reader. We show that the order $<$ is transitive. Suppose that $a < b$ and $b < c$ holds in A_{n+1}. The proof is by induction on $p(a) + p(b) + p(c)$. If $b \leq c^\circ$ for some c°, then $a < c^\circ$ holds by induction hypothesis, so $a < c$. Otherwise $b < c$ in $F(A_n)$, which is a linear order (by induction on n), and $b^\circ < c$ for all b°. If $a \leq b^\circ$ then $a < c$ by induction hypothesis. If $a < b$ in $F(A_n)$ and $a^\circ < b$ for all a° then $a < c$ in $F(A_n)$ and $a^\circ < c$ by induction hypothesis, so $a < c$. In all cases, we have $a < c$.

Second we prove that e_n is strictly order-preserving by induction on n. The base case $n = 0$ is obvious. We verify that $a < a'$ in A_{n+1} implies $\overline{a} < \overline{a}'$ in A_{n+2} where $\overline{a} = e_{n+1}(a)$ and $\overline{a}' = e_{n+1}(a')$ by inner induction on $p(a) + p(a')$. Suppose that $a < a'$ is derived by the condition *(2)* of Def. 2.19. Then we have $\overline{a} < \overline{a}'$ in FA_{n+1}, since e_n is strictly order-preserving by induction hypothesis on n, thus so is Fe_n. Furthermore, if $(X, z) \xrightarrow{f} (A_n, a)$ is a weak normal form, Lemma 1.5, (ii) induces that $(X, z) \xrightarrow{e_n f} (A_{n+1}, \overline{a})$ is a weak normal form. Hence, for every c° in the image of $X \xrightarrow{e_n f} A_{n+1} \xrightarrow{e_{n+1}} A_{n+2}$, there is a° in the image of $X \xrightarrow{f} A_n \xrightarrow{e_n} A_{n+1}$ such that $c^\circ = e_{n+1}(a^\circ)$. Applying inner induction hypothesis to $a^\circ < a'$, we have $c^\circ < \overline{a}'$. Therefore $\overline{a} < \overline{a}'$ holds. We use Lemma 1.5, (i) for the case that $a < a'$ is derived the condition *(1)*. □

We have shown that, in Thm. 2.10. the divisibility ordering is a well-quasi-order if the involving functor preserves well-quasi-orders. A similar theorem holds for the recursive path ordering. The proof is parallel to the one of that theorem, and is omitted (see [31]).

2.21 Theorem

Let F be a lifting of an analytic functor to the category of linear orders.

If F sends well-orders to well-orders, the recursive path ordering on μF is a well-order.

Example: (i) Since the product on linear orders is non-commutative, we have two natural recursive path orderings $\mu X.\, 1 + XA$ and $\mu X.\, 1 + AX$ on the set List A of finite lists of members of A. The former is the monadic path ordering [55, 46], and the latter compares the lengths of lists first and inverse lexicographic order second. This may be regarded as the total degree-lexicographic ordering on monomials for non-commutative variables.

(ii) The recursive path ordering on $\mu X.\, 1 + X^2 + X$ is the lexicographic path ordering on the set of terms generated by a constant a, a binary function symbol f, and a unary function symbol g, with precedence $a \prec f \prec g$. One can employ this ordering to prove the termination of the rewriting system of free groups, a well-known application of the Knuth-Bendix completion algorithm:

$$\begin{array}{lll@{\qquad}lll}
(xy)z & \to & x(yz) & e^{-1} & \to & e \\
xe & \to & x & (x^{-1})^{-1} & \to & x \\
ex & \to & x & x(x^{-1}y) & \to & y \\
xx^{-1} & \to & e & x^{-1}(xy) & \to & y \\
x^{-1}x & \to & e & (xy)^{-1} & \to & y^{-1}x^{-1} .
\end{array}$$

The unit e is a constant, the multiplication is a binary function, and the inverse is a unary function. By the precedence $e \prec \times \prec (\cdot)^{-1}$, we can easily see that $\sigma l > \sigma r$ for each rule $l \to r$ and each ground substitution σ.

(iii) The recursive path ordering on $\mu X.\, 1 + X^3 + \exp X$ is equal to the recursive path ordering with status [44] on the set of terms generated by a constant a, a tertiary function symbol f, and a function symbol g of variable arity, with precedence $a \prec f \prec g$.

All of these examples are well-orders (for the first example, provided that A is a well-order), as derived from the following observation: The involving functor $F : \mathbf{LO} \to \mathbf{LO}$ makes sense also as a functor on the category $\mathbf{QO}$ of quasi-orders, and it sends well-quasi-orders to well-quasi-orders. For each linear order A, the linear order $F(A)$ as the image of the functor on $\mathbf{LO}$ is a linearization of the partial-order $F(A)$ as the image of the functor on $\mathbf{QO}$. Since a linearization of a well-partial-order is a well-order, we can conclude that if A is a well-order then $F(A)$ is a well-order. Hence Thm. 2.21 implies that the recursive path ordering on $\mu X.\, F(X)$ is a well-order.

We defined the class of algebras in Def. 2.12 and well-partial-orders on them. They behave also as well-orders by considering the recursive path ordering. Namely, if we define the linear order on ηA for each algebra A with its canonical substitution η as follows, the linear order turns out to be a well-order.

(i) In $\eta(A + B)$, it holds that $\iota a < \iota a'$ if $a < a'$ holds in ηA. Likewise $\iota' b < \iota' b'$ holds if $b < b'$ holds in ηB. Moreover $\iota a < \iota' b$ for every $a \in \eta A$ and every $b \in \eta B$.
(ii) In $\eta(A \times B)$, it holds that $\langle a, b\rangle < \langle a', b'\rangle$ if either $b < b'$ holds in ηB or $b = b'$ and $a < a'$ in ηA.
(iii) In $\eta(\mu X_n.\, B)$, for terms $a = \gamma_n b$ and $a' = \gamma_n b'$, it holds that $a < a'$ if either of the following holds: *(1)* there is a subterm $a'^{\circ} \subset a'$ such that $a \leq a'^{\circ}$. *(2)* $b < b'$ holds in ηB and furthermore $a^{\circ} < a'$ holds for all subterms $a^{\circ} \subset a$.

Below we determine the supremum of the order types of the well-orders on A which ranges over all closed algebras.

Logical Complexity

We have a class of well-ordering defined on each closed algebra as discussed above. We show that the order types of these well-orderings surpass any ordinals smaller than the proof-theoretic ordinal $\theta\Omega_\omega 0$ of $(\Pi^1_1\text{-CA})_0$ [10].

Takeuti defined ordinal diagrams [61] as a system of ordinal notation to prove the consistency of a fragment of second order arithmetic. We give a slightly adapted definition of the ordinal diagram. Later we show that it is nothing but the recursive path ordering on an initial algebra.

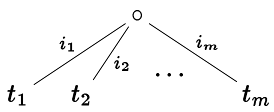
2.22 Definition
Let n be a positive integer.

An *ordinal diagram* is a member of $O(n,1)$ inductively generated by the following:

(i) A pair (i,t) is an ordinal diagram for $i < n$, provided t is an ordinal diagram.
(ii) A finite multiset of ordinal diagrams of the form (i,t) is an ordinal diagram. We write the multiset $(i_1,t_1) \sharp (i_2,t_2) \sharp \cdots \sharp (i_m,t_m)$, and 0 for the empty multiset. We identify a singleton multiset of (i,t) with (i,t) itself.

Remark: The empty diagram 0 satisfies $t = 0 \sharp t$. This differs from the original definition where $0 \sharp t$ behaves as the successor of t. Takeuti defined $O(n,k)$ in general, but we need the case $k = 1$ only.

It is helpful to regard an ordinal diagram $t = (i_1,t_1) \sharp (i_2,t_2) \sharp \cdots \sharp (i_m,t_m)$ as an edge-labeled tree



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recursively unfolding $t_1, t_2, \ldots, t_n$. The empty ordinal diagram 0 is regarded as the tree consisting of a single node.

An *i-section* of an ordinal diagram t is a subdiagram u subject to the following condition: if u occurs as

$$\begin{array}{lcll} t & = & \cdots \sharp (i_0,t_1) \sharp \cdots & \\ t_k & = & \cdots \sharp (i_k,t_{k+1}) \sharp \cdots & \text{for } k = 1,2,\ldots,m-1 \\ t_m & = & \cdots \sharp (i_m,u) \sharp \cdots & \end{array}$$

for some $m \geq 1$, then the labels should satisfy $i_0, i_1, \ldots, i_{m-1} > i$ and moreover $i_m = i$. Equivalently, an i-section is defined as a subtree u where the edges on the path from the root of t to the root of u have labels $i_0, i_1, \ldots, i_m$ in this order satisfying $i_0, i_1, \ldots, i_{m-1} > i$ and moreover $i_m = i$.

We define the linear order $<_0$ on the set $O(n,1)$ of ordinal diagrams. To define this order, we need auxiliary ordering $<_1, <_2, \ldots, <_{n-1}$ and $<_\infty$.

2.23 Definition (of $<_0, <_1, \ldots, <_{n-1}, <_\infty$)
On the set $O(n,1)$ of ordinal diagrams, we define $<_i$ for natural numbers $i < n$ as well as $<_\infty$ in the following way:

(i) $s <_i t$ holds iff either of the following two conditions is satisfied: *(1)* $s \leq_i t^\circ$ for some i-section t° of t, or *(2)* $s^\circ <_i t$ for all i-section s° of s and furthermore $s <_{i+1} t$ holds. If $i = n-1$, we set $i+1$ to be ∞.

(ii) $s <_\infty t$ is defined first for ordinal diagrams of the form $s = (i, s')$ and $t = (j, t')$, in which case $s <_\infty t$ iff either $i < j$, or both $i = j$ and $s' <_i t'$. Then this $<_\infty$ is extended to multisets by the multiset ordering induced by this.

Remark: In the definition above, we apply the multiset ordering only to $<_\infty$ while Takeuti did to all of $<_i$ and $<_\infty$ in the original definition. According to this modification, we extend the definition of i-sections to all ordinal diagrams, not only to those of the form (j, s).

We prove that the ordinal diagrams and the orderings on them are exactly the recursive path orderings on the iterated initial algebra $\mu X_1 \mu X_2 \cdots \mu X_n.\, \exp(X_1 + X_2 + \cdots + X_n)$. Namely, we put

$$\begin{aligned} D_1 &= \mu X_1.\, D_2[X_1] \\ D_2[X_1] &= \mu X_2.\, D_3[X_1, X_2] \\ &\vdots \\ D_n[X_1, X_2, \ldots, X_{n-1}] &= \mu X_n.\, D_\infty[X_1, X_2, \ldots, X_{n-1}, X_n] \\ D_\infty[X_1, X_2, \ldots, X_{n-1}, X_n] &= \exp(X_1 + X_2 + \cdots + X_{n-1} + X_n)\ . \end{aligned}$$

The canonical substitution is given by $\eta = [D_1/X_1][D_2/X_2]\cdots[D_n/X_n]$. We denote the partial substitution $[D_{k+1}/X_{k+1}]\cdots[D_n/X_n]$ by $\eta^{(k)}$. The members of ηD_∞ are multisets $\{\overline{\gamma}_{i_1} t_1, \overline{\gamma}_{i_2} t_2, \ldots, \overline{\gamma}_{i_m} t_m\}$ where $\overline{\gamma}_i t$ is the shorthand of $\gamma_i \gamma_{i+1} \cdots \gamma_n t$. We identify these multisets with ordinal diagrams

$$(i_1 - 1, t_1) \sharp (i_2 - 1, t_2) \sharp \cdots \sharp (i_m - 1, t_m).$$

The decreasing by 1 is simply because of the difference of the conventions of labeling. An empty multiset $\{\,\}$ is identified with the ordinal diagram 0. We show first that i-sections are characterized by weak normal forms.

2.24 Lemma
The partial substitution $\eta^{(i+1)} D_\infty$ is an analytic functor in X_{i+1} as well as in $X_1, X_2, \ldots, X_i$. We fix the latter variables, and put this functor $F(X_{i+1})$. Let

t be a member of ηD_∞.

$\overline{\gamma}_{i+1}u$ is in the image of f for a weak normal form $(Y, z) \xrightarrow{f} (\eta X_{i+1}, t)$ in $\mathrm{el}(F)$ if and only if u is an i-section of t.

(*Proof*) We can characterize the image of f of a weak normal form by the subterm relation $\subseteq$ analogously to Def. 2.13 for algebras. The condition on labels i_k in definition of i-sections corresponds to the condition on cardinalities in definition of the subterm relation. □

2.25 Lemma

Let s and t be ordinal diagrams.

(i) *$s <_\infty t$ holds iff $s < t$ by the multiset ordering on ηD_∞.*

(ii) *$s <_{i-1} t$ holds iff $\overline{\gamma}_i s < \overline{\gamma}_i t$ by the recursive path ordering on ηD_i, for each $i = 1, 2, \ldots, n$.*

(*Proof*) By definition and Lemma 2.24. □

Therefore the principal ordering $<_0$ on ordinal diagrams is exactly the recursive path ordering on the initial algebra $\mu X_1 \mu X_2 \cdots \mu X_n.\ \exp(X_1 + X_2 + \cdots + X_n)$. The characterization by the analytic functor clarifies the reason we need auxiliary ordering $<_1, <_2, \ldots, <_{n-1}$ and $<_\infty$. Moreover, the rather syntactic definition of i-sections is justified by weak normal forms.

Since the operation $\exp(\cdot)$ is not a part of the syntax of algebras, we embed the recursive path ordering on $\mu X_1 \mu X_2 \cdots \mu X_n.\ \exp(X_1 + X_2 + \cdots + X_n)$ to the recursive path ordering on

$$\mu X_1\, 1 + \mu X_2.\, X_1 + \mu X_3.\, X_2 + \cdots \mu X_{n-1}.\, X_{n-2} + \mu X_n.\, X_{n-1} + {X_n}^2.$$

Namely we define the algebra E_1 by the following:

$$\begin{array}{lcl}
E_1 & = & \mu X_1.\ 1 + E_2[X_1] \\
E_2[X_1] & = & \mu X_2.\ X_1 + E_3[X_2] \\
E_3[X_2] & = & \mu X_3.\ X_2 + E_4[X_3] \\
 & \vdots & \\
E_{n-1}[X_{n-2}] & = & \mu X_{n-1}.\ X_{n-2} + E_n[X_{n-1}] \\
E_n[X_{n-1}] & = & \mu X_n.\ E_\infty[X_{n-1}, X_n] \\
E_\infty[X_{n-1}, X_n] & = & X_{n-1} + {X_n}^2.
\end{array}$$

We denote the canonical substitution $[E_1/X_1][E_2/X_2]\cdots[E_n/X_n]$ by η' to distinguish it from the canonical substitution associated to D_i above.

The mapping $(\cdot)^\flat : \eta D_\infty \to \eta' E_\infty$ is defined as follows: Let $t = \{\overline{\gamma}_{k_1} t_1, \overline{\gamma}_{k_2} t_2, \ldots, \overline{\gamma}_{k_m} t_m\}$ be a member of ηD_∞. We assume without loss of generality that $\overline{\gamma}_{k_1} t_1 \leq$

$\overline{\gamma}_{k_2} t_2 \le \cdots \le \overline{\gamma}_{k_m} t_m$ in the poset $\eta(X_1 + X_2 + \cdots + X_n) = \eta D_1 + \eta D_2 + \cdots + \eta D_n$. In particular, it holds that $k_1 \le k_2 \le \cdots \le k_m$ since the right component is prior by definition of sum $+$ of linear orders. Let us write $t = t' \cup \{\overline{\gamma}_{k_m} t_m\}$. Then we define

$$t^\flat \;=\; \langle \gamma_n {t'}^\flat, \overline{\overline{\gamma}}_{k_m} {t_m}^\flat \rangle \qquad (\text{where } \overline{\overline{\gamma}}_l = \gamma_n \cdots \gamma_{l+1} \gamma_l \gamma_{l+1} \cdots \gamma_n)$$

which is in the right component of $\eta' E_\infty = \eta' E_{n-1} + (\eta' E_n)^2$. As for the empty multiset $\{\}$, we put $\{\}^\flat = \gamma_{n-1} \cdots \gamma_2 \gamma_1 *$ in the left component.

2.26 Lemma
If $\overline{\gamma}_k t$ is a subterm of $\overline{\gamma}_k u$ in ηD_k, then $\overline{\gamma}_k t^\flat$ is a subterm of $\overline{\gamma}_k u^\flat$ in $\eta' E_k$, for every $k = 1, 2, \ldots, n$.

2.27 Lemma
(i) *If $t < u$ holds in ηD_∞, then $t^\flat < u^\flat$ holds in $\eta' E_\infty$.*
(ii) *If $\overline{\gamma}_k t < \overline{\gamma}_k u$ holds in ηD_k, then $\overline{\gamma}_k t^\flat < \overline{\gamma}_k u^\flat$ holds in $\eta' E_k$, for every $k = 1, 2, \ldots, n$.*

The proof is by induction on construction of terms. By this lemma, we have a strictly order-preserving map from D_1 into E_1. Therefore the following corollary holds:

2.28 Corollary
The order type of the algebra E_1 is greater than or equal to the order type of the set D_1 of ordinal diagrams.

Arai proved that, using the well-order $O(n, 1)$ of ordinal diagrams, Feferman's system ID_{n-1} of iterated inductive definition is consistent [3]. We refer the reader to [18] for the general theory of systems ID_n. Although we modify definition of the ordinal diagrams, we can apply the same proof after appropriate adjustment. For example, the original proof uses $t \sharp 0$ as the successor of the ordinal diagram t, while $t \sharp 0 = t$ in our definition. So we should replace it by $t \sharp 1$ where 1 is the second least element $\{\overline{\gamma}_1\{\}\}$ in $O(n, 1)$. Hence, by Cor. 2.28, the order type of the algebra E_1 is greater than or equal to the proof theoretical ordinal of ID_{n-1}.

It is well-known that the proof-theoretical ordinal $|(\Pi^1_1\text{-CA})_0| = \theta\Omega_\omega 0$ is the supremum of the proof-theoretical ordinals of ID_n for $n = 1, 2, \ldots$. Therefore, if we let $|A|$ denote the order type of the well-ordering on the algebra A, we have the inequation

$$|(\Pi^1_1\text{-CA})_0| \le \sup |A|$$

where A ranges over all closed algebras.

Next we show the reverse inequation of ordinals. To this end, it suffices to prove that the well-orderedness of each algebra is derived in system $(\Pi^1_1\text{-CA})_0$ of second order arithmetic.

System $(\Pi^1_1\text{-CA})_0$ [58] of second order arithmetic is given as follows. The language is that of the usual arithmetic with set variables X. The axioms are those of Peano arithmetic where the induction is restricted to the following form:

$$\forall X.\, X(0)\ \&\ (\forall x.\, X(x) \to X(x')) \to \forall x.\, X(x).$$

Moreover this system has the axiom scheme of Π^1_1-comprehension, which is the source of the name of the system:

$$\exists X\, \forall x.\, \varphi(x) \leftrightarrow X(x)$$

where $\varphi(x)$ is a Π^1_1-formula that does not contain X but may contain other set parameters. We note that the induction principle is applied to only Π^1_1-predicates.

We show that we can carry out in system $(\Pi^1_1\text{-CA})_0$ the proof of Thm. 2.10 asserting that the divisibility ordering is a well-quasi-order, provided that the involving functor F can be encoded in the language of arithmetic. This is the case if F is the functor occurring as an algebra in Def. 2.12.

2.29 Lemma
System $(\Pi^1_1\text{-CA})_0$ proves the following: If F is an endofunctor on **QO** *preserving inclusions and is a lifting of an analytic functor, and if in addition F preserves well-quasi-orders, then μF is a well-quasi-order.*

(*Proof*) Towards contradiction, we assume that there is an infinite bad sequence g in A. Let T be the tree of the finite initial segments of all infinite bad sequences in A. Namely, $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ is in T iff there is $f \in A^\omega$ such that f is bad and $a_i = f_i$ for all $i = 0, 1, \ldots, n-1$. This formula is Σ^1_1 in A. So we can assert the existence of T using the Π^1_1-comprehension axiom, from which Σ^1_1-comprehension is derivable. We define the subtree T' of T as follows: $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ is in T' iff it is in T and, moreover, for every $\langle b_0, b_1, \ldots, b_{k-1}\rangle$ in T and every $i < \min\{n, k\}$ such that $a_0 = b_0, a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$, it holds that $p(a_i) \leq p(b_i)$. The definition of T' is arithmetic in T, so exists provably in $(\Pi^1_1\text{-CA})_0$. The least number principle implies that T' is non-empty and every path in T' is infinite. We note that the involving set on which the least number principle is applied is arithmetic in T'. Let us take a path h in T'. Since every path in T' is infinite, it suffices to take the leftmost path that can be defined arithmetically in T'. Then h is an infinite bad sequence such that $\langle p(h_0), p(h_1), \ldots\rangle$ is minimal lexicographically among all infinite bad sequences.

Now we define S as the set of the immediate subterms of all h_i. As shown earlier, S is a well-partial-order, which can be formalized easily. So induction hypothesis implies $F(S)$ is a well-partial-order, contradicting that h is bad. $\square$

2.30 Corollary
Let A be an algebra with its canonical substitution η.

System $(\Pi^1_1\text{-CA})_0$ proves that the partial order $\trianglelefteq_A$ on ηA is a well-partial-order.

(*Proof*) This is obtained by applying the previous lemma iteratively. $\square$

We note this corollary asserts that, if we fix an algebra, system $(\Pi^1_1\text{-CA})_0$ proves that the associated partial order is a well-partial-order. We cannot prove in system $(\Pi^1_1\text{-CA})_0$ that every algebra is a well-partial-order with respect to the associated partial order. Namely there is no uniform proof of well-partial-orderedness applied to all algebras in system $(\Pi^1_1\text{-CA})_0$.

2.31 Corollary
Let A be an algebra with its canonical substitution η.

System $(\Pi^1_1\text{-CA})_0$ proves that the linear order $<_A$ on ηA is a well-order.

(*Proof*) The linear order $<_A$ is a linearization of the partial order $\trianglelefteq_A$. So $<_A$ is a well-order. Obviously this argument can be carried out in system $(\Pi^1_1\text{-CA})_0$. $\square$

The proof-theoretical ordinal of a logical system is the least ordinal the well-orderedness of which cannot be proved within the system. Hence the corollary induces the inequation $|A| < |(\Pi^1_1\text{-CA})_0|$ where we recall that $|A|$ is the order type of the linear order $<_A$ on the closed algebra A. Therefore

$$\sup |A| \leq |(\Pi^1_1\text{-CA})_0|.$$

The reverse inequation has already been verified above. So we can conclude the following.

2.32 Theorem
The supremum of the order types $|A|$ of the linear orders on algebras A equals the proof-theoretical ordinal $|(\Pi^1_1\text{-CA})_0| = \theta\Omega_\omega 0$.

It is known that a large part of mathematics can be formalized in system $(\Pi^1_1\text{-CA})_0$. So we have fairly strong well-partial-orders and well-orders by the divisibility ordering and the recursive path ordering.

3 Fixpoint, Trace, and Inversion

Relation between lambda calculi and cartesian closed categories is well-known [43]. The lambda calculus is introduced in mathematical logic as a syntax of type theory by Church. Later it is employed as a foundation of functional programming language in computer science, and developed in both theory and practice. The link between lambda calculus and categories follows after Lambek's and Lawvere's work, accompanying the development of the theory of categorical logic. We refer the reader to [2, 43] for these subjects.

We verified that category $\mathbf{CAAcc}_{\mathrm{NF}}$ of complete atomic accessible categories and normal functors is cartesian closed (Cor. 1.25). Hence we can form a model of simply typed lambda calculus in this category.

Category $\mathbf{CAAcc}_{\mathrm{NF}}$ shares many good properties with the categories of partial orders used in domain theory [26], although the former is not of partial orders. For example, we pointed out in §1 the similarity of categories $\mathbf{Set}^A$ of presheaves to complete atomic Boolean algebras. The most important is that we can solve domain equations in category $\mathbf{CAAcc}_{\mathrm{NF}}$. Theorem 1.24 shows that the exponential in this category is given by $\mathbf{Set}^{\exp B \times C}$, interpreting function type $B \Rightarrow C$. A remarkable point is that, although B is the negative occurrence in the type, the corresponding interpretation $\exp B \times C$ is a covariant functor. In fact, this is an analytic functor in B and C. Hence one can solve domain equations by the initial algebra construction, or by the terminal algebra construction (each analytic functor F preserves ω-inverse limits [4]).

In particular, from a solution of the domain equation $A \cong \exp A \times A$, we can form a model of the untyped lambda calculus in $\mathbf{Set}^A$ [23]. This object of $\mathbf{CAAcc}_{\mathrm{NF}}$ satisfies the isomorphism $\mathbf{Set}^A \cong [\mathbf{Set}^A, \mathbf{Set}^A]_{\mathrm{NF}}$. Hence one can interpret terms of the lambda calculus as formal power series in $\operatorname{card} A$ variables. We show in another paper [30] some properties of this model. In particular, if A is an initial algebra of the analytic functor $\exp(\text{-}) \times \text{-}$, the formal power series interpreting the terms have finite coefficients only. So they are regarded as formal power series in non-negative integer coefficients.

In this paper, we give a model of system PCF [50, 49] of typed lambda calculus in category $\mathbf{CAAcc}_{\mathrm{NF}}$. This system is a variation of simply typed lambda calculus augmented with elementary arithmetic and Boolean operations as well as the fixpoint combinator.

The interpretation of the fixpoint combinator is especially interesting. We give a concrete formula computing the interpretation of the fixpoint combinator. To this end, we pass through the intuitionistic linear logic with the fixpoint combinator and its model using categorical trace by Joyal, Street and Verity [40]. It is shown by Hasegawa that giving a trace in a cartesian category is equal to giving a fixpoint operator [27]. Hence the computation of a fixpoint operator is reduced to the computation of trace subject to certain axioms.

A fixpoint operator in the context of formal power series has already appeared in the literature. It is known as the Lagrange-Good inversion formula [21]. The formula for formal power series in a single variable dates back to the end of eighteenth century, and is attributed to Lagrange. Many celebrated mathematicians tried to extend the formula to several variables, and obtained partial solutions. The currently known formula is finally established by Good. The traditional proof of the formula is the use of residues in analysis [24, 33]. Recently several new proofs are produced, employing enumerative combinatorics. They include [37, 21, 16, 11]. Here we give a new proof of the Lagrange-Good inversion formula based on the interpretation of the fixpoint combinator of lambda calculus. This is an application of the ideas developed in the theory of lambda calculus in theoretical computer science to pure mathematics.

PCF

We define system *PCF* of typed lambda calculus with fixpoint combinator as well as arithmetic and Boolean operations. The types σ of PCF are given by the following Backus-Naur form:

$$\sigma \quad ::= \quad \iota \mid o \mid \sigma \Rightarrow \sigma.$$

The type ι is regarded as the type of natural numbers, and the type o as that of Boolean values. The terms M are given by the following syntax:

$$\begin{array}{rcl} M & ::= & x \mid MM \mid \lambda x^{\sigma}.\, M \mid \mathsf{fix}\, M \\ & & \mid \mathsf{succ}\, M \mid \mathsf{pred}\, M \mid \mathsf{zero?}\, M \\ & & \mid \mathsf{cond}\, MMM \mid \mathsf{t} \mid \mathsf{f} \mid n\ . \end{array}$$

Here x is a variable from a fixed countable set, and n ranges over the set of natural numbers $0, 1, \ldots$. The typing rules are obvious and we omit them. We consider the following reduction rules:

$$\begin{array}{lcllcl} (\lambda x^{\sigma}.\, M)N & \to & M[N/x] & & & \\ \mathsf{fix}\, M & \to & M(\mathsf{fix}\, M) & & & \\ \mathsf{succ}\, n & \to & n+1 & & & \\ \mathsf{pred}\, n+1 & \to & n & \mathsf{pred}\, 0 & \to & 0 \\ \mathsf{zero?}\, n+1 & \to & \mathsf{t} & \mathsf{zero?}\, 0 & \to & \mathsf{f} \\ \mathsf{cond}\, \mathsf{t}\, MN & \to & M & \mathsf{cond}\, \mathsf{f}\, MN & \to & N. \end{array}$$

We give a model of PCF in the category $\mathbf{CAAcc}_{\mathbf{NF}}$. Types are interpreted as sets by the following definition:

$$\begin{aligned}
[\![\iota]\!] &= \omega \\
[\![o]\!] &= 2 \\
[\![\sigma \Rightarrow \tau]\!] &= \exp[\![\sigma]\!] \times [\![\tau]\!]
\end{aligned}$$

where ω is the set of natural numbers and 2 is the set $\{0,1\}$. The interpretation of function type corresponds to Thm. 1.24.

For the interpretation of a term, we define it as a function of the pairs of environments $\Gamma = x_1 : \sigma_1,\ x_2 : \sigma_2,\ \ldots,\ x_n : \sigma_n$ and a term $M : \tau$ such that $\Gamma \vdash M : \tau$ is a correct typing judgement. The interpretation $[\![M]\!]_{\vec{x}}$ is a normal functor from $\mathbf{Set}^{A_1+A_2+\cdots+A_n}$ to $\mathbf{Set}^B$ where $A_i = [\![\sigma_i]\!]$ and $B = [\![\tau]\!]$. The definition of $[\![M]\!]_{\vec{x}}$ is by induction on construction of terms. What is the most interesting is the interpretation of the fixpoint combinator. But we start with easy ones.

The interpretation of the fragment of ordinary typed lambda calculus is induced from the structure of $\mathbf{CAAcc}_{\mathrm{NF}}$ as a cartesian closed category:

$$\begin{aligned}
[\![x_i]\!]_{\vec{x}}(\vec{t}) &= t_i \\
[\![MN]\!]_{\vec{x}}(\vec{t}) &= ev([\![M]\!]_{\vec{x}}(\vec{t}), [\![N]\!]_{\vec{x}}(\vec{t})) \\
[\![\lambda y^\sigma . M]\!]_{\vec{x}}(\vec{t}) &= lam([\![M]\!]_{\vec{x},y}(\vec{t}, \text{-}))
\end{aligned}$$

where $ev : \mathbf{Set}^{\exp A \times B} \times \mathbf{Set}^A \to \mathbf{Set}^B$ is the counit of cartesian closedness and lam is the isomorphism $[\mathbf{Set}^A,\ \mathbf{Set}^B]_{\mathrm{NF}} \xrightarrow{\sim} \mathbf{Set}^{\exp A \times B}$.

The interpretations of arithmetic and Boolean operations are given next. The numerals of type ι are interpreted as the singletons of the corresponding numerals in ω. Namely, $[\![n]\!]$ for each numeral n is the singleton multiset $\{n\}$, which means the presheaf in $\mathbf{Set}^\omega$ carrying n to 1 and all other members of ω to $\emptyset$. Likewise the Boolean values t and f are interpreted as

$$[\![\mathsf{t}]\!] = \{1\}, \quad [\![\mathsf{f}]\!] = \{0\}.$$

The operations succ and pred are interpreted as presheaves in $\mathbf{Set}^{\exp \omega \times \omega}$ carrying

$$\begin{aligned}
[\![\mathsf{succ}]\!] &: [\{n\}, n+1] \mapsto 1 \\
[\![\mathsf{pred}]\!] &: [\{n+1\}, n] \mapsto 1 \\
[\![\mathsf{pred}]\!] &: [\{0\}, 0] \quad\ \mapsto 1
\end{aligned}$$

and taking the value $\emptyset$ for all other members. In other words, for example, $[\![\mathsf{succ}]\!]$ is the normal functor from $\mathbf{Set}^\omega$ to $\mathbf{Set}^\omega$ (so ω-indexed family of formal power series) satisfying that the n-th component is simply monomial x^{n+1}. Likewise the zero-test $\mathsf{zero?}$ is interpreted as a presheaf in $\mathbf{Set}^{\exp \omega \times 2}$ carrying

$$\begin{aligned}
[\![\mathsf{zero?}]\!] &: [\{n+1\}, 1] \mapsto 1 \\
[\![\mathsf{zero?}]\!] &: [\{0\}, 0] \quad\ \mapsto 1
\end{aligned}$$

and taking $\emptyset$ for all other members. Finally cond for type σ is interpreted as a presheaf in $\mathbf{Set}^{\exp 2 \times \exp A \times \exp A \times A}$ for $A = [\![\sigma]\!]$, carrying

$$\begin{array}{lll} [\![\mathsf{cond}]\!] & : & [\{1\}, \{a\}, \{\}, a] \mapsto 1 \\ [\![\mathsf{cond}]\!] & : & [\{0\}, \{\}, \{a\}, a] \mapsto 1 \end{array}$$

for all members $a \in A$, and taking the value $\emptyset$ for all other members of $\exp 2 \times \exp A \times \exp A \times A$.

The interpretation of, for example, succ operates as follows: Let t be a presheaf in $\mathbf{Set}^{\omega}$, regarded as a multiset of members of ω. Then $[\![\mathsf{succ}]\!](t)$ equals the multiset $\{n+1 \mid n \in t\}$. In particular, if $t = [\![n]\!]$, i.e., the singleton $\{n\}$, we have $[\![\mathsf{succ}]\!](t) = [\![n+1]\!]$, validating the reduction $\mathsf{succ}\, n \to n+1$. It is easy to see that all other δ-reductions are sound by the interpretations just given.

What remains is the interpretation of the fixpoint combinator. It is given by initial algebra construction. We recall that, for each normal functor $f : \mathbf{Set} \to \mathbf{Set}$, there is an initial f-algebra μf. This easily generalizes to a normal functor f from $\mathbf{Set}^X$ to $\mathbf{Set}^X$. So the initial algebra constructor $f \mapsto \mu f$ yields a mapping from $\mathbf{Set}^{\exp X \times X}$ into $\mathbf{Set}^X$. We show that this mapping is actually a normal functor. By cartesian closedness of $\mathbf{CAAcc}_{\mathrm{NF}}$, there is a normal functor $ev : \mathbf{Set}^{\exp X \times X} \times \mathbf{Set}^X \to \mathbf{Set}^X$ carrying (f, x) to $f(x)$ as the counit of the adjunction of cartesian closedness. If we generalize Prop. 1.27 to initial algebras in the category $\mathbf{Set}^X$ of presheaves and apply it, we obtain a normal functor from $\mathbf{Set}^{\exp X \times X}$ to $\mathbf{Set}^X$ carrying f to μf. Hence we define the interpretation of the fixpoint combinator by

$$[\![\mathsf{fix}\, M]\!]_{\vec{x}}(\vec{t}) \;=\; \mu([\![M]\!]_{\vec{x}}(\vec{t})),$$

which is a normal functor in $\vec{t}$. (With a slight abuse, we do not distinguish a normal functor $f : \mathbf{Set}^X \to \mathbf{Set}^X$ from the presheaf $\tilde{f} \in \mathbf{Set}^{\exp X \times X}$ of its coefficients, where the notation $\tilde{f}$ is given in section 1 before Thm. 1.24).

3.1 Theorem
The model of PCF *in category* $\mathbf{CAAcc}_{\mathrm{NF}}$ *is sound. Namely, if* $M \to N$, *then* $[\![M]\!]_{\vec{x}} = [\![N]\!]_{\vec{x}}$ *as normal functors.*

Trace

We defined the interpretation of the fixpoint combinator by the initial algebra constructor μ. This is not convenient, however, to compute the actual form of the interpretations of PCF-terms as formal power series. We want to give the concrete form of the initial algebra constructor. To this end, we recall the relation between fixpoint operator and trace.

Categorical trace is introduced by Joyal, Street, and Verity as a generalization of several concepts including the usual trace in linear algebra [40]. It is given as an operation satisfying natural axioms of trace in a balanced monoidal categories [39]. We need only symmetric monoidal categories and, indeed, our main interest is in the case that the monoidal structure is given by the cartesian product.

3.2 Definition

A *traced monoidal category* is a symmetric monoidal category endowed with a family of operations

$$\frac{A \otimes X \xrightarrow{f} B \otimes X}{A \xrightarrow{\mathrm{tr}^X f} B}$$

subject to the following conditions:

(vanishing)	$\mathrm{tr}^I f \;=\; f$	for $A \otimes I \xrightarrow{f} B \otimes I$
	$\mathrm{tr}^X(\mathrm{tr}^Y f) \;=\; \mathrm{tr}^{X \otimes Y} f$	for $A \otimes X \otimes Y \xrightarrow{f} B \otimes X \otimes Y$
(superposing)	$\mathrm{tr}^X(1 \otimes f) \;=\; 1 \otimes \mathrm{tr}^X f$	for $A \otimes X \xrightarrow{f} B \otimes X$
(yanking)	$\mathrm{tr}^X c \;=\; 1$	for $X \otimes X \xrightarrow{c} X \otimes X$
(left-tightening)	$\mathrm{tr}^X(f(g \otimes 1)) \;=\; (\mathrm{tr}^X f)g$	for $A \otimes X \xrightarrow{f} B \otimes X$ and $A' \xrightarrow{g} A$
(right-tightening)	$\mathrm{tr}^X((g \otimes 1)f) \;=\; g(\mathrm{tr}^X f)$	for $A \otimes X \xrightarrow{f} B \otimes X$ and $B \xrightarrow{g} B'$
(sliding)	$\mathrm{tr}^X((1 \otimes g)f) \;=\; \mathrm{tr}^Y(f(1 \otimes g))$	for $A \otimes X \xrightarrow{f} B \otimes Y$ and $Y \xrightarrow{g} X$

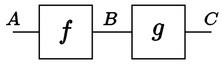
where $c = c_X$ is the symmetry. We omitted canonical isomorphisms, which should be clear from the context. For instance, the right hand side of the first rule of vanishing should be $\rho_B^{-1} \circ f \circ \rho_A$ with canonical isomorphisms $\rho_A : A \to A \otimes I$ and ρ_B.

Remark: In the balanced monoidal category, we modify the axiom of yanking by the following equations:

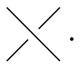
$$(\mathrm{tr}^X c)\theta^{-1} \;=\; 1_X \;=\; (\mathrm{tr}^X c^{-1})\theta$$

where $c : X \otimes X \to X \otimes X$ is the crossing and $\theta : X \to X$ is the twisting [40].

The axioms of traced monoidal category are simulated by graphs. A morphism is drawn as a directed graph where the vertices are labeled by primitive morphisms and the edges are labeled by objects. For example, the composite of two morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ is given by the cascade of two graphs:



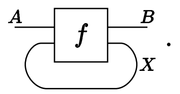
where the directions on edges are omitted. The symmetry $c : A \otimes B \to B \otimes A$ is given by the crossing



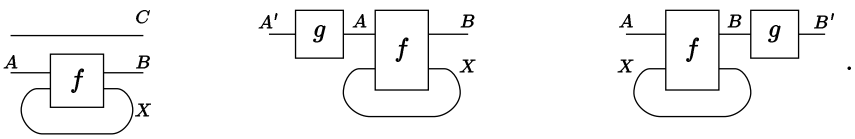
Since we are interested in only symmetric monoidal categories, which string is upper to the other is irrelevant. If one deals with a braided monoidal category, two ways of laying two strings must be distinguished. The naturality is simply moves of links. For instance, the naturality of the crossing c amounts to the equation between two graphs

meaning $c \circ (f \otimes g) = (g \otimes f) \circ c$. We note that these two graphs denote the topologically same graph.

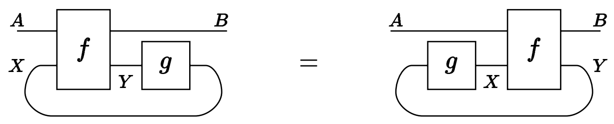
The trace $A \xrightarrow{\mathrm{tr}^X f} B$ is denoted by a loop connecting two edges of X:

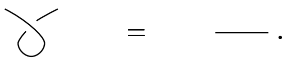


The axioms of trace monoidal category are transformed into graphs so that the both sides of the equalities do not change the topological structure of graphs, except two vanishing rules. The superposing and the left and right tightening rules are drawn by the following graphs respectively:

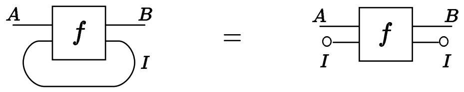


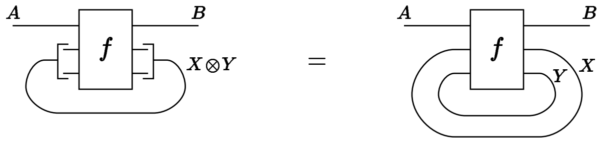
Both sides of these rules are denoted by the same graphs. For example, the superposing rule means that trace after tensor equals tensor after trace. The sliding rule and the yanking rule are more interesting. They amount to the following equalities between graphs:





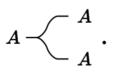
Both sides of two vanishing rules do not give the topologically same graphs. If we insist on drawing them in the form of figures, they may be written as





In this paper, we handle these rules implicitly. Namely, we ignore the loop on the unit I and identify a loop on tensor with decomposed two loops.

We consider the traced monoidal categories where the monoidal structures are cartesian products. For cartesian product, there are diagonal maps $\Delta_A : A \to A \times A$ and unique morphisms $!_A : A \to 1$ to the terminal object. Conversely, if a monoidal category has natural transformations $\Delta_A : A \to A \otimes A$ and $!_A : A \to I$, then $\otimes$ is cartesian product and I is a terminal object. We denote Δ_A diagramatically by



We define a fixpoint operator in a cartesian category.

3.3 Definition
A *fixpoint operator* in a cartesian category is an operation $(\cdot)^\dagger$

$$\frac{A \times X \xrightarrow{f} X}{A \xrightarrow{f^\dagger} X}$$

of morphisms, natural in A and dinatural in X, this operation subject to the condition that $f^\dagger$ is equal to the composite $A \xrightarrow{\Delta_A} A \times A \xrightarrow{1 \times f^\dagger} A \times X \xrightarrow{f} X$

As a condition on a fixpoint operator, we define Bekič's formula, which asserts that a fixpoint of a binary function is equal to iteration of fixpoints in one of the arguments fixing the other.

3.4 Definition

We consider a cartesian category with fixpoint operator $(\cdot)^\dagger$. Given two morphisms $A \times X \times Y \xrightarrow{f} X$ and $A \times X \times Y \xrightarrow{g} Y$, we put $h : A \times X \to X$ as $A \times X \xrightarrow{\Delta_{A\times X}} A \times X \times A \times X \xrightarrow{1_{A\times X}\times g^\dagger} A \times X \times Y \xrightarrow{f} X$ where Δ is the diagonal.

Bekič's formula is the equality $\langle f, g\rangle^\dagger = \langle h^\dagger, g^\dagger\langle 1_A, h^\dagger\rangle\rangle$ between morphisms from A to $X \times Y$.

Remark: It would be easier to understand, if we informally write h as $h(a, x) = f(a, x, g^\dagger(a, x))$ where a and x are parameters from A and X.

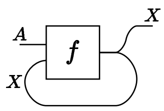
M. Hasegawa proved that, in a cartesian category, giving a trace is equivalent to giving a fixpoint operator satisfying Bekič's formula [27].

3.5 Theorem

Let $\mathbf{C}$ *be a cartesian category.*

The category $\mathbf{C}$ *is a traced cartesian category iff* $\mathbf{C}$ *has a fixpoint operator satisfying Bekič's formula.*

We give a sketch of a proof of $(\Rightarrow)$. For a morphism $A \times X \xrightarrow{f} X$, the fixpoint operator $f^\dagger : A \to X$ is given by trace as shown diagrammatically in the following:



The proof that $f^\dagger$ is a fixpoint of f is given as in Fig. 3.6. The first equation is the naturality of diagonal. The second is the sliding of f, and the third is the naturality of crossing. Finally, the yanking concludes that $f^\dagger$ is the fixpoint of f.

We leave to the reader the proof of Bekič's formula. It amounts to derive the following equality between two graphs.

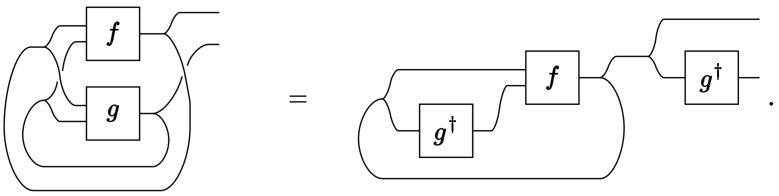
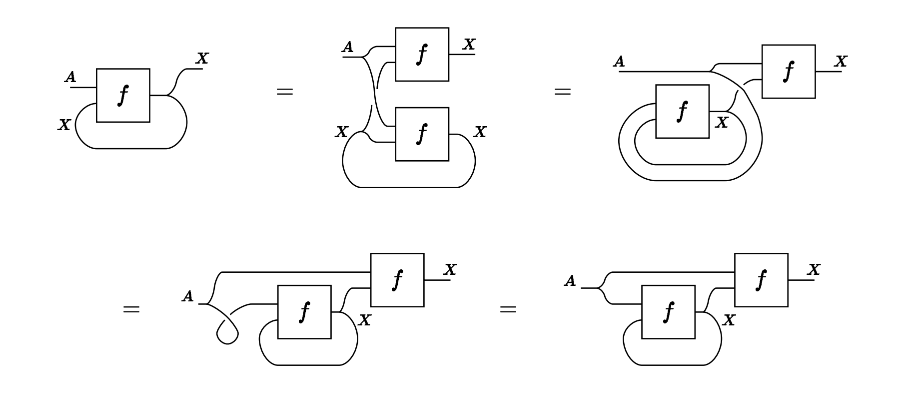


Figure 3.6 *Fixpoint combinator from trace*



Intuitionistic Linear Logic

By the observation above, in order to find a fixpoint in a cartesian category, we only have to find a trace. So we want to define a trace in the cartesian category $\mathbf{CAAcc}_{\mathrm{NF}}$ of complete atomic accessible categories and normal functors. To this end, it is informative to consider the intuitionistic linear logic. We can refine the model of PCF by $\mathbf{CAAcc}_{\mathrm{NF}}$ into a model of intuitionistic linear logic. In this model, the interpretations of terms are given by a kind of matrices. Hence an analogy to linear algebra helps to find the trace we are looking for.

We define the *intuitionistic linear logic*, which is given as a system of typed calculus [6]. The types A are generated by the following form:

$$A \quad := \quad \alpha \mid A \otimes A \mid I \mid A \multimap A \mid\ !A$$

where α ranges over atomic types. For example, for the system to correspond to PCF, we may let α be either ι or o.

A typing judgement is, as usual, of the form $\Gamma \vdash e : B$ where e is a term, B is a type, and Γ is an environment $x_1 : A_1,\ x_2 : A_2,\ \ldots\ ,\ x_n : A_n$ that is an assignment of types to a sequence of pairwise distinct variables. The derivation rules for typing judgements are given in Tab. 3.7, which defines the terms of intuitionistic linear logic at the same time.

The reduction rules for the term calculus of intuitionistic linear logic are not completely established. See [8], for example. It does not matter which ones we take, for miscellaneous rules. The core rules are the β-reductions given by the following six:

Table 3.7 *Term assignment for linear natural deduction*

$$\frac{}{x : A \vdash x : A}$$

$$\frac{\Gamma,\ x : A \vdash e : B}{\Gamma \vdash (\lambda x^A.\, e) : (A \multimap B)} \qquad \frac{\Gamma \vdash e : (A \multimap B) \qquad \Delta \vdash f : A}{\Gamma, \Delta \vdash (\mathsf{apply}\, e\ \mathsf{to}\ f) : B}$$

$$\frac{}{\vdash * : I} \qquad \frac{\Gamma \vdash e : I \qquad \Delta \vdash f : A}{\Gamma, \Delta \vdash (\mathsf{let}\, e\ \mathsf{be}\ *\ \mathsf{in}\ f) : A}$$

$$\frac{\Gamma \vdash e : A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash (e \otimes f) : (A \otimes B)} \qquad \frac{\Gamma \vdash e : (A \otimes B) \qquad \Delta,\ x : A,\ y : B \vdash f : C}{\Gamma, \Delta \vdash (\mathsf{let}\, e\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f) : C}$$

$$\frac{\Gamma \vdash e : (!A)}{\Gamma \vdash (\mathsf{derelict}\, e) : A} \qquad \frac{\Gamma \vdash e : (!A) \qquad \Delta \vdash f : B}{\Gamma,\ \Delta \vdash (\mathsf{discard}\, e\ \mathsf{in}\ f) : B}$$

$$\frac{\Gamma \vdash e : (!A) \qquad \Delta,\ x :(!A),\ y :(!A) \vdash f : B}{\Gamma,\ \Delta \vdash (\mathsf{copy}\, e\ \mathsf{as}\ x, y\ \mathsf{in}\ f) : B}$$

$$\frac{\Gamma_i \vdash e_i : (!A_i) \quad (i = 1, \ldots, n) \qquad x_1 :(!A_1),\ \ldots,\ x_n :(!A_n) \vdash f : B}{\Gamma_1,\ \ldots,\ \Gamma_n \vdash (\mathsf{promote}\, e_1, \ldots, e_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ f) : (!B)}$$

$$\begin{array}{lcl}
\mathsf{apply}(\lambda x^A.\, e)\ \mathsf{to}\ f & \to & f[e/x] \\
\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ f & \to & f \\
\mathsf{let}\, d \otimes e\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f & \to & f[d/x, e/y] \\
\mathsf{derelict}(\mathsf{promote}\, \vec{e}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ f) & \to & f[\vec{e}/\vec{x}] \\
\mathsf{discard}(\mathsf{promote}\, \vec{e}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ f)\ \mathsf{in}\ g & \to & \mathsf{discard}\, \vec{e}\ \mathsf{in}\ g \\
\mathsf{copy}(\mathsf{promote}\, \vec{e}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ f)\ \mathsf{as}\ y, z\ \mathsf{in}\ g & \to & \mathsf{copy}\, \vec{e}\ \mathsf{as}\ \vec{v}, \vec{w}\ \mathsf{in}\ g[c/y, d/z].
\end{array}$$

In the last rule, we put c to be $(\mathsf{promote}\, \vec{v}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ f)$ and d to be $(\mathsf{promote}\, \vec{w}\ \mathsf{for}\ \vec{x}\ \mathsf{in}\ f)$. Here we use shorthands

$$\begin{array}{rcl}
\mathsf{discard}\, e_1, e_2, \ldots, e_n\ \mathsf{in}\ g & = & \mathsf{discard}\, e_1\ \mathsf{in}\ (\mathsf{discard}\, e_2\ \mathsf{in} \cdots (\mathsf{discard}\, e_n\ \mathsf{in}\ g) \cdots) \\
\mathsf{copy}\, e_1, e_2, \ldots, e_n\ \mathsf{as}\ \vec{v}, \vec{w}\ \mathsf{in}\ g & = & \mathsf{copy}\, e_1\ \mathsf{as}\ v_1, w_1\ \mathsf{in}\ (\mathsf{copy}\, e_2\ \mathsf{as}\ v_2, w_2\ \mathsf{in} \\
& & \qquad \cdots (\mathsf{copy}\, e_n\ \mathsf{as}\ v_n, w_n\ \mathsf{in}\ g) \cdots)
\end{array}$$

where in the latter the vector of variables $\vec{v} = v_1, v_2, \ldots, v_n$ and $\vec{w} = w_1, w_2, \ldots, w_n$ are used.

There is a standard translation of simply typed lambda calculus into intuitionistic linear logic. A type $A \Rightarrow B$ of typed lambda calculus is translated into the type $!A \multimap B$ of intuitionistic linear logic. We denote this translation by $A \leadsto A^*$. Accordingly, we have the translation of typing judgements as

$$x_1 : A_1\ ,\ x_2 : A_2\ ,\ \dots\ ,\ x_n : A_n\ \vdash\ e : A$$
$$\leadsto\ x_1 : !A_1^*\ ,\ x_2 : !A_2^*\ ,\ \dots\ ,\ x_n : !A_n^*\ \vdash\ e^* : B^*$$

for an appropriate translation $e \leadsto e^*$ of terms (to be precise, definition e^* depends also on the environment).

A model of intuitionistic linear logic is given by the category $\mathbf{C}$ fulfilling the following structures: The multiplicative fragment $\otimes, I$ and $\multimap$ are interpreted by a symmetric monoidal closed category [45]. The exponential ! is interpreted as a symmetric monoidal functor [17], which is given as a triple $(!, \tilde{\varphi}, \varphi_0)$ where $! : \mathbf{C} \to \mathbf{C}$ is a functor, $!A \otimes !B \xrightarrow{\tilde{\varphi}_{A,B}} !(A \otimes B)$ is a natural transformation, and $I \xrightarrow{\varphi_0} !I$ is a morphism. To interpret discard and copy, we assume that each object of the form $!A$ is endowed with a commutative comonoid structure $(!A, e_A, d_A)$ where $!A \xrightarrow{e_A} I$ and $!A \xrightarrow{d_A} !A \otimes !A$ are monoidal natural transformations. To interpret derelict and promote, we assume that the functor ! takes part of the comonad $(!, \epsilon, \delta)$ where $!A \xrightarrow{\epsilon_A} A$ and $!A \xrightarrow{\delta_A} !!A$ are monoidal natural transformations. Moreover, we need several coherence conditions to make this model sound. See, for example, [8].

We modify the model of PCF in the category $\mathbf{CAAcc}_{\mathrm{NF}}$ to construct a model of intuitionistic linear logic. In this model, morphisms should interpret linear terms. So we need the following definition:

3.8 Definition
A *linear normal functor* from $\mathbf{Set}^A$ to $\mathbf{Set}^B$ is a functor preserving all pullbacks and all colimits. The category $\mathbf{CAAcc}_{\mathrm{LNF}}$ of complete atomic accessible categories and the ismorphism classes of linear normal functors is induced as a subcategory of $\mathbf{CAAcc}_{\mathrm{NF}}$.

Alternatively, linear normal functors are those normal functors $\mathbf{Set}^A \xrightarrow{f} \mathbf{Set}^B$ where, for every normal form (X, a) in $\mathrm{el}(f_b)$ for a member $b \in B$, the underlying finitely presentable object $X \in \mathbf{Set}^A$ corresponds to a singleton in $\exp A$. Hence, if we denote by $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{LNF}}$ the category of linear normal functors and cartesian natural transformations, we have the categorical equivalence $[\mathbf{Set}^A, \mathbf{Set}^B]_{\mathrm{LNF}} \cong \mathbf{Set}^{A\times B}$.

A presheaf in $\mathbf{Set}^{A\times B}$ is regarded as a matrix with the columns indexed by the members of A and the rows indexed by the members of B such that each entry is a set. If we have two matrices $M \in \mathbf{Set}^{A\times B}$ and $N \in \mathbf{Set}^{B\times C}$, the composite of the corresponding linear normal functors is represented by multiplication of matrices NM where the entry of index $(a,c) \in A \times C$ is the set $\sum_{b \in B} N[b,c]M[a,b]$. Here coproduct in $\mathbf{Set}^{A\times C}$ is denoted by $\sum$ and cartesian product by concatenation.

We show that the category $\mathbf{CAAcc}_{\mathbf{LNF}}$ forms a model of intuitionistic linear logic. First we define the symmetric monoidal closed structure. Tensor of $\mathbf{Set}^A$ and $\mathbf{Set}^B$ is given by $\mathbf{Set}^{A\times B}$, and unit I by $\mathbf{Set}^1$ where 1 is a singleton. The right adjoint of tensor is given also by product as $\mathbf{Set}^A \multimap \mathbf{Set}^B = \mathbf{Set}^{A\times B}$.

The monoidal functor to interpret the exponential ! is defined as follows: On objects, $!A$ is the set $\exp A$ of all finite multisets of members of A. On morphisms, we define the following operation associating the matrix $\wp M \in \mathbf{Set}^{\exp A \times \exp B}$ to a matrix $M \in \mathbf{Set}^{A\times B}$. First we note the categorical equivalence $[\mathbf{Set}^A, \mathbf{Set}]_{\mathbf{NF}} \cong \mathbf{Set}^{\exp A}$. The mapping $g \mapsto g \circ {}^tM$ defines a functor from $[\mathbf{Set}^A, \mathbf{Set}]_{\mathbf{NF}}$ to $[\mathbf{Set}^B, \mathbf{Set}]_{\mathbf{NF}}$ where ${}^tM$ is the usual transpose of the matrix M. This functor is linear (although the proof is not so trivial as it looks, since we identify certain cartesian natural transformations). Hence it determines a linear normal functor $\wp M$ from $\mathbf{Set}^{\exp A}$ to $\mathbf{Set}^{\exp B}$. By definition, it is obvious that $\wp$ is functorial, preserving identities and composition. This construction $\wp M$ appears in the tensor representation in a polynomial ring [56].

In the level of matrices, we have the following direct description of $\wp M$. Suppose we are given a matrix M in $\mathbf{Set}^{A\times B}$. For each $\gamma = \{a_1, a_2, \ldots, a_m\} \in \exp A$ and each $\delta = \{b_1, b_2, \ldots, b_n\} \in \exp B$, the entry of $\wp M$ at the γ-th column and the δ-th row is given as, if $m = n$,

$$\wp M[\gamma,\delta] \;=\; \sum_{\langle a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)}\rangle} M[a_{p(1)}, b_1]M[a_{p(2)}, b_2]\cdots M[a_{p(n)}, b_n]$$

where the summation is over all linear orderings $\langle a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)}\rangle$ of members of γ, identifying two orderings determined by p and q in S_n if it holds that $a_{p(i)} = a_{q(i)}$ for all $i = 1, 2, \ldots, n$. If $m \neq n$, then $\wp M[\gamma, \delta]$ is always $\emptyset$. One may write as follows instead: Let $\Sigma\gamma$ be the underlying set of γ distinguishing two a_i's even if they are the same as members of A. That is to say, $\Sigma\gamma$ is the set $f_!(\gamma)$ where $\mathbf{Set}^A \xrightarrow{f_!} \mathbf{Set}$ is associated to a unique function $A \xrightarrow{f} 1$ to a singleton 1. Furthermore, let $(\Sigma\gamma, \Sigma\delta)/\mathrm{Aut}(\gamma)$ be the quotient set of all bijections from $\Sigma\gamma$ onto $\Sigma\delta$ divided by the equivalence class determined by the action of the group $\mathrm{Aut}(\gamma)$ of all automorphisms on γ in $\mathbf{Set}^A$. Then $\wp M[\gamma, \delta]$ may write

$$\wp M[\gamma,\delta] \;=\; \sum_{k \in (\Sigma\gamma, \Sigma\delta)/\mathrm{Aut}(\gamma)} \; \prod_{b \in \Sigma\delta} M[k^{-1}(b), b].$$

Remark: The isomorphism $\wp(MN) \cong \wp M \wp N$ of matrices holds in the category $\mathbf{Set}^{\exp A \times \exp C}$ if $MN \in \mathbf{Set}^{A\times C}$. However, the isomorphism is not canonical.

Table 3.9 *Functions for natural transformations in linear category*

f^*	corresponding function f
φ_0	$\exp 1 \to 1$ $\{*, *, \ldots, *\} \mapsto *$
$\tilde{\varphi}_{A,B}$	$\exp(A \times B) \to \exp A \times \exp B$ $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \ldots, \langle a_n, b_n\rangle\}$ $\mapsto \langle\{a_1, a_2, \ldots, a_n\}, \{b_1, b_2, \ldots, b_n\}\rangle$
ϵ_A	$A \to \exp A$ $a \mapsto \{a\}$
δ_A	$\exp(\exp A) \to \exp A$ $\{\{a_{11}, \ldots, a_{1n_1}\}, \{a_{21}, \ldots, a_{2n_2}\}, \ldots, \{a_{m1}, \ldots, a_{mn_m}\}\}$ $\mapsto \{a_{11}, \ldots, a_{1n_1}, a_{21}, \ldots, a_{2n_2}, \ldots, a_{m1}, \ldots, a_{mn_m}\}$
e_A	$1 \to \exp A$ $* \mapsto \{\}$
d_A	$\exp A \times \exp A \to \exp A$ $\langle\{a_1, a_2, \ldots, a_m\}, \{a'_1, a'_2, \ldots, a'_n\}\rangle$ $\mapsto \{a_1, a_2, \ldots, a_m, a'_1, a'_2, \ldots, a'_n\}$

If we work in the 2-categorical setting, the operation $\wp$ is not even a pseudo-functor, although this problem does not occur since we work with categories, identifying isomorphic ones.

The natural transformations involved in linear category are given as the inverse image f^* of appropriate function f. The inverse image $f^* : \mathbf{Set}^B \to \mathbf{Set}^A$ is a linear normal functor, and its matrix $M \in \mathbf{Set}^{B \times A}$ satisfies the condition that $M[b, a]$ equals 1 if $f(a) = b$; otherwise equals 0. For instance, the morphism $\tilde{\varphi} : !A \otimes !B \multimap !(A \otimes B)$ is the functor f^* in $[\mathbf{Set}^{\exp A \times \exp B}, \mathbf{Set}^{\exp(A \times B)}]_{\mathrm{LNF}}$ corresponding to the function $\exp(A \times B) \xrightarrow{f} \exp A \times \exp B$ carrying $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \ldots, \langle a_n, b_n\rangle\}$ to the pair of $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$. The functions to define natural transformations are given in Tab. 3.9.

The category $\mathbf{CAAcc}_{\mathrm{NF}}$ of normal functors is equivalent to the coKleisli category over the category $\mathbf{CAAcc}_{\mathrm{NF}}$ of linear normal functors by the comonad $\exp(\cdot)$. One can prove this assertion by direct computation, using the concrete form of $\wp M$ given above.

Trace of Normal Functors

As observed above, the model of intuitionistic linear logic has similarity to linear algebras, although the entries of matrices are sets rather than numbers. So we can define the trace of a linear normal functor $\mathbf{Set}^A \xrightarrow{f} \mathbf{Set}^A$ by the diagonal sum $\sum_{a\in A} M[a,a]$ where M is the matrix in $\mathbf{Set}^{A\times A}$ associated to f. With this definition, the category $\mathbf{CAAcc}_{\mathrm{LNF}}$ of linear normal functors turns out to be a traced monoidal category.

However, what we want to have is the trace in the category $\mathbf{CAAcc}_{\mathrm{NF}}$ of normal functors. A normal functor $\mathbf{Set}^A \to \mathbf{Set}^A$ corresponds to a matrix in $\mathbf{Set}^{\exp A\times A}$. We cannot take the diagonal sum, since this is not a square matrix. In the following, we show how to modify the trace of linear normal functors to the trace of normal functors.

A straightforward idea is the following. If we have a normal functor $A \xrightarrow{f} A$, that is, a linear normal functor $!A \overset{f}{\multimap} A$, we have the promotion $!A \overset{pf}{\multimap} !A$, corresponding to a square matrix in $\mathbf{Set}^{\exp A\times \exp A}$. Hence we can take the diagonal sum.

More generally, if a normal functor in $[\mathbf{Set}^{A+X}, \mathbf{Set}^{B+X}]_{\mathrm{NF}}$ is given, we may write it a pair of $!A\otimes !X \overset{h}{\multimap} B$ and $!A\otimes !X \overset{f}{\multimap} X$, employing the terminology of intuitionistic linear logic. Promoting the latter, we have $pf : !A\otimes !X \multimap !X$. Hence $h\otimes pf$ preceded by canonical morphism from $!A\otimes !X$ to $!A\otimes !X\otimes !A\otimes !X$ yields a linear map from $!A\otimes !X$ to $B\otimes !X$. We define $\sigma_h(f)$ as the diagonal sum of this linear map with respect to $!X$. Writing down everything in normal functors, we have the following: The promotion $!A\otimes !X \xrightarrow{pf} !X$ corresponds to the normal functor in $[\mathbf{Set}^{A+X}, \mathbf{Set}^{\exp X}]_{\mathrm{NF}}$ such that, for each $\gamma \in \exp X$, the γ-th formal power series is given by $f(a,x)^\gamma$. Hence the normal functor in $[\mathbf{Set}^{A+X}, \mathbf{Set}^{B\times \exp X}]_{\mathrm{NF}}$ given by $h\otimes pf$ preceded by the comultiplication on $!A\otimes !X$ corresponds to the system of formal power series $h_b(a,x)f(a,x)^\gamma$ where (b,γ) ranges over $B\times \exp X$. Hence, taking the diagonal sum with respect to $\gamma\in \exp X$, we obtain the following definition.

3.10 Definition (of $\sigma_h(f)$)
Let $\mathbf{Set}^{A+X} \xrightarrow{f} \mathbf{Set}^X$ and $\mathbf{Set}^{A+X} \xrightarrow{h} \mathbf{Set}^B$ be normal functors.

The normal functor $\sigma_h(f)$ from $\mathbf{Set}^A$ to $\mathbf{Set}^B$ is defined by $\sum [x^\gamma]h(a,x)f(a,x)^\gamma$ where the summation is over all $\gamma\in\exp X$. Here the notation $[x^\gamma]g(x)$ denotes the coefficient of x^γ in power series $g(x)$.

Unfortunately, this $\sigma_h(f)$ does not satisfy the axioms of traced monoidal categories. We give an improved definition later. Before that, we exploit several properties of $\sigma_h(f)$.

Remark: (i) We may regard the coefficients $\tilde{f}[(\alpha,\gamma),x]$ for $(\alpha,\gamma)\in \exp A\times \exp X$ and $x\in X$ as indeterminates. Let R denote the ring of all polynomials in these indeterminates with integer coefficients. If the coefficients of $h(a,x)$ are all finite, then $\sigma_h(f)$ may be regarded as a formal power series over the ring $R[[a]]$.

(ii) To prove equality $f = g$ between normal functors in $(\mathbf{Set}^A, \mathbf{Set}^B)_{\mathrm{NF}}$, it is sufficient and necessary to prove equalities $\tilde{f}[\gamma, b] = \tilde{g}[\gamma, b]$ between coefficients for every (γ, b) in $\exp A \times B$. In particular, if we know, in some way, that all coefficients are finite, we may regard $f(x)$ and $g(x)$ to be families of formal power series in $\mathbb{Z}[[x]]$, and we may apply an analytic or algebraic method to the formal power series ring (e.g, division which exists in certain cases) to obtain the equality as formal power series, thus obtaining the equalities between coefficients.

3.11 Lemma
Let $h, h' : \mathbf{Set}^{A+X} \to \mathbf{Set}^B$ and $\mathbf{Set}^{A+X} \to \mathbf{Set}^X$ be normal functors.

The equality $\sigma_{h\times h'}(f) \times \sigma_1(f) = \sigma_h(f) \times \sigma_{h'}(f)$ holds, where $h \times h'$ and 1 are normal functors from $\mathbf{Set}^{A+X}$ to $\mathbf{Set}^B$, the former carrying the argument (a, x) to the product $h(a, x) \times h'(a, x)$ and the latter to the terminal presheaf 1 irrelevant of the argument.

(*Proof*) Let z be a sequence of fresh variables of the same length as x. We verify $\sigma_{h\times h'}(zf) \times \sigma_1(zf) = \sigma_h(zf) \times \sigma_{h'}(zf)$ as formal power series in a and z. We note that $\sigma_h(zf)$ is linear in h. Hence it suffices to prove the case where h is a monomial x^β. The coefficient of z^γ in $\sigma_{x^\beta}(zf) \times \sigma_{h'}(zf)$ equals

$$\sum ([x^\delta]x^\beta f(a,x)^\delta)([x^{\delta'}]h'(a,x)f(a,x)^{\delta'})$$

where the summation is over all pairs (δ, δ') satisfying $\delta + \delta' = \gamma$. Furthermore, we can assume $\beta \subseteq \delta$ as multisets. Then

$$\begin{aligned} [x^\delta]x^\beta f(a,x)^\delta &= [x^{\delta-\beta}]f(a,x)^\delta \\ [x^{\delta'}]h'(a,x)f(a,x)^{\delta'} &= [x^{\delta'+\beta}]x^\beta h'(a,x)f(a,x)^{\delta'}. \end{aligned}$$

Therefore the coefficient of z^γ in $\sigma_h(zf) \times \sigma_{h'}(zf)$ equals the coefficient of z^γ in $\sigma_{x^\beta\times h'}(zf) \times \sigma_1(zf)$. Finally, if we put $z = 1$, the lemma is derived. □

As mentioned above, the diagonal sum $\sigma_h(f)$ does not satisfy the axioms of traced monoidal categories. So we "normalize" it as in the following definition. We verify that, with this $\tau_h(f)$, the category $\mathbf{CAAcc}_{\mathbf{NF}}$ turns out to be a traced cartesian category.

3.12 Definition (of $\tau_h(f)$)
Let $\mathbf{Set}^{A+X} \xrightarrow{h} \mathbf{Set}^B$ and $\mathbf{Set}^{A+X} \xrightarrow{f} \mathbf{Set}^X$ be normal functors.

The family $\tau_h(f)(a)$ of formal power series is defined by $\sigma_h(f)(a)/\sigma_1(f)(a)$ where the division notation means that the i-th series of the numerator is divided by the i-th series of the denominator.

We may regard all coefficients $\tilde{f}[(\alpha,\gamma),x]$ of $f(a,x)$ as indeterminates, and we let R be the ring of all polynomials over integers in these indeterminates. Then the normal functor $f(a,x)$ may be regarded to lie in the ring $R[[a,x]]$ of the formal power series. If g is of the form $z\cdot f(a,x)$ with $z\in\mathbf{Set}^X$, the denominator

$$\sigma_1(g)(a,z) \;=\; \sum_{\gamma\in\exp X} z^\gamma[x^\gamma]f(a,x)^\gamma$$

of $\tau_h(g)(a,z)$ is of the shape $1+P(a,z)$ where $P(a,z)$ is a formal power series in the ring $R[[a,z]]$ with no constant term. Noticing the formal power series of this form is invertible for multiplication, the expression $\tau_h(g)(a,z)$ makes sense as an element of the ring $R[[a,z]]$ (supposed the coefficients of $h(a,x)$ are finite). We postpone the verification that $\tau_h(g)(a)$ is meaningful for all g. This will be proved by observing that only the polynomials of non-negative coefficients in R are involved. For the moment, we deal with only the case where g is of the form $zf(a,x)$.

The next lemma shows that all $\tau_h(f)$ is computed if we know the case where h is the projection from $\mathbf{Set}^{A+X}$ onto $\mathbf{Set}^A$. So we introduce the following notation:

Notation: We write $\tau x.\,f(a,x)=\tau_h(f)(a,x)$ in case that $\mathbf{Set}^{A+X}\xrightarrow{h}\mathbf{Set}^A$ is the projection. This operator τ binds the variable succeeding, so the α-convertible expressions are identified.

3.13 Lemma
Let $\mathbf{Set}^{A+X}\xrightarrow{h}\mathbf{Set}^B$ and $\mathbf{Set}^{A+X}\xrightarrow{f}\mathbf{Set}^X$ be normal functors.

The equality $\tau_h(f)(a,x)=h(a,\ \tau x.\,f(a,x))$ holds.

(*Proof*) Since $\tau_h(f)$ is linear in h, it suffices to verify the case where h is a monomial $x^\beta=x_{i_1}x_{i_2}\cdots x_{i_n}$. By the previous lemma, $\sigma_{x^\beta}(f)\times\sigma_1(f)^{n-1}=\sigma_{x_{i_1}}(f)\times\sigma_{x_{i_2}}(f)\times\cdots\times\sigma_{x_{i_n}}(f)$. Hence, we have $\tau_{x^\beta}(f)=\sigma_{x_{i_1}}(f)\times\sigma_{x_{i_2}}(f)\times\cdots\times\sigma_{x_{i_n}}(f)/\sigma_1(f)^n=\tau_{x_{i_1}}(f)\times\tau_{x_{i_2}}(f)\times\cdots\times\tau_{x_{i_n}}(f)$, that is, $\tau(f)^\beta$. $\square$

We start the proof that $\tau_h(f)$ satisfies the axioms of traced monoidal categories. The axiom of traced monoidal category translates into the following equations: First of all, tightening and superposing are direct consequences of Lemma 3.13. The rest turns out to be

(vanishing) $\tau(x,y).\,(f(x,y),F(x,y))$
$= (\tau x.\,f'(x),\ \tau y.\,F(\tau x.\,f'(x),\,y))$
where $f'(x)\triangleq f(x,\ \tau y.\,F(x,y))$

(sliding) $g(\tau y.\,f(g(y))) \;=\; \tau x.\,g(f(x))$

(yanking) $\tau y.\,x \;=\; x$.

In the original axiom of sliding, f may depend on the parameter from A whereas g may not. However, the extended sliding rule where also g may depend on A is derived from the axioms of the traced cartesian category. Hence the fixpoint equation

$$\tau x.\, g(x) \;=\; g(\tau x.\, g(x))$$

is an immediate consequence of the extended sliding rule by setting f to be an identity, irrelevant of whether g depends on parameters.

We verify the equality of vanishing. Let us put $f'(x) \triangleq f(x,\ \tau y.\, F(x,y))$ and $h'(x) \triangleq h(x,\ \tau y.\, F(x,y))$. We verify the following equation:

$$\sigma_h(f,F) \;=\; h'(\tau x.\, f'(x)) \times \sigma_1(F)(\tau x.\, f'(x)) \times \sigma_1(f').$$

The left hand side is by definition equal to $\sum_\gamma [x^\gamma] \sum_\delta [y^\delta] h f^\gamma F^\delta$ where γ ranges over $\exp X$ and δ over $\exp Y$. The sum $\sum [y^\delta] h f^\gamma F^\delta$ is equal to the definition of $\sigma_{hf^\gamma}(F)(x)$, which turns out to equal the multiplication $\tau_{hf^\gamma}(F)(x) \times \sigma_1(F)(x)$. Since Lemma 3.13 yields

$$\tau_{hf^\gamma}(F)(x) \;=\; h(x, \tau y.\, F(x,y)) \times f(x, \tau y.\, F(x,y))^\gamma,$$

$\sigma_h(f,F)$ above equals $\sum [x^\gamma] h'(x) f'(x)^\gamma \sigma_1(F)(x)$. The last is by definition equal to $\sigma_{h'\cdot\sigma_1(F)}(f')$. We write this $\tau_{h'\cdot\sigma_1(F)}(f') \cdot \sigma_1(f')$. A further application of Lemma 3.13 implies that the left factor equals

$$h'(\tau x.\, f'(x)) \;\times\; \sigma_1(F)(\tau x.\, f'(x)).$$

Hence the claimed equation follows. Now we consider $\tau_h(f,F)$, the definition of which is $\sigma_h(f,F)/\sigma_1(f,F)$. We recall that we deal with only the case where the involving formal power series are members of a ring, which is an integral domain. Hence the cancellation of non-zero factors holds, yielding $\tau_h(f,F) = h'(\tau x.\, f'(x))$, that is,

$$h(\,\tau x.\, f'(x),\ \tau y.\, F(\,\tau x.\, f'(x),\ y)).$$

Taking projections as h, the due equation of vanishing is derived. The idea of this proof comes from [12].

Next we verify the equality of sliding. First we recall the following. Each normal functor $\mathbf{Set}^X \xrightarrow{f} \mathbf{Set}^Y$ is regarded as a linear map $!X \multimap Y$. The promotion $pf : !X \multimap !Y$ corresponds to the matrix M_{pf} in $\mathbf{Set}^{\exp X \times \exp Y}$ given by $M_{pf}[\gamma, \delta] = [x^\gamma] f(x)^\delta$. If, in addition, we have $h : !X \multimap B$, the linear map $(h \otimes pf) : !X \multimap (B \otimes !Y)$ corresponds to the matrix $M_{h\otimes pf}$ in $\mathbf{Set}^{\exp X \times (B \times \exp Y)}$ defined by $M_{h\otimes pf}[\gamma, (b, \delta)] = [x^\gamma] h_b(x) f(x)^\delta$ where b is a member of B.

It suffices to consider h of type $!X \multimap I$, i.e., a normal functor from $\mathbf{Set}^X$ to $\mathbf{Set}$. We compute $\sigma_h(g \circ f) = \sum_\gamma M_{h \otimes p(g \circ f)}[\gamma, \gamma]$ where the comultiplication on $!X$ is omitted after $h \otimes p(g \circ f)$. Since the term in the sum equals $[x^\gamma]h(x)g(f(x))^\gamma$ and $g(f(x))^\gamma = \sum_\delta f(x)^\delta [y^\delta] g(y)^\gamma$, we have

$$\sigma_h(g \circ f) = \sum_{\gamma,\delta} [x^\gamma y^\delta] h(x) f(y)^\delta g(x)^\gamma.$$

(We remark the last is equal to $\sigma_h(g, f)$ where the endofunctor (g, f) on $\mathbf{Set}^{X+Y}$ is defined by $(x, y) \mapsto (g(y), f(x))$.) On the other hand, we want to compute $\sigma_{h\circ g}(f \circ g) = \sum_\delta M_{(h \circ pg)\otimes p(f\circ g)}[\delta, \delta]$ where the comultiplication is omitted. Since the linear map $(h \circ pg) \otimes p(f \circ g)$ from $!Y$ to $!X$ is equal to the composite $(h \otimes pf) \circ pg$ (again comultiplication is omitted), the term in the sum equals $\sum_\gamma M_{h\otimes pf}[\gamma, \delta] M_{pg}[\delta, \gamma]$. It is easy to see that also this is equal to $\sigma_h(g, f)$. Therefore we have $\sigma_h(g \circ f) = \sigma_{h\circ g}(f \circ g)$. Since h is arbitrary, we have also $\sigma_1(g \circ f) = \sigma_1(f \circ g)$. Hence the equality $\tau_h(g \circ f) = \tau_{h \circ g}(f \circ g)$ holds. If we take projections as h, we can infer $\tau x.\, g(f(x)) = g(\tau y.\, f(g(y)))$. This verifies the equality of sliding.

It is easy to show the soundness of yanking. For the first and second projection $\pi_1, \pi_2 : \mathbf{Set}^{X+X} \rightrightarrows \mathbf{Set}^X$, we have $\tau y.\, x = \tau_{\pi_2}(\pi_1)(x) = x$. Hence the trace of the symmetry equals an identity.

Summarizing the observations above, we obtain the following theorem.

3.14 Theorem
The category $\mathbf{CAAcc}_{\mathrm{NF}}$ *is a traced cartesian category where the trace is given by* $\tau_h(f)$.

(*Proof*) We have proved above that $\tau_h(f)$ satisfies the axiom of trace if all coefficients of f are indeterminates and f has the shape $z \cdot f(x)$ with fresh indeterminates z. As proven in Thm. 3.15 below, this case implies that $\tau_h(f)$ is a normal functor, namely, that all involved coefficients are non-negative. Hence the proof above gives even a valid verification of equalities between normal functors. Finally, if we set $z = 1$ and substitute arbitrary coefficients for indeterminates of f, we can conclude that the axioms of trace are satisfied for all normal functors.

Lagrange-Good Inversion

By Thm. 3.14, we can induce the fixpoint operator from trace. The following theorem asserts that the induced fixpoint operator is actually the least fixpoint operator.

3.15 Theorem
Let $f : \mathbf{Set}^{A+X} \to \mathbf{Set}^X$ be a normal functor.

$\tau x.\, f(a,x)$ coincides with the initial algebra $\mu x.\, f(a,x)$. In particular, $\tau x.\, f(a, x)$ is a normal functor.

(*Proof*) The argument of the preceding subsection implies that, if we regard all coefficients of f as indeterminates and we consider $z \cdot f(a,x)$ with new indeterminates $z \in \mathbf{Set}^X$, then $\tau x.\, zf(a,x)$ makes sense as formal power series and gives a fixpoint $x = z \cdot f(a,x)$.

We prove that the fixpoint $x = z \cdot f(a,x)$ is unique. Let us define $g^0(a,z) = 0$ and $g^{n+1}(a,z) = z \cdot f(a, g^n(a,z))$. We claim that, if $x = h(a,z)$ is a fixpoint $x = z \cdot f(a,x)$, then the equation $[z^\gamma]h(a,z) = [z^\gamma]g^n(a,z)$ between coefficients holds for all $\gamma \in \exp X$ satisfying $\operatorname{card} \Sigma\gamma \leq n$. For $n = 0$, the constant term of h must be 0 by the equation $h = zf(a,h)$. For induction step, if the claim holds for n, then $[z^\gamma]f(a, h(a,z)) = [z^\gamma]f(a, g^n(a,z))$ also holds for every γ such that $\operatorname{card} \Sigma\gamma \leq n$, since the coefficient of z^γ depends only on those terms of powers of γ' such that $\operatorname{card} \Sigma\gamma' \leq n$ in h and g^n. Multiplying by z and noticing $h = zf(a,z)$, we have $[z^\delta]h(a,z) = [z^\delta]g^{n+1}(a,z)$ for all δ of cardinality $n+1$ or less. So the claim is verified. Hence $h(a,z)$ must equal $\varinjlim g^n(a,z)$, that is, $\mu x.\, zf(a,x)$, showing that the fixpoint is unique.

By uniqueness of fixpoints, we must have $\tau x.\, zf(a,x) = \mu x.\, zf(a,x)$. In particular, $\tau x.\, zf(a,x)$ is a normal functor, since so is the initial algebra. Thus we can put $z = 1$, concluding $\tau x.\, f(a,x) = \mu x.\, f(a,x)$. $\square$

We can write down the fixpoint by the formal power series as follows. For each normal functor $\mathbf{Set}^X \xrightarrow{h} \mathbf{Set}$, we have the formula

$$h(f^\dagger(a)) \;=\; \frac{\sum_\gamma [x^\gamma]h(x) f(a,x)^\gamma}{\sum_\gamma [x^\gamma] f(a,x)^\gamma}$$

where a and x are parameters from $\mathbf{Set}^A$ and $\mathbf{Set}^X$.

As a special case of the formula above yielding fixpoints, we consider the solution x of the equation $x = z \cdot g(x)$ where x and z are vectors of disjoint variables of the same length and $g(x)$ is a formal power series. Namely we want to find the fixpoint $f^\dagger(z)$ if we put $f(z,x) = z \cdot g(x)$. This special case is particularly interesting, since it can be used to find the inverse of formal power series. Let us consider a system of formal power series $z = k(x)$. If we put $f(z,x) = zx/k(x)$ (that is, $g(x) = x/k(x)$), then the fixpoint $f^\dagger(z)$ gives the compositional inverse of $z = k(x)$, provided that it exists.

The Lagrange-Good inversion is the formula to find the solution of this special case $x = zg(x)$. By substituting $zg(x)$ for $f(a,x)$ in the general formula above for fixpoints, we have the following form

$$h(f^\dagger(z)) \;=\; \frac{\sum_\gamma z^\gamma [x^\gamma] h(x) g(x)^\gamma}{\sum_\gamma z^\gamma [x^\gamma] g(x)^\gamma}.$$

Although this formula is different from the standard Lagrange-Good inversion formula occurring in the literature, we can verify that this is equal to the standard one, applying Jacobi's residue formula. We recall Jacobi's residue formula. Let $F_1, F_2, \ldots, F_n$ be formal Laurent series in n variables of the shape $F_i(x_1, x_2, \ldots, x_n) = a_i {x_1}^{b_{i1}} {x_2}^{b_{i2}} \cdots {x_n}^{b_{in}} +$ (higher degree terms). Then, for an arbitrary Laurent series $h(x_1, x_2, \ldots, x_n)$, the formula

$$\det(b_{ij}) \operatorname{Res} h(x) \;=\; \operatorname{Res}\left(h(F(x)) \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)} \right)$$

holds, where the residue $\operatorname{Res}(f(x_1, x_2, \ldots, x_n))$ is defined as the coefficient of $(x_1 x_2 \cdots x_n)^{-1}$ in Laurent series f. Following [21], we show $\sum_\gamma z^\gamma [x^\gamma] g(x)^\gamma = \det(E - M(z, f^\dagger(z)))^{-1}$ where $M(z,x)$ is the square matrix $(z_i(\partial g_i(x)/x_j))$, if we can restrict the involving formal power series to have finite coefficients. First we take the partial derivatives of $f^\dagger(z) = zg(f^\dagger(z))$, obtaining

$$\frac{\partial f_i^\dagger(z)}{\partial z_k} \;=\; \delta_{ik} g_i(f^\dagger(z)) + z_i \sum_j \frac{\partial g_i}{\partial x_j}(f^\dagger(z)) \frac{\partial f_j^\dagger(z)}{\partial z_k}$$

Namely we have

$$\sum_j \frac{\partial f_i^\dagger(z)}{\partial z_k} \left(\delta_{jk} - z_i \frac{\partial g_i}{\partial x_j}(f^\dagger(z)) \right) \;=\; \delta_{ik} g_i(f^\dagger(z)).$$

Since the left hand side is the multiplication of matrices, taking the determinants of both sides, $\det(\partial f_i^\dagger(z)/\partial z_k) \det(E - M(z, f^\dagger(z))) = g(f^\dagger(z))^{\vec{1}}$ is derived, where $z^{\vec{1}} = z_1 z_2 \cdots z_n$ for $z = z_1, z_2, \ldots, z_n$. Hence $[z^\gamma] \det(E - M(z, f^\dagger(z)))^{-1} = [z^\gamma] \det(\partial f_i^\dagger(z)/\partial z_j)/g(f^\dagger(z))^{\vec{1}}$. The latter equals

$$\operatorname{Res} \frac{1}{z^{\gamma+\vec{1}} g(f^\dagger(z))^{\vec{1}}} \det\left(\frac{\partial f_i^\dagger(z)}{\partial z_j} \right) = \operatorname{Res} \frac{g(f^\dagger(z))^\gamma}{f^\dagger(z)^{\gamma+\vec{1}}} \det\left(\frac{\partial f_i^\dagger(z)}{\partial z_j} \right).$$

But this is equal to $\operatorname{Res} g(x)^\gamma / x^{\gamma+\vec{1}}$, that is, $[x^\gamma] g(x)^\gamma$ by Jacobi's residue formula, noticing that $f_i^\dagger(z)$ has the form $z_i +$ (higher term) as observed from the shape of the equation $f^\dagger(z) = zg(f^\dagger(z))$. So $[z^\gamma] \det(E - M(z, f^\dagger(z)))^{-1} = [x^\gamma] g(x)^\gamma$ holds as claimed. Therefore the standard Lagrange-Good inversion formula

$$\frac{h(f^\dagger(z))}{\det(E - M(z, f^\dagger(z)))} = \sum_\gamma z^\gamma [x^\gamma] h(x) g(x)^\gamma.$$

is derived.

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