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# Categorical Data Types in Parametric Polymorphism

*Dedicated to Prof. Satoru Takasu on the occasion of his sixtieth birthday*

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The categorical data types in models of second order lambda calculus are studied. We prove that Reynolds parametricity is a sufficient and necessary condition for the categorical data types to fulfill the universal properties.

## 1. Introduction

The second order (polymorphic) lambda calculus has two origins. The first is in logic. Girard introduced the system $F$ to solve the consistency problem of analysis [Girard 1972, Girard 1971]. System $F$ may be regarded as the proof system of intuitionistic second order logic through the Curry-Howard isomorphism, and the second order lambda calculus we mention here is the version corresponding to intuitionistic second order propositional logic.

Prawitz showed [Prawitz 1965] that in intuitionistic second order logic the connectives implication and universal quantification are sufficient to represent all other connectives as, for example,

$$
\begin{aligned}
\perp &= \forall X.X \\
A \wedge B &= \forall X((A \Rightarrow B \Rightarrow X) \Rightarrow X) \\
A \vee B &= \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X) \\
\exists X.F(X) &= \forall Y(\forall X(F(X) \Rightarrow Y) \Rightarrow Y)
\end{aligned}
$$

(the representation of $A \vee B$ goes back to Russell). Here one natural question arises. What do these representations mean when seen from the second order lambda calculus through Curry-Howard isomorphism? In particular, we would like to know whether they mean anything natural in the models of the calculus. One might naturally conjecture, for example, $A \vee B$ would represent a coproduct of $A$ and $B$ in some appropriate categorical framework. This is false, however, since only the weak universal conditions [Mac Lane 1971] are fulfilled in general: $A \vee B$ represents only a weak coproduct. Our main objective is to look for a necessary and sufficient condition that makes $A \vee B$ a coproduct, $A \wedge B$ a product, and so on.

The second origin of the second order lambda calculus is in computer science. Let $F$ be a map of the collection of types into itself. A family of terms $\{M_X : F(X)\}_{X \in \mathit{Types}}$, where *Types* means the collection of types, is called a *polymorphic collection*. The simplest and most often used example is the family of identities $\{\lambda x^X x : X \Rightarrow X\}_{X \in \mathit{Types}}$. Another example is $\{\lambda x^X J_{AX}(x) : X \Rightarrow A\} = X \in \mathit{Types}$, due to Girard [Girard 1971]. See the introductory part of Section 3 for the definition of $J_A X$.

Strachey proposed a distinction between parametric polymorphism and ad hoc polymorphism. Intuitively a polymorphic collection is said to be *parametric* if all its members have a uniform definition, independent of the indexing type $X$. Otherwise it is called *ad hoc*. The first example above (the family of identities) is parametric, since every identity has the uniform derivation

$$\frac{x : X \vdash x : X}{\vdash \lambda x^X x : X \Rightarrow X\,.}$$

On the contrary, the second example is ad hoc since it is defined by case analysis of the type. Reynolds developed the polymorphic lambda calculus for exploring the parametric polymorphism in harmony with the simply typed lambda calculus, and later it was observed that Reynolds' system is essentially equivalent to Girard's $F$. The algorithmic property of parametric polymorphism differs from that of ad hoc polymorphism. As proved by Girard, the reduction of any term in the second order lambda calculus halts into a unique normal form, while as soon as a polymorphic collection $J = J_{XY}$ is added, the system has the term which allows an infinite reduction sequence (see Section 3).

Here again one natural question arises. What does parametricity mean in models? Reynolds introduces a notion of semantical parametricity in pursuing a full set-theoretical model [Reynolds 1983]. We can regard a subset of the product of two sets $A$ and $B$ as a binary relation between $A$ and $B$. After extending the operations $\Rightarrow$ and $\forall$ on types to those on the class of relations, we say $\{M_A : F(A) \mid A \in \mathit{Type}\}$ is parametric if it satisfies $(M_A, M_B) \in F(r)$ for any $A$, $B$, and relation $r \subseteq A \times B$. Reynolds hoped parametricity could be used to avoid the cardinality conflict induced by the fullness of the (candidate of) model. Though his attempt has been rejected by himself in the end [Reynolds 1984, Reynolds and Plotkin 1990], his idea of parametricity is very useful for our purpose. We will extend the model framework of Bruce, Meyer and Mitchell [Bruce et al. 1990] by supplying relations so that parametricity is definable.

Now armed with parametricity, we go back to the first question: what is the condition which makes $A \vee B$ a coproduct, etc. Surprisingly parametricity happens to be necessary and sufficient. In a sequence of theorems, which constitute the main part of this paper, we will show that $A \vee B$ is a coproduct iff it is parametric; $\bot$ is an initial object iff it is parametric; $A \wedge B$ is a product such that $\text{-} \wedge B$ is a left adjoint of $B \Rightarrow \text{-}$ iff it is parametric, and so on.

Other categorical constructions, such as initial and terminal fixed points of endofunctors, are also discussed. Let $F : \mathbf{C} \to \mathbf{C}$ be an endofunctor and $F$-**Alg** is the category of $F$-algebra whose objects are pairs $(f, A) = F(A) \xrightarrow{f} A$ and whose arrows

$k : (f, A) \to (g, B)$ are arrows $k \in \mathbf{C}(A, B)$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(k)"] \arrow[d, "f"'] & F(B) \arrow[d, "g"] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$

commutes. An initial object of $F$-**Alg**, denoted by $(\gamma, \mu F)$, provides a fixed point $F(\mu F) \cong \mu F$ in the category $\mathbf{C}$ [Lambek 1968, Smyth and Plotkin 1982]. So we call $\mu F$ an initial fixed point of $F$.

Every "free" algebraic type without generators is representable in the second order lambda calculus [Böhm and Berarducci 1985]. For example, the type of natural numbers constructed from 0 and $s$ (successor) is represented as $N = \forall X (X \Rightarrow (X \Rightarrow X) \Rightarrow X)$. Or, if we can use the type of terminal object $\top$ and a coproduct $+$, it is also written $\forall X (((\top + X) \Rightarrow X) \Rightarrow X)$. This sort of construction generalizes to the form $\forall X ((F(X) \Rightarrow X) \Rightarrow X)$ where $F$ is an endofunctor. The case $F(X) = (X \Rightarrow B) \Rightarrow B$ where $B$ is the type of boolean values $\forall X (X \Rightarrow X \Rightarrow X)$ is used to prove the non-existence of the full set-theoretical model by Reynolds [Reynolds 1984]. Hence the representations in the style of $\forall X ((F(X) \Rightarrow X) \Rightarrow X)$ are useful, but they are, in general, only weakly initial pre-fixed points, and weakness sometimes causes some strange properties. For example, with $N = \forall X (X \Rightarrow (X \Rightarrow X) \Rightarrow X)$, there is a representation of addition $\lambda x^N \lambda y^N x + y : N \Rightarrow N \Rightarrow N$ (in fact any second order provable total function is representable [Girard 1972, Statman 1981, Girard et al. 1989]), but $\lambda x^N \lambda y^N x + y \neq \lambda x^N \lambda y^N y + x$ in general. The assertion that $\mu F$ is an initial fixed point, roughly speaking, corresponds to the induction principle [Lehmann and Smyth 1981]. And we have $\lambda x^N \lambda y^N x + y = \lambda x^N \lambda y^N y + x$ if $N$ is an initial fixed point. Again under parametricity, we will show $\forall X ((F(X) \Rightarrow X) \Rightarrow X)$ is an initial fixed point of $F$.

Dually a terminal object of the category of $F$-coalgebras $F$-**Coalg** provides a terminal fixed point of $F$. Terminal fixed points are used to define lazy types such as streams. A weak representation of terminal fixed point, $\exists X ((X \Rightarrow F(X)) \times X)$, is obtained by Wraith and the author independently [Hasegawa 1989, Wraith 1989]. With parametricity, $\exists X ((X \Rightarrow F(X)) \times X)$ is a terminal fixed point of $F$.

In Section 5, we give two examples of parametric models which admits all of these categorical constructions. One is a trivial complete Heyting algebra model, in which the interpretations of any terms with the same type are collapsed. The other is a modified per model on natural numbers. Almost the same model is given in [Bainbridge et al. 1990] and it is proved that the model has initial fixed points of syntactically defined endofunctors. In our case, all internal functors which lie in the effective topos have initial and terminal fixed points.

## 2. Syntax of Second Order Lambda Calculus

We briefly review the syntax of *second order (polymorphic) lambda calculus*. See for instance [Fortune et al. 1983, Breazu-Tannen and Coquand 1988, Girard et al. 1989] for more information. The calculus is written $\lambda^{\forall}(\Sigma)$ where $\Sigma$ is a *polymorphic signature* $(\mathcal{K}, \mathcal{C})$ whose meaning and conditions are given below.

Let us first define *types*. An element of $\mathcal{K}$ is a *type constructor* denoted by $F, G, \ldots$, which has a fixed arity. In particular, a type constructor with the arity 0 is called a *type constant*. A countably infinite set of type variables is provided, and those type variables are denoted by $X, Y, \ldots$. We use $\Gamma, \Delta, \ldots$ for finite sets of mutually distinct type variables, though they are written in the style of sequences, as $\Gamma = X_1, X_2, \ldots$. Types are denoted by $\sigma, \tau$ and so on.

A *type judgement* is of the form

$$\Gamma \vdash \sigma$$

where $\Gamma$ is a list of type variables which include all free type variables in $\sigma$. The axiom and the rules of type judgement are as follows;

(Ty proj) $$\Gamma \vdash X \qquad (X \text{ in } \Gamma)$$

(Ty const) $$\frac{\Gamma \vdash \sigma_1 \quad \cdots \quad \Gamma \vdash \sigma_n}{\Gamma \vdash F(\sigma_1, \cdots, \sigma_n)}$$

(Ty $\Rightarrow$) $$\frac{\Gamma \vdash \sigma \qquad \Gamma \vdash \tau}{\Gamma \vdash \sigma \Rightarrow \tau}$$

(Ty $\forall$) $$\frac{\Gamma, X \vdash \sigma}{\Gamma \vdash \forall X.\sigma}$$

We do not distinguish between two types which differ only in bound type variables. It may appear that there is no reason to restrict the arguments of $F$ to type variables or type constants, but the restriction will turn out to be important in semantical considerations.

Next *terms* are defined. Let $\mathcal{C}$ be a given collection of *individual constants* denoted by $a$, $b$, $c$, .... To each element $a$ of $\mathcal{C}$, a *closed* type Type($a$) is associated. Countably many *individual variables*, denoted by $x$, $y$, ..., are provided. Terms are denoted by $M$, $N$, ....

A *syntactic type assignment* has the form

$$\Gamma; x_1 : \sigma_1, \cdots, x_n : \sigma_n$$

where $x_1$, ..., $x_n$ are mutually distinct and $\Gamma \vdash \sigma_1$, ..., $\Gamma \vdash \sigma_n$ are all well-defined type-judgements. $x_1 : \sigma_1, \cdots, x_n : \sigma_n$ is again regarded as a finite set though written as sequences. We use $\Theta$ for an abbreviation of $x_1 : \sigma_1, \cdots, x_n : \sigma_n$.

A *term judgement* is of the form

$$\Gamma; \Theta \vdash M : \tau$$

where $\Gamma \vdash \tau$ is, and $\Gamma \vdash \sigma$ for each $x : \sigma \in \Theta$ is, well-formed, and all free individual variables in $M$ occur in $\Theta$. The axioms and rules are as follows;

(te proj) $$\Gamma;\Theta \vdash x:\sigma \qquad (x:\sigma \in \Theta)$$

(te const) $$\Gamma;\Theta \vdash a:\mathrm{Type}(a)$$

where $\Gamma$ is arbitrary, since $\mathrm{Type}(a)$ is assumed to be closed.

(te $\Rightarrow$I) $$\frac{\Gamma;\Theta, x:\sigma \vdash M:\tau}{\Gamma;\Theta \vdash \lambda x^{\sigma}.M:\sigma \Rightarrow \tau}$$

(te $\Rightarrow$E) $$\frac{\Gamma;\Theta \vdash M:\sigma \Rightarrow \tau \qquad \Gamma;\Theta \vdash N:\sigma}{\Gamma;\Theta \vdash MN:\tau}$$

(te $\forall$I) $$\frac{\Gamma, X;\Theta \vdash M:\tau}{\Gamma;\Theta \vdash \Lambda X.M:\forall X.\tau}$$

where X does not appear as free type variables in $\Theta$.

(te $\forall$E) $$\frac{\Gamma;\Theta \vdash M:\forall X.\tau}{\Gamma;\Theta \vdash M\{\sigma\}:\tau[X:=\sigma]}$$

where $\sigma$ is such that $\Gamma \vdash \sigma$ holds ($\tau[X:=\sigma]$ is meta-notation for a substitution). In logical contexts, weak systems with the range of $\sigma$ restricted are often considered. Since (te $\forall$E) is the comprehension rule, asserting the existence of sets defined by formulas, the strengths of systems are sensitive to the class where $\sigma$ ranges. In this paper, we consider two systems. The one is the system $\lambda^{\forall}(\Sigma)$ with no restriction on the range of $\sigma$, called simply a second order lambda calculus or a second order lambda calculus *with full comprehension*, to stress the full power. The other is the system where the range of $\sigma$ is constrained to type variables and type constants. We call the latter system a *basic* second order lambda calculus or a second order lambda calculus with basic comprehension, in the symbol $b\lambda^{\forall}(\Sigma)$. For the basic second order lambda calculus, it is appropriate to modify the type judgement rules in one point:

**Remark 2.1.** In the basic second order lambda calculus $b\lambda^{\forall}(\Sigma)$, the type judgement rule (Ty const) is constrained as follows:

(Ty const) $$\frac{\Gamma \vdash \sigma_1 \quad \cdots \quad \Gamma \vdash \sigma_n}{\Gamma \vdash F(\sigma_1, \cdots, \sigma_n)}$$

where each $\sigma_i$ is a *type variable* or a *type constant.* □

This restriction allows us to interpret $F$ as a kind of functor, and to obtain fixed points of $F$ in later sections

In the next section, we will give a general framework of models for the calculus with basic comprehension. For the system with full comprehension, some modifications are necessary (Remark 3.5).

As usual, the next four reductions are defined (with implicit $\alpha$ conversions, i.e. differences of bound variables);

($\beta$) $$(\lambda x^{\sigma}.M)N \rhd M[x:=N]$$

$(\eta)$ $\lambda x^\sigma.Mx \rhd M$

(Type $\beta$) $(\Lambda X.M)\{\sigma\} \rhd M[X := \sigma]$

(Type $\eta$) $\Lambda X.M\{X\} \rhd M$

Equality between terms is the least congruence (equivalence) relation including these four reductions. See [Breazu-Tannen and Coquand 1988, Pitts 1987] for complete axiomatization, in which supplementary equations are allowed to be added. In this paper, however, only the minimal equality is used in the syntactic level.

We will end this section by giving three syntactic results. Head normal form is used in 3.20.

**Definition 2.2.** (i) A term $M$ *is* a *head normal form*, if $M$ is of the form

$$M = l_1 \cdots l_m.KL_1 \cdots L_n$$

where $l_i$ is either $\lambda x^\sigma$ or $\Lambda X$ according to the type of $M$, $K$ is either an individual variable or an individual constant, and $L_i$ is either a term or a type.

(ii) A term $M$ *has* a head normal form, if $M = M'$ for some $M'$, and $M'$ is a head normal form. □

**Theorem 2.3.** (i) $\lambda^\forall(\Sigma)$ *($b\lambda^\forall(\Sigma)$) satisfies Church-Rosser property.*

(ii) *Every term in* $\lambda^\forall(\Sigma)$ *($b\lambda^\forall(\Sigma)$) is strongly normalizable.*

(iii) *Every term in* $\lambda^\forall(\Sigma)$ *($b\lambda^\forall(\Sigma)$) has a head normal form.*

*Proof.* For $\lambda^\forall(\Sigma)$, it is straightforward to obtain proofs of (i) and (ii) from the arguments given in, for example, [Girard et al. 1989, Fortune et al. 1983, Breazu-Tannen and Coquand 1988]. For $b\lambda^\forall(\Sigma)$, we must check that basic comprehension remains basic after reductions.

As for (iii), prove, by induction on the construction of a term $M$, that $M$ is not normal or is otherwise a head normal form. Then the normal form of $M$, which is ensured to exist by (i) and (ii), is a head normal form. □

## 3. Semantics

One very simple semantics for second order lambda calculus is a complete Heyting algebra (shorthand, cHa) [Fourman and Scott 1979]. A *cHa* is a complete lattice $\Omega$ such that the $\wedge, \bigvee$-distributive law

$$A \wedge (\bigvee_{i\in I} B_i) = \bigvee_{i\in I} (A \wedge B_i)$$

holds for any $A \in \Omega$ and $\{B_i \mid i \in I\} \subseteq \Omega$. A cHa $\Omega$ has exponentiation $B \Rightarrow C$ such that

$$A \wedge B \leq C \quad \text{iff} \quad A \leq B \Rightarrow C\,,$$

defined by $B \Rightarrow C = \bigvee\{X \in \Omega \mid X \wedge B \leq C\}$. Translating $\Rightarrow$ and $\forall$ of the syntax into $\Rightarrow$ and $\bigwedge$ in a cHa $\Omega$, we can define an interpretation of second order lambda calculus in an obvious way, where terms are interpreted as the order between two elements of the cHa. Therefore any terms that have the same type degenerates to the same interpretation. While this situation is not desirable, modeling in ordered sets is still useful as the guide of understanding properties of general models (categorical constructs are often the generalizations of lattice-theoretic ones; see, for example, [Lambek 1968]). In cHa's, superma and infima are redefined using $\Rightarrow$ and $\bigwedge$:

$$
\begin{aligned}
\bot &= \bigwedge_{X\in\Omega} X \\
A\wedge B &= \bigwedge\{X \mid A \le (B\Rightarrow X)\} \\
&= \bigwedge_{X\in\Omega} ((A\Rightarrow B\Rightarrow X)\Rightarrow X) \\
A\vee B &= \bigwedge\{X \mid A\le X \text{ and } B\le X\} \\
&= \bigwedge_{X\in\Omega} ((A\Rightarrow X)\Rightarrow(B\Rightarrow X)\Rightarrow X) \\
\bigvee_{i\in I} A_i &= \bigwedge\{Y \mid \forall i\in I.A_i\le Y\} \\
&= \bigwedge_{Y\in\Omega} ((\bigwedge_{i\in I}(A_i\Rightarrow Y))\Rightarrow Y)\,.
\end{aligned}
$$

It would be desirable that, even in general models, we can define some limits and colimits in the formulas similar to the above representations of superma and infima. Unfortunately it is not the case in general (see Section 4.1) and we thus want to look for a good condition for models to have such limits and colimits. The condition we provide hereinafter is not only necessary and sufficient, but also intrinsic to the nature of polymorphic lambda calculus, that is, *parametricity.*

As mentioned in Introduction, second order lambda calculus could be regarded as a language for parametric polymorphism. By Theorem 2.3 (iii), every closed term $M$ of a universal type $\forall X.F(X)$ has a head normal form $\Lambda X.N[X]$ where $[X]$ designates the occurrences of $X$ in $N$. So the derivation of term judgement for the head normal form of $M$ ends with

$$\frac{X;\vdash N[X]:F(X)}{;\vdash \Lambda X.N[X]:\forall X.F(X)\,.}$$

Therefore $M$ is a family whose components $N[X]$ have a uniform definition with parameter $X$, namely it is a parametric data.

An example of ad hoc polymorphism, which is not definable in the second order lambda calculus, is

$$J_{YX} \rhd \begin{cases} \lambda x^X x & \text{if } Y = X \\ \lambda x^X 0_Y & \text{otherwise} \end{cases}$$

due to Girard [Girard 1971] (for each type $A$, there exists a constant term $0_A$ of type $A$ in some variants of system $F$). Using $J_{YX}$, we can construct an (even weakly) non-normalizing term

$$(\Lambda X.J_{XI}(\lambda x^{\bot}x\{I\}x))\{I\}(\Lambda X.J_{XI}(\lambda x^{\bot}x\{I\}x))$$

where $\bot = \forall X.X$ and $I = \bot\Rightarrow\bot$. Note that, erasing the information of types and putting $J_{XI} = \lambda x.x$, we have a non-normalizing untyped $\lambda$-term which is $\beta$-equal to $(\lambda x.xx)(\lambda x.xx)$. When we mention to "parametric" models, we expect they reject at least the interpretation of the family of $J_{YX}$.

In the rest of this section, we propose a generalization of Reynolds' parametricity [Reynolds 1983] to the model framework of Bruce-Meyer-Mitchell. This new interpretation includes a class of binary relations to measure how the members of polymorphic data are related each other. In [Wadler 1989], Wadler proposed a similar generalization. He defines models to be the same as Bruce-Meyer-Mitchell's. After that, environments are extended to binary relations in order to take parametricity into consideration. Only the differences between his approach and ours are on the point that his model has relations outside but ours implicate them. In most cases both approaches induce the same effects, as straightforward applications of Reynolds' idea to BMM models. One advantage of our model is, however, that Girard's coherence space model turns out to be a model only when implicating relations (see below). It is verified in a later section (Section 4) that Reynolds' parametricity is a necessary and sufficient condition for the existence of limits and colimits mentioned above.

### *3.1. Parametric Semantics*

Recall the definition of categories. A category $\mathbf{C}$ consists of a class of objects and a class of arrows with an domain map $d_0$ and a codomain map $d_1$ from the arrows to the objects and an identity map $id$ from the objects to the arrows with $d_0 \circ id = d_1 \circ id = 1$. Moreover, for any composable pair of arrows, i.e., pair $(f, g)$ where $d_0 \circ f = d_1 \circ g$, a composition $m(f, g) = f \circ g$ equipped with appropriate equations.

Removing composition from categories, we get to the the next definition. A *relation frame* $\mathbf{T}$, or *r-frame* for short, is $(\mathcal{U}, \mathcal{R})$ where $\mathcal{U}$ and $\mathcal{R}$ are some classes, with a domain map and a codomain map $d_0, d_1 : \mathcal{R} \rightrightarrows \mathcal{U}$ and an identity map $id : \mathcal{U} \to \mathcal{R}$ such that $d_0 \circ id = d_1 \circ id = 1_{\mathcal{U}}$. We write $r : A \multimap B$ for $d_0(r) = A$ and $d_1(r) = B$. An element of $\mathcal{U}$ is called an object, and an element of $\mathcal{R}$ is called a (*binary*) *relation*, which is used in an abstract sense, i.e., does not mean a true relation in $\mathbf{Set}$. We use $\mathrm{Obj}(\mathbf{T})$ and $\mathrm{Rel}(\mathbf{T})$ to refer to $\mathcal{U}$ and $\mathcal{R}$ respectively.

Let $\mathbf{T} = (\mathcal{U}, \mathcal{R})$ and $\mathbf{T}' = (\mathcal{U}', \mathcal{R}')$ be r-frames. An *r-frame morphism* $F : \mathbf{T} \to \mathbf{T}'$ is a pair of maps $F : \mathcal{U} \to \mathcal{U}'$ and $F : \mathcal{R} \to \mathcal{R}'$ (we denote both by the same $F$, following the convention of category theory) such that, for every $r : A \multimap B$ in $\mathcal{R}$, we have $F(r) : F(A) \multimap F(B)$. Similarly an $n$-ary r-frame morphism ($n \geq 2$) is defined. Note $F$ does not necessarily preserve identities. It is said that $\mathbf{T}$ is an *r-subframe* of $\mathbf{T}'$ if $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{R} \subseteq \mathcal{R}'$.

Suppose $F$ and $G$ are r-frame morphisms from $\mathbf{T}$ to $\mathbf{T}'$, and $q$ is a map from $\mathrm{Rel}(\mathbf{T})$ to $\mathrm{Rel}(\mathbf{T}')$. Say $q$ is a *relation of r-frame morphisms* $F$ and $G$, and write $q : F \multimap G$, if $q(r) : F(A) \multimap G(B)$ for all $r : A \multimap B$ in $\mathcal{R}$. The identity relation of an r-frame morphism $F$ is $F$ itself. Namely $id_F = F : F \multimap F$. Thence we obtain an r-frame $\mathbf{R}(\mathbf{T}, \mathbf{T}')$ where $\mathrm{Obj}(\mathbf{R}(\mathbf{T}, \mathbf{T}'))$ is the collection of all r-frame morphisms from $\mathbf{T}$ to $\mathbf{T}'$, and $\mathrm{Rel}(\mathbf{R}(\mathbf{T}, \mathbf{T}'))$ is the collection of all relations of r-frame morphisms. Similarly we can define $\mathbf{R}(\mathbf{T}^n, \mathbf{T}')$.

A simply typed lambda calculus has an intuitively clear interpretation defining $[\![\sigma \Rightarrow \tau]\!]$ as the set of all set-theoretic functions on $[\![\sigma]\!]$ to $[\![\tau]\!]$. In contrast, it was proved that a second order lambda calculus has no full set-theoretic model, by Reynolds [Reynolds 1984, Reynolds and Plotkin 1990], and therefore we should be careful and eager to capture the models. Until now, two frameworks of models of second order lambda calculi are proposed; one is Bruce-Meyer-Mitchell models [Bruce et al. 1990], and the other is Seely's PL categories [Seely 1987]. The former retains the flavors of set-theoretic structures, carefully avoiding so many restrictions imposed as to fall into Reynolds paradox, and the latter is a purely categorical framework.

Our results are concerned with categorical properties, so PL category was used in the preceding paper [Hasegawa 1989]. However extensionality (well-pointedness) makes the condition simple. Indeed the one proposed in [Hasegawa 1989] is regarded as the parametricity condition imposed on the extensionalization of a PL category. Furthermore the parametric models we know until now are all extensional. Therefore we adopt Bruce-Meyer-Mitchell models as a foundational framework, and introduce their parametric extensions. As in [Breazu-Tannen and Coquand 1988], first we define interpretations, which do not respect reductions, and later define models by adding more conditions, which retrieve the soundness for reductions. The reason we are interested in the notion of interpretation is that the collection of closed types and closed terms is not a model but merely an interpretation.

**Remark 3.1.** The following definition is only applicable to a basic system $b\lambda^{\forall}(\Sigma)$, and some modification is necessary to make it fit with a system with full comprehension (see Remark 3.5). $\square$

Let $\mathbf{T} = (\mathcal{U}, \mathcal{R})$ be given r-frame and $\mathbf{T}_0 = (\mathcal{U}, \mathcal{R}_0)$ an r-subframe of $\mathbf{T}$. Suppose also given r-frames $[\mathbf{T}_0^n \to \mathbf{T}]$ for each $n \geq 0$. Each $[\mathbf{T}_0^n \to \mathbf{T}]$ is an r-subframe of $\mathbf{R}(\mathbf{T}_0^n, \mathbf{T})$. In case $n = 0$, $\mathbf{T}_0^0$ is regarded to be an r-frame $\mathbf{1}$ which consists of one object $\mathbf{1}$ and one relation $id_1 : 1 \twoheadrightarrow 1$. Namely an object of $[\mathbf{T}_0^0 \to \mathbf{T}]$ is a pair of an object $A \in \mathcal{U}$ and a relation $r : A \twoheadrightarrow A$. We force some conditions on $[\mathbf{T}_0^n \to \mathbf{T}]$ $(n \geq 0)$ to interpret all types. The reader who is interested in details is referred to Appendix 6.

An interpretation $I$ of type constructors is supposed to be given, so that

$$I(F) \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$$

where $n$ is the arity of $F$. And two relation maps

$$\Rightarrow : \mathbf{T}^2 \to \mathbf{T}$$

$$\forall : [\mathbf{T}_0 \to \mathbf{T}] \to \mathbf{T}$$

should be also provided. From $\forall : [\mathbf{T}_0 \to \mathbf{T}] \to \mathbf{T}$, we obtain an r-frame morphism $\tilde{\forall} : [\mathbf{T}_0 \to \mathbf{T}] \to [\mathbf{T}_0^0 \to \mathbf{T}]$: $\tilde{\forall} F$ is a pair $(\forall F, \forall id_F : F \twoheadrightarrow F)$ and $\tilde{\forall}(q : F \twoheadrightarrow G)$ is defined to be $\forall q$. As abbreviations, we write $F \Rightarrow G$ for $\Rightarrow \circ (F, G)$, and $\tilde{\forall} F$ for $\lambda X_1 \cdots X_n . \tilde{\forall}(F(X_1, \ldots, X_n, \text{-}))$ for $F \in \mathrm{Obj}\,[\mathbf{T}_0^{n+1} \to \mathbf{T}]$.

$[\![\Gamma \vdash \sigma]\!]$ is inductively defined as an object of $[\mathbf{T}_0^{|\Gamma|} \to \mathbf{T}]$ where $|\Gamma|$ means the number of type variables in $\Gamma$.

(Ty proj) $$[\![\Gamma \vdash X]\!] = P_i^{|\Gamma|}$$

when $X$ is the $i$-th element of $\Gamma$.

(Ty const) $$[\![\Gamma \vdash F(\sigma_1, \ldots, \sigma_n)]\!] = \lambda X_1 \cdots X_{|\Gamma|}.I(F)([\![\Gamma \vdash \sigma_1]\!](\underline{X}) \\ , \ldots, [\![\Gamma \vdash \sigma_n]\!](\underline{X}))$$

where $X_i$ in the RHS stands for both a member of $\mathcal{U}$ and of $\mathcal{R}_0$. Note that $[\![\Gamma \vdash \sigma_i]\!](\underline{r}) \in \mathcal{R}_0$ for $\underline{r} \in \mathcal{R}_0^n$ since $\sigma_i$ is constrained to a type variable or a type constant. Indeed, in the former case, $[\![\Gamma \vdash X_i]\!](\underline{r}) = r_i$, and in the latter case, $[\![\Gamma \vdash A]\!](\underline{r}) = id_{I(A)}$.

(Ty $\Rightarrow$) $$[\![\Gamma \vdash \sigma \Rightarrow \tau]\!] = [\![\Gamma \vdash \sigma]\!] \Rightarrow [\![\Gamma \vdash \tau]\!]$$

(Ty $\forall$) $$[\![\Gamma \vdash \forall X.\sigma]\!] = \tilde{\forall}[\![\Gamma, X \vdash \sigma]\!]$$

Precisely speaking, this definition has ambiguity, since $\Gamma$ is a set, not a sequence. From now on, we will regard $\Gamma$ as a sequence ordered as is written, and consider only those properties which are invariant under exchanges of arguments.

**Example 3.2.** We assume knowledge of [Girard 1986, Girard et al. 1989]. Let $\mathcal{U}$ be the collection of all coherence spaces and $\mathcal{R}$ the collection of all rigid embeddings. $\mathbf{T}_0 = \mathbf{T}$. $\mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$ consists of all functors preserving filtered colimits and pullbacks. We have something more to say for $\mathrm{Rel}\,[\mathbf{T}^n \to \mathbf{T}]$. Let $\tau : F \to G$ be a natural transformation where $F, G \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$. From it we can construct $|\tau| : F \rightarrowtriangle G$ as, for $n$ rigid embeddings $\underline{e} : \underline{A} \to \underline{B}$,

$$|\tau|(\underline{e}) \stackrel{def}{=} G(\underline{e}) \circ \tau_{\underline{A}} \quad (= \tau_{\underline{B}} \circ F(\underline{e}))\,.$$

The operation $|\,|$ is one to one, because, if $\tau \neq \tau'$, say $\tau_A \neq \tau'_A$, then

$$|\tau|(\underline{id_A}) = G(\underline{id_A}) \circ \tau_{\underline{A}} = \tau_{\underline{A}} \neq \tau'_{\underline{A}} = G(\underline{id_A}) \circ \tau'_{\underline{A}} = |\tau'|(\underline{id_A})\,,$$

thus, $|\tau| \neq |\tau'|$. Identifying $\tau$ with $|\tau|$, we put $\mathrm{Rel}\,[\mathbf{T}^n \to \mathbf{T}]$ as the collection of all natural transformations $\tau : F \to G$ such that for any $n$ rigid embeddings $\underline{e} : \underline{A} \to \underline{B}$,

(*) $$\begin{array}{ccc} F(\underline{A}) & \xrightarrow{\alpha_{\underline{A}}} & G(\underline{A}) \\ {\scriptstyle F(\underline{e})}\downarrow & & \downarrow{\scriptstyle G(\underline{e})} \\ F(\underline{B}) & \xrightarrow[\alpha_{\underline{B}}]{} & G(\underline{B}) \end{array}$$ is a pullback diagram.

$\Rightarrow$ is just the same as that in [Girard et al. 1989]. $\forall$ is $\Pi$ of [Girard et al. 1989] for $F \in \mathrm{Obj}\,[\mathbf{T} \to \mathbf{T}]$, and for $\tau : F \to G$ in $\mathrm{Rel}\,[\mathbf{T} \to \mathbf{T}]$, $\forall\tau : \Pi F \to \Pi G$ is a rigid embedding which satisfies $\forall\tau(A, \alpha) = (A, \tau_A(\alpha))$ for $(A, \alpha) \in \|\mathcal{T}r(F)\|$. It is straightforward to check $[\mathbf{T}^n \to \mathbf{T}]$ $(n \geq 0)$ satisfy all conditions in Appendix 6. $\square$

**Proposition 3.3.** *For $\Gamma, Y \vdash \tau$ and $\Gamma \vdash \sigma$ where $\sigma$ is either a type variable or a type constant,*

$$[\![\Gamma \vdash \tau[Y := \sigma]\,]\!](\underline{X}) = [\![\Gamma, Y \vdash \tau]\!]\,(\underline{X}, [\![\Gamma \vdash \sigma]\!]\,(\underline{X}))$$

*where $X$ stands for both $A \in \mathcal{U}$ and $r \in \mathcal{R}_0$.*

*Proof.* Straightforward induction on the construction of $\Gamma, Y \vdash \tau$. Note that since $\sigma$ is either a type variable or a type constant, $[\![\Gamma \vdash \sigma]\!](\underline{r})$ belongs to $\mathcal{R}_0$ again. □

Next let us settle domains to interpret terms. To each $A \in \mathcal{U}$ is associated a possibly empty set $D_A$ (we often write just $A$ for $D_A$). To each $r : A \twoheadrightarrow B$ in $\mathcal{R}$ is associated $D_r$ (often written $r$) a subset of $D_A \times D_B$, in such a way that identities in $\mathcal{R}$ are mapped into identity relations. In other word, these data determines an r-frame morphism preserving identities, $D : \mathbf{T} \to \mathbf{Rel}$ where $\mathbf{Rel}$ is an r-frame consisting of all sets and all set-theoretical binary relations. To $F \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$ and $q \in \mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$, an r-frame morphism and a relation of r-frame morphisms are associated in an obvious manner, by the same names or by $D_F$ and $D_q$.

Let $(A \to B)$ denote a function space in an ordinary set-theoretical sense, and $\Pi X \in \mathcal{U}.F(X)$ the collection of all $\mathcal{U}$-indexed families f such that $f(A) \in F(A)$ for all $A \in \mathcal{U}$.

We suppose that $\Phi^{\Rightarrow} = \{\Phi^{\Rightarrow}_{A,B} | A, B \in \mathcal{U}\}$ is given where

$$\Phi^{\Rightarrow}_{A,B} : D_{A \Rightarrow B} \to (D_A \to D_B)\,,$$

and $\Phi^{\forall} = \{\Phi^{\forall}_F \,|\, F \in \mathrm{Obj}\,[\mathbf{T}_0 \to \mathbf{T}]\}$ where

$$\Phi^{\forall}_F : D_{\forall F} \to \Pi X \in \mathcal{U}.D_{F(X)}\,.$$

We also suppose that an interpretation $I$ for individual constants $a \in \mathcal{C}$ is given so that $I(a) \in D_{[\![\vdash \mathbf{Type}(a)]\!]}$.

Interpretation of a term judgement $[\![\Gamma; \Theta \vdash M : \tau]\!]$ is an element of

$$\Pi \underline{X} \in \mathcal{U}^{|\Gamma|}.([\![\Gamma \vdash \mathrm{Type}\,\Theta]\!](\underline{X}) \to [\![\Gamma \vdash \tau]\!](\underline{X}))$$

where, for $\Theta = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, $[\![\Gamma \vdash \mathrm{Type}\,\Theta]\!](\underline{X}) = [\![\Gamma \vdash \sigma_1]\!](\underline{X}) \times \cdots \times [\![\Gamma \vdash \sigma_n]\!](\underline{X})$. $[\![\Gamma; \Theta \vdash M : \tau]\!]$ is given below, not as an inductive definition, but as the conditions the interpretations should satisfy. Intuitively $\lambda x^{\sigma}.M$ is interpreted as an ordinary function through $\Phi^{\Rightarrow}$, and $\Lambda X.M$ as a $\mathcal{U}$-indexed family through $\Phi^{\forall}$.

**Condition 3.4.** $[\![\Gamma; \Theta \vdash M : \tau]\!]$ should fulfill the following condition;

- (te proj)

$$[\![\Gamma; \Theta \vdash x_i : \sigma_i]\!](\underline{A})(\underline{a}) = a_i\,.$$

- (te const)

$$[\![\Gamma; \Theta \vdash c : \mathrm{Type}(c)]\!](\underline{A})(\underline{a}) = I(c)$$

  where $c \in \mathcal{C}$.
- (te $\Rightarrow$I) The interpretation of $\Gamma; \Theta \vdash \lambda x^{\sigma} M : \sigma \Rightarrow \tau$ should satisfy the next equation, for all $b \in [\![\Gamma \vdash \sigma]\!](\underline{A})$,

$$\begin{aligned}&\Phi^{\Rightarrow}_{B,C}([\![\Gamma; \Theta \vdash \lambda x^{\sigma} M : \sigma \Rightarrow \tau]\!](\underline{A})(\underline{a}))(b)\\ &\quad = [\![\Gamma; \Theta, x : \sigma \vdash M : \tau]\!](\underline{A})(\underline{a}, b)\end{aligned}$$

where $B = \llbracket \Gamma \vdash \sigma \rrbracket(\underline{A})$ and $C = \llbracket \Gamma \vdash \tau \rrbracket(\underline{A})$.

- (te $\Rightarrow$E)

$$\llbracket \Gamma; \Theta \vdash MN : \tau \rrbracket(\underline{A})(\underline{a}) = \Phi^{\Rightarrow}_{B,C}(\llbracket \Gamma; \Theta \vdash M : \sigma \Rightarrow \tau \rrbracket(\underline{A})(\underline{a})) \\ (\llbracket \Gamma; \Theta \vdash N \rrbracket(\underline{A})(\underline{a}))$$

  where $B = \llbracket \Gamma \vdash \sigma \rrbracket(\underline{A})$ and $C = \llbracket \Gamma \vdash \tau \rrbracket(\underline{A})$.

- (te $\forall$I) The interpretation of $\Gamma; \Theta \vdash \Lambda X.M : \forall X.\tau$ should satisfy the following equation, for all $B \in \mathcal{U}$,

$$\Phi^{\forall}_{F}(\llbracket \Gamma; \Theta \vdash \Lambda X.M : \forall X.\tau \rrbracket(\underline{A})(\underline{a}))(B) \\ = \llbracket \Gamma, X; \Theta \vdash M : \tau \rrbracket(\underline{A}, B)(\underline{a})$$

  where $F = \llbracket \Gamma, X \vdash \tau \rrbracket(\underline{A}, -)$. Note that since $\Theta$ does not include $X$ as free type variables, $\llbracket \Gamma, X \vdash \mathrm{Type}\,\Theta \rrbracket(\underline{A}, B) = \llbracket \Gamma \vdash \mathrm{Type}\,\Theta \rrbracket(\underline{A})$ and thus this $\underline{a}$ is an element of both of them.

- (te $\forall$E) (for basic comprehension)

$$\llbracket \Gamma; \Theta \vdash M\{Y\} : \tau[X := Y] \rrbracket(\underline{A})(\underline{a}) \\ = \Phi^{\forall}_{F}(\llbracket \Gamma; \Theta \vdash M : \forall X.\tau \rrbracket(\underline{A})(\underline{a}))(B)$$

  where $F = \llbracket \Gamma, X \vdash \tau \rrbracket(\underline{A}, -)$ and $Y$ occurring in $\Gamma$ corresponds to $B$ in $\underline{A}$. And

$$\llbracket \Gamma; \Theta \vdash M\{C\} : \tau[X := C] \rrbracket(\underline{A})(\underline{a}) \\ = \Phi^{\forall}_{F}(\llbracket \Gamma; \Theta \vdash M : \forall X.\tau \rrbracket(\underline{A})(\underline{a}))(I(C))$$

  where $C \in \mathcal{K}$ with arity 0. Proposition 3.3 ensures that the both sides of these equations are in the same domain. □

Again notice that (te $\Rightarrow$I) and (te $\forall$I) states only the conditions $\llbracket \Gamma; \Theta \vdash M : \tau \rrbracket$ should fulfill. Furthermore, to make the conditions correct statements, what follows is necessary.

1 $\llbracket \Gamma; \Theta, x : \sigma \vdash M : \tau \rrbracket(\underline{A})(\underline{a}, -) \in \mathrm{Range}(\Phi^{\Rightarrow}_{B,C})$, as a function of $D_B$, where $B = \llbracket \Gamma \vdash \sigma \rrbracket(\underline{A})$ and $C = \llbracket \Gamma \vdash \tau \rrbracket(\underline{A})$.

2 $\llbracket \Gamma, X; \Theta \vdash M : \tau \rrbracket(\underline{A}, -)(\underline{a}) \in \mathrm{Range}(\Phi^{\forall}_{F})$ where $F = \llbracket \Gamma, X \vdash \tau \rrbracket(\underline{A}, -)$, whenever $\Theta$ does not include $X$ as free type variables.

**Remark 3.5.** In order to interpret a system with full comprehension, some modifications are necessary, as follows;

- $\mathbf{T}_0 = \mathbf{T}$.
- $[\mathbf{T}_0^n \to \mathbf{T}]$ $(n \geq 0)$ are closed under compositions: if $F_1, \ldots, F_m \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$ and $G \in \mathrm{Obj}\,[\mathbf{T}^m \to \mathbf{T}]$, then $G \circ (F_1, \ldots, F_m) \in \mathrm{Obj}\,[\mathbf{T}^n \to \mathbf{T}]$.
- For each $A \in \mathcal{U}$, a pair $(A, id_A : A \multimap A)$ is a member of $\mathrm{Obj}\,[\mathbf{T}_0^0 \to \mathbf{T}]$.
- 0., 2. and 3. in Appendix 6 are then redundant.

Then (te $\forall$E) is interpreted as

$$\llbracket \Gamma; \Theta \vdash M\{\sigma\} : \tau[X := \sigma] \rrbracket(\underline{A})(\underline{a}) = \Phi^{\forall}_{F}(\llbracket \Gamma; \Theta \vdash M : \forall X.\tau \rrbracket(\underline{A})(\underline{a})) \\ (\llbracket \Gamma \vdash \sigma \rrbracket(\underline{A}))\,.$$

The main results in section 4 will be stated for the models of the system with basic comprehension, but are still applicable in most part to the models of the full comprehension system. See Remark 4.8. □

**Definition 3.6.** A *second order relational interpretation*, or an *interpretation* in short, of $b\lambda^{\forall}(\Sigma)$ (or $\lambda^{\forall}(\Sigma)$ with some modifications above) is a tuple

$$\xi = (\mathbf{T}, \mathbf{T}_0, [\mathbf{T}_0^n \to \mathbf{T}]\,(n \geq 0), \Rightarrow, \forall, D, \Phi^{\Rightarrow}, \Phi^{\forall}, I, [\![\,]\!])$$

such that the conditions stated above all hold. □

**Remark 3.7.** (i) Distinguishing objects in the signature $(\mathcal{K}, \mathcal{C})$ from their interpretation is boring. Henceforth we often suppress the use of $I$, writing $F$ for $I(F)$ and $c$ for $I(c)$.

(ii) The way we define an interpretation of terms is such as $[\![\Gamma; \Theta \vdash M : \tau]\!](\underline{A})(\underline{a}) = \cdots$, where $a \in D_{[\![\Gamma \vdash \mathbf{Type}\,\Theta]\!]}(\underline{A})$. But it is somehow misleading, since $D_{[\![\Gamma \vdash \mathbf{Type}\,\Theta]\!](\underline{A})}$ may be empty. In such cases, $[\![\Gamma; \Theta \vdash M : \tau]\!](\underline{A})$ is considered to be an empty map. This shows that, in the interpretation of $\Gamma; \Theta \vdash M : \tau$, the existence of $\Theta$ is very influential [Meyer et al. 1987]. If $[\![\Gamma \vdash \sigma]\!]$ is empty,

$$[\![\Gamma; \Theta, x : \sigma \vdash M : \tau]\!] = [\![\Gamma; \Theta, x : \sigma \vdash N : \tau]\!]$$

for all possible $M$ and $N$, thus especially for those which do not include $x$ as free individual variables. Nevertheless in general,

$$[\![\Gamma; \Theta \vdash M : \tau]\!] \neq [\![\Gamma; \Theta \vdash N : \tau]\!]\,.$$

(iii) An interpretation does not respect equality. Even if $M = N$ syntactically (i.e. in the sense of $\beta\eta$-equality),

$$[\![\Gamma; \Theta \vdash M : \tau]\!] \neq [\![\Gamma; \Theta \vdash N : \tau]\!]$$

in general. Because of this fact, an interpretation is not termed a "model". □

**Definition 3.8.** An interpretation $\xi$ is a *second order relational model*, or a *model* in short, if $\Phi^{\Rightarrow}_{A,B}$ for $A, B \in \mathcal{U}$ and $\Phi^{\forall}_{F}$ for $F \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$ are all one to one. □

Note that if $\Phi^{\Rightarrow}_{A,B}$ and $\Phi^{\forall}_{F}$ are all one to one, then in the interpretation of (te $\Rightarrow$I) and (te $\forall$I), $[\![\Gamma; \Theta \vdash \lambda x^{\sigma} M : \sigma \Rightarrow \tau]\!](\underline{A})(\underline{a})$ and $[\![\Gamma; \Theta \vdash \Lambda X.M : \forall X.\tau]\!](\underline{A})(\underline{a})$ are uniquely determined, if they exist. $[\![\,]\!]$ is hence unique in a model.

Models are worth for the name because they respect equality.

**Proposition 3.9.** *In models, for* $\Gamma; \Theta \vdash M : \tau$ *and* $\Gamma; \Theta \vdash N : \tau$*, if* $M \rhd N$ *then*

$$[\![\Gamma; \Theta \vdash M : \tau]\!] = [\![\Gamma; \Theta \vdash N : \tau]\!]\,.$$

*Proof.* For $\eta$-reduction, the proof is direct from the fact $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ are one to one. For $\beta$-reduction, first prove $[\![\Gamma; \Theta \vdash M[x{:=}L] : \tau]\!](\underline{A})(\underline{a}) = [\![\Gamma; \Theta, x : \sigma \vdash M : \tau]\!](\underline{A})(\underline{a}, [\![\Gamma; \Theta \vdash L : \sigma]\!](\underline{A})(\underline{a}))$ by induction, and the proof is immediate. □

The reader might be aware that so far $\mathcal{R}$ had no essential effects in the interpretation of, especially, terms. It plays a major role in what follows.

For each $a \in \forall F$, $\Phi^{\forall}_{F}(a)$ is an element of $\Pi X^{\mathcal{U}}.F(X)$, namely, a $\mathcal{U}$-indexed family such that $\Phi^{\forall}_{F}(a)(A) \in F(A)$ for $A \in \mathcal{U}$. In this sense, $a$ can be considered as a polymorphic data, and $\forall F$ as a polymorphic data type. Nothing further said, however, $a$ might be representing simply a family whose components are mutually irrelevant, far from parametric polymorphism. Thence we introduce the notion of parametricity.

Before that, we define a new notation. We have used the round head arrow notation $r : A \multimap B$ to represent a binary relation from $A$ to $B$. When $A$ and $B$ are sets and $r$ is a set-theoretical binary relation (i.e., $r \subseteq A \times B$), we write $r : a \mapsto b$ for $(a, b) \in r$.

**Definition 3.10.** (i) Let $F$ be an object of $[\mathbf{T}_0 \to \mathbf{T}]$. $a \in \forall F$ is *parametric*, if for all $r : A \multimap B$ in $\mathcal{R}_0$,

$$\Phi^{\forall}_F(a)(A) \xmapsto{F(r)} \Phi^{\forall}_F(a)(B)\,.$$

(If $\Phi^{\forall}$ is natural (defined below), this condition is equal to $\forall id_F : a \mapsto a$. Note that $\forall id_F$ is not always an identity relation since $\forall$ may not preserve identities.)

(ii) $\forall F$ is *parametric*, if all $a \in \forall F$ is parametric. (If $\Phi^{\forall}$ is natural, this condition is equal to $id_{\forall F} \subseteq \forall id_F$.)

(iii) An interpretation $\xi$ is *parametric*, if for all $F \in \mathrm{Obj}\,[\mathbf{T}_0 \to \mathbf{T}]$, $\forall F$ is parametric. $\square$

Intuitively a $\mathcal{U}$-indexed family $a \in \forall F$ is parametric if for any relation $r$ of $A$ to $B$, the member of index $A$ is related to that of index $B$ via $F(r)$. Let us consider the particular case that $F$ is represented as $[\![\Gamma \vdash \sigma]\!]$. When $a \in \forall X(G(X) \Rightarrow H(X))$, we wish to say the member of index $A$ in $G(A) \Rightarrow H(A)$ is related to that of index $B$ in $G(B) \Rightarrow H(B)$ by $G(r) \Rightarrow H(r)$ iff these two members send all elements related by $G(r)$ into those related by $H(r)$. When $a \in \forall X \forall Y.G(X,Y)$, the member of index $A$ in $\forall Y.G(A,Y)$ should be also a parametric $\mathcal{U}$-indexed family. Thereby we will naturally arrive at Definition 3.12.

**Definition 3.11.** (i) For $r, s \in \mathcal{R}$ such that $r : A \multimap B$, $s : A' \multimap B'$, $r \to s$ is a relation $(A \to B) \multimap (A' \to B')$, such that $r \to s : f \mapsto f'$ if and only if $s : f(a) \mapsto f'(a')$ for all $r : a \mapsto a'$.

(ii) For $q : F \multimap F'$ a relation of r-frame morphisms in $\mathrm{Rel}\,[\mathbf{T}_0 \to \mathbf{T}]$, $\Pi X.q(X)$ is a relation $\Pi X \in \mathcal{U}.F(X) \multimap \Pi X \in \mathcal{U}.F'(X)$ such that $\Pi X.q(X) : f \mapsto f'$ if and only if $q(r) : f(A) \mapsto f'(A')$ for all $r : A \multimap A' \in \mathcal{R}_0$. $\square$

**Definition 3.12.** (Naturality condition)

(i) $\Phi^{\Rightarrow}$ is *natural*, if for any $r : A \multimap A'$, $s : B \multimap B'$ in $\mathcal{R}$,

$$a \xmapsto{r \Rightarrow s} a' \quad \text{if and only if} \quad \Phi^{\Rightarrow}_{A,B}(a) \xmapsto{r \to s} \Phi^{\Rightarrow}_{A',B'}(a')$$

where $a \in A \Rightarrow B$, $a' \in A' \Rightarrow B'$.

(ii) $\Phi^{\forall}$ is *natural*, if for any $q : F \multimap F'$ in $\mathrm{Rel}\,[\mathbf{T}_0 \to \mathbf{T}]$,

$$a \xmapsto{\forall q} a' \quad \text{if and only if} \quad \Phi^{\forall}_F(a) \xmapsto{\Pi X.q(X)} \Phi^{\forall}_{F'}(a')$$

where $a \in \forall F$ and $a' \in \forall F'$.

(iii) An interpretation $\xi$ is *natural*, if $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ satisfy their naturality condition. $\square$

In a natural interpretation $\xi$, each member of $[\![\Gamma; \Theta \vdash M : \tau]\!]$ is related in the sense of the following theorem, which is called Abstraction Theorem in [Reynolds 1983], and Parametricity Theorem in [Wadler 1989].

**Theorem 3.13.** *[Abstraction Theorem]*
*Let $\xi$ be a natural interpretation, and suppose all constant $a$ in $\mathcal{C}$ satisfies $[\![\vdash \mathrm{Type}(a)]\!] : I(a) \mapsto\!\!\Rightarrow I(a)$. Then for any $\Gamma;\Theta \vdash M : \tau$, and $\underline{r} \in \mathcal{R}_0^{|\Gamma|}$ ($r : A \twoheadrightarrow B$),*

$$[\![\Gamma;\Theta \vdash M : \tau]\!](\underline{A}) \xmapsto{[\![\Gamma\vdash \mathrm{Type}\,\Theta]\!](\underline{r}) \to [\![\Gamma\vdash\tau]\!](\underline{r})} [\![\Gamma;\Theta \vdash M : \tau]\!](\underline{B}) . \qquad \square$$

**Remark 3.14.** The theorem makes sense even if $\Gamma$ is empty. $[\![\vdash \tau]\!]$ is an object of $[\mathbf{T}_0^0 \to \mathbf{T}]$, then is a pair of members of $\mathcal{U}$ and of $\mathcal{R}$. Consider the case $\tau = \forall X.\sigma$, for example. Then the relation part of $[\![\vdash \tau]\!]$ is

$$[\![\vdash \tau]\!] = \tilde{\forall}(id_{[\![X\vdash\sigma]\!]}) \in \mathcal{R} .$$

Since $\forall$ does not necessarily preserve identities, even in this case the theorem is not trivial. $\square$

**Remark 3.15.** We hope to show with mild assumption that no parametric element of $\forall X \forall Y (X \Rightarrow Y)$ behaves as the family of $J_{YX}$. Suppose $A$ has at least two elements, $B$ is non-empty, and $r : A \twoheadrightarrow B$ is a relation such that there is at least one pair $(a, b)$ satisfying $r : a \mapsto\!\!\Rightarrow b$ where $a$ is not the interpretation of $0_A$. If $\Lambda X \Lambda Y.J_{YX}$ is parametric in a natural interpretation, then

$$id_A : J_{AA}(a) \mapsto\!\!\Rightarrow J_{AB}(b)$$

holds. Since $J_{AA}(a) = a$ and $J_{AB}(b) = 0_A$, this contradicts to the assumption for $a$. $\square$

Recall that r-frame morphisms do not always preserve identities. But if $\xi$ is a parametric natural model, the syntactically representable r-frame morphisms preserve identities.

**Proposition 3.16.** (Identity Extension Lemma)
*Suppose that $\xi$ is a parametric natural model, and that $I$ associates an r-frame morphism preserving identities to every $F \in \mathcal{K}$. Then every r-frame morphism represented as $[\![\Gamma \vdash \sigma]\!](\text{-})$ preserves identities, that is, for any $\underline{A} \in \mathcal{U}^n$, it holds that $[\![\Gamma \vdash \sigma]\!](\underline{id_A}) = id_{[\![\Gamma\vdash\sigma]\!](\underline{A})}$.*
$\square$

For the proof, we refer the reader to [Reynolds 1983].

**Corollary 3.17.** *Let $\xi$ be a natural parametric model, and $\xi'$ a restriction of $\xi$ induced by*

$$\mathrm{Obj}\,[\mathbf{T}_0'^n \to \mathbf{T}'] = \{F \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}] \,|\, F \text{ preserves identities}\}$$

*for all $n \geq 0$. Then $\xi'$ is also a natural parametric model.* $\square$

Hence if $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ are defined so as to be extensional and natural, it does no harm to choose only identity-preserving r-frame morphisms, when constructing a parametric model.

**Example 3.18.** (continued) For a coherence space $A$, $D_A$ is the set of the elements of $A$. $D_e$, for a rigid embedding $e : A \to B$, is a binary relation defined as

$$(a, b) \in D_e \quad \text{iff} \quad a = e^-(b)$$

for $a \in D_A$ and $b \in D_B$. *Ap*$(f, a)$ and *Ext*$(t, A)$ in [Girard 1986] are $\Phi^{\Rightarrow}_{A,B}(f)(a)$ and $\Phi^{\forall}_F(t)(A)$ Both *Ap* and *Ext* are one to one, so this interpretation is a model.

$\Phi^{\Rightarrow}$ is not natural. For $e : A \to B$ and $e' : A' \to B'$, $(e \Rightarrow e')^-(f) = e'^- \circ f \circ e^+$ where $f \in A \Rightarrow B$ with a little abuse of $f : A \to B$ with $Tr(f) \in A \Rightarrow B$, whereas in order that $(e \Rightarrow e')^-(f) \mapsto f$ by $(e \to e')$ it is necessary that $e'^- \circ f = (e \Rightarrow e')^-(f) \circ e^-$. But the second equation is not always deduced from the first, since $e^+ \circ e^- \neq id_B$ in general. Hence $e \Rightarrow e' : (e \Rightarrow e')^-(f) \mapsto f$ does not imply $(e \to e') : (e \Rightarrow e')^-(f) \mapsto f$.

In contrast, $\Phi^{\forall}$ is natural. For every $a \in \forall F$, by the mutilation property, $a\{A\} = F(e)^-(a\{B\})$ for all $e : A \to B$ (Theorem 2.9. in [Girard 1986]). Moreover for $\tau \in \mathrm{Rel}\,[\mathbf{T} \to \mathbf{T}]$ such that $\tau : F \to G$, it can be shown that for $a \in \forall F$ and $b \in \forall G$,

$$a = (\forall\tau)^-(b) \quad\Longleftrightarrow\quad a\{A\} = \tau_A^-(b\{A\}) \quad \text{for all } A,$$

using the condition (*) in 3.2. Then

$$\begin{aligned} a \xmapsto{\forall\tau} b \quad &\Longleftrightarrow \quad a = (\forall\tau)^-(b) \\ &\Longleftrightarrow \quad a\{A\} = \tau_A^- \circ G(e)^-(b\{B\}) = |\tau|(e)(b\{B\}) \\ &\qquad\qquad\qquad \text{for all } e : A \to B, \\ &\Longleftrightarrow \quad a \xmapsto{\Pi X.|\tau|(X)} b \end{aligned}$$

The mutilation property shows also this model is parametric.

In summary, the coherence space interpretation is a parametric model, in which $\Phi^{\forall}$ is natural, but $\Phi^{\Rightarrow}$ is not. □

**Remark 3.19.** The example above shows the coherence spaces are made into a parametric model, while they are not a model in the Bruce-Meyer-Mitchell sense, pointed out in [Coquand et al. 1989, Bruce et al. 1990] as the suggestion of Moggi. The key point is that our framework allows types with free type variables to be interpreted not simply as functions, but as maps with more structures, such as functors preserving filtered colimits and pullbacks in this case. □

**Example 3.20.** Consider the collection of all closed types of the second order lambda calculus. To each closed type is associated the set of closed terms of the type. Then we naturally have the closed types/terms interpretation $\xi_0$, taking as the class of relations $\mathcal{R}$ the set of all binary relations. $\xi_0$ is an example of parametric natural interpretation. In fact, all terms of universal types are parametric, as proved immediately from Abstraction Theorem [Wadler 1989]. However $\xi_0$ is not a model, since the expansion map $\Phi^{\Rightarrow}$ is not one-to-one: take two terms $M = \lambda x^{\forall X.X} x$ and $M' = \lambda x^{\forall X.X} x\{\forall X.X\}$, both of which have type $\forall X.X \Rightarrow \forall X.X$. By 2.3 (iii), we can easily prove there is no closed terms of type $\forall X.X$. Therefore $\Phi^{\Rightarrow}_{(\forall X.X)(\forall X.X)}(M)$ and $\Phi^{\Rightarrow}_{(\forall X.X)(\forall X.X)}(M')$ are identical as an empty map, proving $\Phi^{\Rightarrow}_{(\forall X.X)(\forall X.X)}$ is not one-to-one. The theorems in the next section, which prove the universality of data types, are applied only for parametric natural models, not for interpretation. Indeed Theorem 4.2 shows $\forall X.X$ is an initial object in the category obtained from parametric models, but in the parametric interpretation $\xi_0$ the type $\forall X.X$ is not initial as $M$ and $M'$ above refutes. □

## 4. Representations of Categorical Data Types

In this section, we fix an interpretation $\xi$ which is a *natural model.*

*4.1. Categorification of Models*

A category $\mathbf{C}_\xi$ is constructed from $\xi$. Our aim is to show if $\xi$ is parametric then $\mathbf{C}_\xi$ has categorical constructions, such as finite products, finite coproducts, and initial and terminal fixed points of some endofunctors on $\mathbf{C}_\xi$. Conversely if such constructions are defined by the formulas of second order lambda calculus that are given below, then the formulas are parametric with respect to a class of relations.

**Definition 4.1.** $\mathbf{C}_\xi$ is a category defined as follows: an object is just an element of $\mathcal{U}$ and a homset is put as $\mathbf{C}_\xi(A,B) = D_{A\Rightarrow B}$. An identity arrow $id_A : A \to A$ is $[\![X;\vdash \lambda x^X x : X \Rightarrow X]\!](A)$, and a composition $g \circ f$ for $f : A \Rightarrow B$ and $g : B \Rightarrow C$ is $[\![X,Y,Z\,;\, u{:}X \Rightarrow Y, v{:}Y \Rightarrow Z \vdash \lambda x^X v(ux) : X \Rightarrow Z]\!](A,B,C)(f,g)$. □

Remark that we have here defined identities and compositions using the language $\lambda^\forall(\Sigma)$ and $[\![\,]\!]$. It is our main technique to obtain categorical properties.

Before showing $\mathbf{C}_\xi$ is really a category, we would introduce some conventions. For simplicity we suppose that the signature $\Sigma$ is extended to contain all symbols, one for each object and for each arrow of the category $\mathbf{C}_\xi$. For example, we write $[\![\lambda x^A x]\!]$ and $[\![\lambda x^A g(fx) : A \Rightarrow B]\!]$ for the identity and the composition in the definition above (further we here omitted some less important parts, as "$;\vdash$").

**Proposition 4.2.** $\mathbf{C}_\xi$ *is a category.*

*Proof.* It is easy to show that identities and compositions fulfill the necessary equations. For example, for $f : A \Rightarrow B$, $f \circ id_A$ is $[\![\lambda x^A f((\lambda y^A.y)x)]\!]$, which reduces to $[\![f]\!]$ by $\beta\eta$-reductions. □

The construction of $\mathbf{C}_\xi$ as well as that of $[\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$ in Definition 4.10 below is based on the same idea as the classifying hyperdoctrine [Pitts 1987], in which open terms are used to define arrows (for example $[\![x : A \vdash x : A]\!]$ for identity). The classifying hyperdoctrine is defined from a theory of second order lambda calculus. Starting from a model, we can proceed likewise by first considering the theory of the model (with the fully extended signature as above), to obtain categories equivalent to the categories $\mathbf{C}_\xi$ and $[\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$.

Let $M(a)$ and $N(a)$ be some terms which have the same type and possibly contain a constant $a$ of type $A$. We notice that if $[\![M(a)]\!] = [\![N(a)]\!]$ for all $a \in A$, then $[\![\lambda x^A M(x)]\!] = [\![\lambda x^A N(x)]\!]$. Similarly if $M(A)$ and $N(A)$ possibly contain a constant $A \in \mathcal{K}$ and $[\![M(A)]\!] = [\![N(A)]\!]$ holds for all $A \in \mathcal{U}$, then $[\![\Lambda X.M(X)]\!] = [\![\Lambda X.N(X)]\!]$. These are immediate from the extensionality of the model $\xi$, and used repeatedly in the arguments below.

To each arrow $f : A \to B$ in $\mathbf{C}_\xi$, we associate, if possible, a relation $|f| : A \rightharpoondown B$ such that $|f| : a \mapsto b$ if and only if $\Phi^{\Rightarrow}_{A,B}(f)(a) = b$. When there exists $|f|$ for all arrows $f$, we denote the collection of such relations by $\mathcal{R}_{Ar}$. In what follows, we are mainly interested in the case $\mathcal{R}_0 = \mathcal{R}_{Ar}$. We will stress that by saying "parametric w.r.t. $\mathcal{R}_{Ar}$".

The next lemma is a basic tool relating diagrams with relations in $\mathcal{R}_{Ar}$.

**Lemma 4.3.** *Suppose* $f : A \to A'$, $g : B \to B'$, $h : A \to B$, *and* $h' : A' \to B'$. *Then the* *diagram*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ A' \arrow[r, "h'"'] & B' \end{tikzcd}$$

*commutes iff* $|f| \Rightarrow |g| : h \mapsto h'$. *A similar result holds, even if the direction of* $f$ *is reversed to* $f : A' \to A$. ☐

Most properties of categories are expressed by universal conditions. For example, a product $A \times B$ is an object such that two projections $\pi : A \times B \to A$ and $\pi' : A \times B \to B$ exist, and for any pair of arrows $f : C \to A$ and $g : C \to B$ there is a "unique" arrow $\langle f, g \rangle : C \to A \times B$ that makes the diagram

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi"'] \arrow[r, "\pi'"] & B \\ & C \arrow[ul, "f"] \arrow[u, "\langle f,g \rangle"] \arrow[ur, "g"'] & \end{tikzcd}$$

commute. Weak universal conditions [Mac Lane 1971] are obtained by removing "unique" from the corresponding universal conditions. For example, a weak product, which is still denoted by $A \times B$, is an object such that with $\pi$, $\pi'$, and $\langle f, g \rangle$, but this time $\langle f, g \rangle$ is not necessarily the only arrow that makes the diagram above commute.

In intuitionistic second order logic, only with the connectives $\Rightarrow$ and $\forall$, all other connectives can be represented as

$$\begin{aligned} A \wedge B &= \forall X((A \Rightarrow B \Rightarrow X) \Rightarrow X) \\ A \vee B &= \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X) \\ \top &= \forall X(X \Rightarrow X) \\ \bot &= \forall X.X \\ \exists X.A(X) &= \forall Y(\forall X(A(X) \Rightarrow Y) \Rightarrow Y) \end{aligned}$$

where $X$ and $Y$ are propositional variables. All these are seen in [Prawitz 1965], except $\top$, which is clearly a tautology. (Here we have ignored the first order part, since our main interest is in propositional logic. For instance, $\exists x.A(x)$ where $x$ is a first order variable is also representable). Comparison with representations of inf and sup given in cHa's (Section 3) would clarify the meaning behind these Prawitz formulas.

Recalling that second order lambda calculus is the proof system of the $\Rightarrow, \forall$-fragments of intuitionistic second order propositional logic through Curry-Howard isomorphism, we can ask one natural question "What the representations above represent in semantics?" One may naturally suppose that, in $\mathbf{C}_\xi$, $A \wedge B$ is a product $A \times B$, $A \vee B$ a coproduct $A + B$, $\top$ a terminal object, $\perp$ an initial object, and $\exists X.A(X)$ a left adjoint of an appropriate substitution functor. It is, however, not the case in general. Only the weak universal conditions are satisfied; $\top$ is a weak terminal object, and $A \wedge B$ is a weak product, for example. Let us consider the coherence space model and we immediately know the corresponding category, which consists of coherence spaces and stable maps, have a weak initial object, but no initial object.

What is the condition for $A \wedge B$ to be a product then? Main results in this article are listed in the following subsections. They say that the representations of Prawitz fulfill the corresponding universal conditions if and only if they are parametric w.r.t. $\mathcal{R}_{Ar}$. For example, $\perp = \forall X.X$ is an initial object if and only if it is parametric w.r.t. $\mathcal{R}_{Ar}$. Hence, in short, our slogan is

$$\text{Universal} \quad \equiv \quad \text{Parametric}\,.$$

*4.2. Finite Products and Finite Coproducts*

**Theorem 4.4.** *i) $\mathbf{C}_\xi$ has a weak initial object $\perp = \forall X.X$.*

*ii) $\perp$ is an initial object iff it is parametric w.r.t. $\mathcal{R}_{Ar}$.*

*Proof.* i) $\perp$ is a weak initial object iff there is an arrow $f_A : \perp \to A$ for all object $A$. Put $f_A = [\![\lambda x^\perp x\{A\}]\!]$.

ii) The proof is similar to, but simpler than, the proof of the next theorem. So we here omit it. $\square$

**Theorem 4.5.** *i) $\mathbf{C}_\xi$ has a weak coproduct $A + B = \forall X((A \Rightarrow X) \Rightarrow (B \Rightarrow X) \Rightarrow X)$ for each objects $A$ and $B$.*

*ii) $A + B$ with two injections given in the proof of i) is a coproduct iff it is parametric w.r.t. $\mathcal{R}_{Ar}$.*

*Proof.* i) $A + B$ is a weak coproduct iff there are two injections $\iota : A \to A + B$ and $\iota' : B \to A + B$, and $[f, g] : A + B \to C$ for $f : A \to C$ and $g : B \to C$ such that $[f, g] \circ \iota = f$ and $[f, g] \circ \iota' = g$. Put

$$\begin{aligned}
\iota &= [\![\lambda x^A \Lambda X \lambda y^{A\Rightarrow X} \lambda z^{B\Rightarrow X} yx]\!] \\
\iota' &= [\![\lambda x^B \Lambda X \lambda y^{A\Rightarrow X} \lambda z^{B\Rightarrow X} zx]\!] \\
[f, g] &= [\![\lambda x^{A+B} x\{C\} fg]\!]\,.
\end{aligned}$$

It is simply by reductions to prove $[f, g] \circ \iota = f$ and $[f, g] \circ \iota' = g$.

ii)($\Rightarrow$) Provided that $A + B$ is a coproduct, a unique arrow $[f, g] : A + B \to C$ for $f : A \to C$ and $g : B \to C$ must be equal to the one in the proof of i). Given $f : C \to C'$, $g : A \to C$, $g : A \to C'$, $h : B \to C$, and $h' : B \to C'$ such that $A \Rightarrow |f| : g \mapsto g'$ and

$B \Rightarrow |f| : h \mapsto h'$, they are made into a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "g'"'] & C \arrow[d, "f"] & B \arrow[l, "h"'] \arrow[dl, "h'"] \\ & C' & \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} A+B \arrow[r, "{[g,h]}"] \arrow[dr, "{[g',h']}"'] & C \arrow[d, "f"] \\ & C' \end{tikzcd}$$

is commutative, thus it follows that $[\![\lambda x^{A+B} f(x\{C\}gh)]\!] = [\![\lambda x^{A+B} x\{C'\}g'h']\!]$. Hence, for all $a \in A+B$, $|f| : [\![a\{C\}gh]\!] \mapsto [\![a\{C'\}g'h']\!]$ holds. Observing $A \Rightarrow |f| : g \mapsto g'$ and $B \Rightarrow |f| : h \mapsto h'$, we obtain the conclusion that $a$ is parametric w.r.t. $\mathcal{R}_{Ar}$.

($\Leftarrow$) First prove $[\iota, \iota'] = id_{A+B}$. For any $f : A \to C$ and $g : B \to C$, it holds that $[f,g] \circ \iota = f$ and $[f,g] \circ \iota' = g$. Thus $A \Rightarrow |[f,g]| : \iota \mapsto f$ and $B \Rightarrow |[f,g]| : \iota' \mapsto g$. Since $A+B$ is parametric w.r.t. $\mathcal{R}_{Ar}$, for any $a \in A+B$, it holds that $|[f,g]| : [\![a\{A+B\}\iota\iota']\!] \mapsto [\![a\{C\}fg]\!]$, i.e., that $[\![a\{A+B\}\iota\iota'\{C\}fg]\!] = [\![a\{C\}fg]\!]$. Since $C$, $f$, $g$, and $a$ are arbitrary, $[\iota, \iota'] = [\![\lambda x^{A+B} x\{A+B\}\iota\iota']\!] = [\![\lambda x^{A+B} x]\!] = id_{A+B}$.

Next for any $h : A+B \to C$ such that $h \circ \iota = f$ and $h \circ \iota' = g$, we prove $h = [f,g]$. Since every $a \in A+B$ is parametric, from $A \Rightarrow |h| : \iota \mapsto f$ and $B \Rightarrow |h| : \iota' \mapsto g$, it follows that $|h| : [\![a\{A+B\}\iota\iota']\!] \mapsto [\![a\{C\}fg]\!]$. Having $[\![a\{A+B\}\iota\iota']\!] = [\![a]\!]$ from $[\iota, \iota'] = id_{A+B}$, we obtain that $[\![h(a)]\!] = [\![a\{C\}fg]\!]$, thus $h = [f,g]$. $\square$

Remark that for the ($\Rightarrow$) part in ii) of the proof above (and similarly in the theorems below), it is sufficient that $\xi$ is parametric w.r.t. $\mathcal{R}_0$ which includes $\mathcal{R}_{Ar}$.

**Theorem 4.6.** *i)* $\mathbf{C}_\xi$ *has a weak terminal object* $\top = \forall X(X \Rightarrow X)$.

*ii)* $\top$ *is terminal iff it is parametric w.r.t.* $\mathcal{R}_{Ar}$.

*iii) In the case of ii),* $A \cong \top \Rightarrow A$ *and* $\top$ *is a generator.*

*Proof.* i) If there is an arrow $t_A : A \to \top$ for all object $A$, then $\top$ is a weak terminal object. Put $t_A = [\![\lambda y^A \Lambda X \lambda x^X x]\!]$.

ii)($\Rightarrow$) First we note $\top$ has exactly one element $[\![\Lambda X \lambda x^X x]\!]$. In fact, for any $a, b \in \top$, take arrows $f = [\![\lambda x^\top a]\!]$ and $g = [\![\lambda x^\top b]\!]$, which are arrows from $\top$ to $\top$. Since $\top$ is terminal, $f = g$, and thus, applying $[\![\Lambda X \lambda x^X x]\!]$, we obtain $a = b$. This shows all elements of $\top$ must coincide. Therefore it suffices to prove $[\![\Lambda X \lambda x^X x]\!]$ is parametric, but it is immediate from Theorem 3.13.

($\Leftarrow$) First prove $[\![\lambda y^\top y\{\top\}(\Lambda X \lambda x^X x)]\!] = id_\top$. For any $b \in B$, putting $g = [\![\lambda x^\top x\{B\}b]\!]$, we have $|g| : [\![\Lambda X \lambda x^X x]\!] \mapsto\!\!\!\Rightarrow b$. Since any $a \in \top$ is parametric, $|g| : [\![a\{\top\}(\Lambda X \lambda x^X x)]\!] \mapsto\!\!\!\Rightarrow [\![a\{B\}b]\!]$. Hence $[\![\lambda y^\top y\{\top\}(\Lambda X \lambda x^X x)]\!] = id_\top$, since $b$, $B$, and $a$ are arbitrary.

Next show $h = t_B$ for any $h : B \to \top$. For $b \in B$, $|t_B| : b \mapsto\!\!\!\Rightarrow [\![\Lambda X \lambda x^X x]\!]$. Since $[\![hb]\!]$ should be parametric, $|t_B| : [\![hb\{B\}b]\!] \mapsto\!\!\!\Rightarrow [\![hb\{\top\}(\Lambda X \lambda x^X x)]\!]$, i.e.,

$$\begin{aligned} [\![\Lambda X \lambda x^X x]\!] &= [\![hb\{\top\}(\Lambda X \lambda x^X x)]\!] \\ &= [\![hb]\!]\,, \qquad \text{by } [\![\lambda y^\top y\{\top\}(\Lambda X \lambda x^X x)]\!] = id_\top . \end{aligned}$$

Since $b$ is arbitrary, $t_B = h$ holds.

iii) The isomorphism $A \to (\top \Rightarrow A)$ is defined by $a \mapsto [\![\lambda x^\top a]\!]$. To prove $\top$ is a generator, take $f, g : A \rightrightarrows B$ such that $fh = gh$ for all $h : \top \to A$. For any $a \in A$, putting $h = [\![\lambda x^\top a]\!]$, we thus have $[\![\lambda x^\top fa]\!] = [\![\lambda x^\top ga]\!]$. Since the correspondence $B \to (\top \Rightarrow B)$ is an isomorphism, $[\![fa]\!] = [\![ga]\!]$. Therefore $f = g$ holds. $\Box$

Given a functor $G : \mathbf{D} \to \mathbf{C}$ and a map $F : \mathrm{Obj}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{D})$, we say $F$ is a *weak left adjoint* of $G$ if and only if there exists $\eta_A : A \to GFA$ and for any $f : A \to GB$ there is at least one arrow $f^* : FA \to B$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GFA \arrow[d, "Gf^*"] \\ & FB \end{tikzcd}$$

commutes. This is the weak universal condition corresponding to the usual definition of adjunction. Note that we have enforced neither $F$ to be a functor nor $\eta$ to be a natural transformation. These properties are automatically induced when $F$ is an actual left adjoint.

**Theorem 4.7.** *i) $\mathbf{C}_\xi$ has a weak product $A \times B = \forall X((A \Rightarrow B \Rightarrow X) \Rightarrow X)$, for which* (-) $\times B$ *is a weak left adjoint of $B \Rightarrow$ (-).*

*ii) $A \times B$ is a product such that (-) $\times B$ is left adjoint of $B \Rightarrow$ (-) with $\eta$ given in the proof of i), iff $A \times B$ is parametric w.r.t. $\mathcal{R}_{Ar}$ for all $A \in \mathrm{Obj}(\mathbf{C}_\xi)$.*

*Proof.* i) Put

$$\begin{aligned} \pi &= [\![\lambda x^{A\times B} x\{A\}(\lambda y^A \lambda z^B y)]\!] : A \times B \to B \\ \pi' &= [\![\lambda x^{A\times B} x\{B\}(\lambda y^A \lambda z^B z)]\!] : A \times B \to B \end{aligned}$$

and for $f : C \to A$ and $g : C \to B$

$$\langle f, g\rangle = [\![\lambda x^C \Lambda X \lambda y^{A \Rightarrow B \Rightarrow X} y(fx)(gx)]\!] : C \to A \times B\,.$$

It is easily seen that $\pi \circ \langle f, g\rangle = f$ and $\pi' \circ \langle f, g\rangle = g$. Therefore $A \times B$ is a weak product. $B \Rightarrow$ (-) $: \mathbf{C}_\xi \to \mathbf{C}_\xi$ is a functor whose arrow function is defined as, for $f : A \to A'$,

$$B \Rightarrow f = [\![\lambda x^{B \Rightarrow A} \lambda y^B f(xy)]\!] : (B \Rightarrow A) \to (B \Rightarrow A')\,.$$

Now put

$$\begin{aligned} \eta_A &= [\![\lambda x^A \lambda y^B \Lambda X \lambda z^{A \Rightarrow B \Rightarrow X} zxy]\!] : A \to (B \Rightarrow (A \times B)) \\ f^* &= [\![\lambda x^{A \times B} x\{C\} f]\!] : A \times B \to C \end{aligned}$$

for $f : A \to (B \Rightarrow C)$. By reductions, it is easily showed that $(B \Rightarrow f^*) \circ \eta_A = f$.

ii)($\Rightarrow$) For any $g : C \to D$, take $e \in A \Rightarrow B \Rightarrow C$ and $f \in A \Rightarrow B \Rightarrow D$ such that $A \Rightarrow B \Rightarrow |g| : e \mapsto\!\!\Rightarrow f$. In the form of a diagram,

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[dr, "g"'] & (B \Rightarrow C) \arrow[d, "B \Rightarrow g"] \\ & (B \Rightarrow D) \end{tikzcd} .$$

By adjunction $(\text{-}) \times B \dashv B \Rightarrow (\text{-})$, we have a commutative diagram

$$\begin{tikzcd} A \times B \arrow[r, "e^*"] \arrow[dr, "f^*"'] & C \arrow[d, "g"] \\ & D \end{tikzcd} .$$

Thus for any $a \in A \times B$, $|g| : [\![a\{C\}e]\!] \mapsto\!\!\Rightarrow [\![a\{D\}f]\!]$, which shows $a$ is parametric w.r.t. $\mathcal{R}_{Ar}$.

($\Leftarrow$) Since $\pi \circ \langle f, g \rangle = f$ and $\pi' \circ \langle f, g \rangle = g$ hold, it suffices to prove $\langle \pi \circ f, \pi' \circ f \rangle = f$. First prove $\pi \circ \eta_A^* = \pi$ and $\pi' \circ \eta_A^* = \pi'$. They are obtained from the fact that $A \times B$ is parametric w.r.t. $\mathcal{R}_{Ar}$ and that $A \Rightarrow B \Rightarrow |\pi| : \eta_A \mapsto\!\!\Rightarrow [\![\lambda x^A \lambda y^B x]\!]$. Next prove $\langle \pi, \pi' \rangle = id_{A \times B}$. For that, take any $f \in A \Rightarrow B \Rightarrow C$ and put $g = [\![\lambda x^{A \times B} f(\pi x)(\pi' x)]\!] : A \times B \to C$. Then $A \Rightarrow B \Rightarrow |g| : \eta_A \mapsto\!\!\Rightarrow f$ is proved simply by reductions. By parametricity, $|g| : \eta_A^* \mapsto\!\!\Rightarrow f^*$, i.e., for all $a \in A \times B$,

$$[\![f(\pi \circ \eta_A^*(a))(\pi' \circ \eta_A^*(a))]\!] = [\![a\{C\}f]\!] .$$

Applying the first half, we obtain $[\![f(\pi a)(\pi' a)]\!] = [\![a\{C\}f]\!]$. Since $f$, $C$, and $a$ are arbitrary, $\langle \pi, \pi' \rangle = [\![\lambda x^{A \times B} \Lambda X \lambda y^{A \Rightarrow B \Rightarrow X} f(\pi x)(\pi' x)]\!] = [\![\lambda x^{A \times B} x]\!] = id_{A \times B}$. Last $\langle \pi, \pi' \rangle \circ f = \langle \pi \circ f, \pi' \circ f \rangle$ is proved by direct calculation. Therefore $\langle \pi \circ f, \pi' \circ f \rangle = f$.

The proof of $(\text{-}) \times B \dashv B \Rightarrow (\text{-})$ is left to the reader. $\square$

**Remark 4.8.** Our theorems have the shape that a data type is universal iff it is parametric w.r.t. $\mathcal{R}_{Ar}$. To deal with the system with full comprehension, we must force r-frames $F$ to have the same domains as the targets, that is, $F : \mathbf{T}^n \to \mathbf{T}$ (Remark 3.5), not simply $F : \mathbf{T}^n_{Ar} \to \mathbf{T}$. Therefore we must modify the definition of parametricity w.r.t. $\mathcal{R}_{Ar}$ as follows: a universal type $\forall F$ is parametric w.r.t. $\mathcal{R}_{Ar}$ iff every $a \in \forall F$ satisfies $F(r) : a\{A\} \mapsto\!\!\Rightarrow a\{B\}$ for all $r : A \multimap B$ that belongs to $\mathcal{R}_{Ar}$. We remind the reader that in the former definition the range of $r$ is not restricted to $\mathcal{R}_{Ar}$, but happens to be equal to $\mathcal{R}_{Ar}$ by the construction of a model. After this modification, the above theorems are all valid, and most theorems below hold too except those involved with universal strong functors, since Theorem 4.21 fails. □

**Corollary 4.9.** *If $\xi$ is parametric w.r.t. $\mathcal{R}_0$ including $\mathcal{R}_{Ar}$, then $\mathbf{C}_\xi$ is a ccc with finite coproducts.* □

### *4.3. Limiting and Colimiting Quantifiers*

The introduction rule of universal quantification is

$$\frac{A \vdash F(X)}{A \vdash \forall X.F(X)}$$

where $A$ does not include $X$ as a free variable. One can consider $A$ in the upper sequent to contain 0 occurrence of the variable $X$ (that is, a dummy variable $X$ is adjoined). This rule corresponds to $;\, x{:}A \vdash \Lambda X.M : \forall X.F(X)$ obtained from $X\,;\, x{:}A \vdash M : F(X)$, in the term derivations of $\lambda^\forall$. Also the converse direction is definable as $X\,;\, x : A \vdash N\{X\} : F(X)$ for $;\, x : A \vdash N : \forall X.F(X)$. This correspondence is isomorphic since $(\Lambda X.M)\{X\} = M$ and $\Lambda X.N\{X\} = N$. From the form of the rule and this isomorphism, one may consider $\forall$ as a right adjoint of the functor which adjoins a dummy variable $X$. After some definitions of categories and functors, it is showed to be true.

The corresponding rule for existential quantification is

$$\frac{F(X) \vdash A}{\exists X.F(X) \vdash A}$$

where $A$ does not include $X$. If $\exists X.F(X)$ is represented as $\forall Y(\forall X(F(X) \Rightarrow Y) \Rightarrow Y)$, we obtain $;y : \exists X.F(X) \vdash y\{A\}(\Lambda X \lambda x^{F(X)} M) : A$ from $X\,;\, x : F(X) \vdash M : A$, and $X\,;\, x{:}F(X) \vdash N[y := \Lambda Y \lambda z^{\forall X(F(X) \Rightarrow Y)} z\{X\}x] : A$ from $;y : \exists X.F(X) \vdash N : A$. In this case, however, the correspondence is not isomorphic since

$$y\{A\}(\lambda x^{F(X)} N[y := \Lambda Y \lambda y^{\forall X(F(X) \Rightarrow Y)} y\{X\}x]) \neq_{\beta\eta} N\,.$$

Thus we cannot expect $\exists$ is always a left adjoint of the functor adjoining a dummy variable. What is necessary for $\exists$ to be the left adjoint is, again, parametricity. The main subject of this subsection is to show $\exists$ is actually a left adjoint in parametric natural models.

**Definition 4.10.** i) For each $n \geq 1$, $[\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$ is the category whose objects are defined by $\mathrm{Obj}\,[\mathbf{C}_\xi^n \to \mathbf{C}_\xi] = \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$ and whose homs are defined by $[\mathbf{C}_\xi^n \to \mathbf{C}_\xi](F, G) = \forall^n \underline{X}.(F(\underline{X}) \Rightarrow G(\underline{X}))$ where $\forall^n \underline{X}$ means $\forall X_1 \cdots \forall X_n$. Identity and composition are defined as

$$\begin{aligned} id_F &= [\![\Lambda^n \underline{X} \lambda x^{F(\underline{X})}.x]\!] \\ f' \circ f &= [\![\Lambda^n \underline{X} \lambda x^{F(\underline{X})} f'\{\underline{X}\}(f\{\underline{X}\}x)]\!] \qquad (f : F \to F' \text{ and } f' : F' \to F'') \end{aligned}$$

where $f\{\underline{X}\}$ means $f\{X_1\} \cdots \{X_n\}$.

ii) For each $n \geq 1$, a functor $\kappa^n : \mathbf{C}_\xi \to [\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$ is defined as; $\kappa^n(A) = \forall^n \underline{X}.A$ and $\kappa^n(f)$ is $[\![\Lambda^n \underline{X}.f]\!]$.

iii) For each $n \geq 1$, a functor $\Pi^n : [\mathbf{C}_\xi^n \to \mathbf{C}_\xi] \to \mathbf{C}_\xi$ is defined as; $\Pi^n(F) = \forall^n \underline{X}.F(\underline{X})$ and $\Pi^n(f) = [\![\lambda x^{\forall^n \underline{X}.F(\underline{X})} \Lambda^n \underline{X}.f\{\underline{X}\}(x\{\underline{X}\})]\!]$ for $f : F \to G$. $\square$

It is easily verified that $[\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$ is a category, and that $\kappa^n$ and $\Pi^n$ are functors. $\kappa^n$ behaves as adjoining dummy variables $X_1, \ldots, X_n$, and $\Pi^n$ is $n$-fold universal quantifiers.

**Proposition 4.11.** *For every $n \geq 1$, $\kappa^n \dashv \Pi^n$.*

*Proof.* The adjunction amounts to saying that for any $A \in \mathrm{Obj}(\mathbf{C}_\xi)$ and $F \in \mathrm{Obj}\,[\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$, there are $\epsilon_F \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](\kappa^n \Pi^n F, F)$ and $f^\Pi \in \mathbf{C}_\xi(A, \Pi^n F)$ for $f \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](\kappa^n A, F)$ satisfying the equations,

$$\begin{aligned} \epsilon_F \circ \kappa^n(f^\Pi) &= f \qquad \text{for } f \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](\kappa^n A, F) \\ (\epsilon_F \circ \kappa^n(g))^\Pi &= g \qquad \text{for } g \in \mathbf{C}_\xi(A, \Pi^n F). \end{aligned}$$

Put $\epsilon_F = [\![\Lambda^n \underline{X}.\lambda x^{\Pi^n \underline{X}.F(\underline{X})} x\{\underline{X}\}]\!]$, and $f^\Pi = [\![\lambda x^A \Lambda^n \underline{X}.f\{\underline{X}\}x]\!]$. $\square$

We say $\kappa^n \dashv \Pi^n$ *at* $F$, if $\epsilon_F$ and $f^\Pi$ in the proof of the above theorem exist only for the specified $F$, satisfying the same equations.

Take $F \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$. For any $\underline{r} : \underline{A} \rightharpoonup \underline{B}$ in $\mathcal{R}_0^n$, it holds that $F(\underline{r}) : F(\underline{A}) \rightharpoonup F(\underline{B})$. So $F$ can be regarded as a generalized diagram in the sense that the vertices of $F$ are $F(\underline{A})$ for $\underline{A} \in \mathcal{U}^n$ and the directed edges are $F(\underline{r}) : F(\underline{A}) \rightharpoonup F(\underline{B})$ for $\underline{r} : \underline{A} \rightharpoonup \underline{B}$ in $\mathcal{R}_0^n$. If $\mathcal{R}_0 = \mathcal{R}_{Ar}$ and $F$ is a functor, i.e., $F$ maps $\mathcal{R}_{Ar}$ into $\mathcal{R}_{Ar}$ preserving identities and compositions, then $F$ is a usual diagram where the edges are arrows $F(\underline{f}) : F(\underline{A}) \to F(\underline{B})$ for $\underline{f} : \underline{A} \to \underline{B}$. We call an arrow $f \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](\kappa^n C, F)$ a *relational cone to $F$* if for all $\underline{r} : \underline{A} \rightharpoonup \underline{B}$ in $\mathcal{R}_0^n$,

$$\begin{tikzcd} & C \arrow[dl, "{[\![f\{\underline{A}\}]\!]}"'] \arrow[dr, "{[\![f\{\underline{B}\}]\!]}"] & \\ F(\underline{A}) \arrow[rr, rightharpoonup, "F(\underline{r})"'] & & F(\underline{B}) \end{tikzcd}$$

commutes, namely $id_C \Rightarrow F(\underline{r}) : f\{\underline{A}\} \mapsto f\{\underline{B}\}$, namely $f$ is parametric.

**Definition 4.12.** Let $F$ be an object of $[\mathbf{T}_0^n \to \mathbf{T}]$. $\Pi^n$ is called a *limiting quantifier* of $F$ if and only if $\kappa^n \dashv \Pi^n$ at $F$ and the counit $\epsilon_F$ is a relational cone. $\square$

The notion of parametricity is easily extended to $n$-fold universal objects such as $\forall^n \underline{X}.F(\underline{X})$. It is said $a \in \forall^n \underline{X}.F(\underline{X})$ is *parametric* iff for all $\underline{r} : \underline{A} \rightharpoonup \underline{B}$ in $\mathcal{R}_0^n$, $F(\underline{r}) : (\Phi^\forall)^n(a)(\underline{A}) \mapsto (\Phi^\forall)^n(a)(\underline{B})$, where $(\Phi^\forall)^n(a)(\underline{A})$ means $\Phi^\forall(\cdots(\Phi^\forall(a)(A_1))\cdots)(A_n)$. But this extension is superfluous, since $a \in \forall^n \underline{X}.F(\underline{X})$ is parametric in the $n$-fold sense iff $a$ is parametric in the old sense, that is, $a \in \forall X_1(\forall^{n-1}\underline{X}.F(\underline{X}))$ is parametric.

**Theorem 4.13.** *Suppose $F$ is an object of $[\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$. For each $n \geq 1$, $\Pi^n$ is a limiting quantifier of $F$ iff $\forall^n \underline{X}.F(\underline{X})$ is parametric.*

*Proof.* $\kappa^n \dashv \Pi^n$ (thus $\kappa^n \dashv \Pi^n$ at $F$) always holds, as proved in Proposition 4.11 (hence in the definition of a limiting quantifier, the condition $\kappa^n \dashv \Pi^n$ at $F$ is superfluous, but it is included to maintain the duality to the colimiting quantifier defined later). To say $\forall^n \underline{X}.F(\underline{X})$ is parametric is just equivalent to say $\epsilon_F$ is a relational cone. $\square$

If $\Pi^n$ is a limiting quantifier of $F$ then $\Pi^n F$ may be regarded to be a limit of relational cones to $F$. Consider $(\kappa^n \downarrow F)$, the category of relational cones to $F$. It is easy to see that if $\Pi^n$ is a limiting quantifier of $F$ then $\epsilon_F : \forall^n \underline{X}(\Pi^n F \Rightarrow F(\underline{X}))$ is a limiting (i.e. initial) relational cone in the category $(\kappa^n \downarrow F)$. To see the converse direction, observe that the parametricity of $\epsilon_F$ implies the parametricity of $\forall^n \underline{X}.F(\underline{X})$, and the latter implies the parametricity of $\forall^n \underline{X}(C \Rightarrow F(\underline{X}))$ for all $C$.

Let $\Sigma^n$ be a map $\mathrm{Obj}\,[\mathbf{C}_\xi^n \to \mathbf{C}_\xi] \to \mathrm{Obj}(\mathbf{C}_\xi)$. $\Sigma^n \dashv \kappa^n$ holds if $\eta_F \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](F, \kappa^n \Sigma^n F)$ and $f^\Sigma \in \mathbf{C}_\xi(\Sigma^n F, A)$ for $f \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](F, \kappa^n)$ exist with the equations

$$\begin{aligned} \kappa^n(f^\Sigma) \circ \eta_F &= f && \text{for } f \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](F, \kappa^n A) \\ (\kappa^n(g) \circ \eta_F)^\Sigma &= g && \text{for } g \in \mathbf{C}_\xi(\Sigma^n F, A). \end{aligned}$$

If $\eta_F$ and $f^\Sigma$ exist only for the specified $F$, we say $\Sigma^n \dashv \kappa^n$ *at* $F$. It is equivalent to say there is a natural bijection

$$\mathbf{C}_\xi(\Sigma^n F, A) \overset{\varphi_A}{\cong} [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](F, \kappa^n A)$$

for the specified $F$. We call $f \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](F, \kappa^n C)$ a *relational cone from* $F$ if $f$ is parametric.

Consider the category $(F \downarrow \kappa^n)$ of relational cones from $F$. $\Sigma^n$ is said to be a colimiting quantifier of $F$ if and only if $\eta_F \in \forall^n \underline{X}(F(\underline{X}) \Rightarrow \Sigma^n F)$ is a colimiting relational cone from $F$ (i.e. a terminal object of $(F \downarrow \kappa^n)$). Namely,

**Definition 4.14.** Let $F$ be an object of $[\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$. $\Sigma^n : \mathrm{Obj}\,[\mathbf{C}_\xi^n \to \mathbf{C}_\xi] \to \mathrm{Obj}(\mathbf{C}_\xi)$ is called a *colimiting quantifier* of $F$ iff $\Sigma^n \dashv \kappa^n$ at $F$ and the unit $\eta_F : F \to \kappa^n \Sigma^n F$ is a relational cone. $\square$

**Theorem 4.15.** *Let* $\Sigma^n : \mathrm{Obj}\,[\mathbf{C}_\xi^n \to \mathbf{C}_\xi] \to \mathrm{Obj}(\mathbf{C}_\xi)$ *be defined by* $\Sigma^n(F) = \forall Y(\forall^n \underline{X}( F(\underline{X}) \Rightarrow Y) \Rightarrow Y)$.

*i)* $\Sigma^n$ *is a weak left adjoint of* $\kappa^n$.

*ii) If* $\Sigma^n F$ *and* $\eta_F$ *in the proof of i) are parametric w.r.t.* $\mathcal{R}_{Ar}$, *then* $\Sigma^n$ *is a colimiting quantifier of* $F$.

*iii) Suppose* $F$ *preserves identities. If* $\Sigma^n$ *is a colimiting quantifier of* $F$ *with the unit of* $\Sigma^n \dashv \kappa^n$ *given in i), then* $\Sigma^n F$ *is parametric w.r.t.* $\mathcal{R}_{Ar}$.

*Proof.* i) Put

$$\begin{aligned} \eta_F &= [\![\Lambda^n \underline{X} \lambda x^{F(\underline{X})} \Lambda Y \lambda y^{\forall^n \underline{X}(F(\underline{X}) \Rightarrow Y)} y\{\underline{X}\} x]\!] \\ f^\Sigma &= [\![\lambda x^{\Sigma^n(F)} x\{C\} f]\!] \end{aligned}$$

where $f \in [\mathbf{C}_\xi^n \to \mathbf{C}_\xi](F, \kappa^n C)$ i.e. $f \in \forall^n \underline{X}(F(\underline{X}) \Rightarrow C)$. We have to check $\kappa^n(f^\Sigma) \circ \eta_F = f$. It is straightforward by reductions.

ii) Notice that the assumption that $\eta_F$ is parametric implies the parametricity of $\forall^n \underline{X}(F(\underline{X}) \Rightarrow C)$ for all $C$. Indeed, $F(\underline{r}) \Rightarrow id_C : [\![(\kappa^n(f^\Sigma) \circ \eta_F)\{\underline{A}\}]\!] \rightharpoonup [\![(\kappa^n(f^\Sigma) \circ \eta_F)\{B\}]\!]$ easily follows from the assumption. Then use i).

We show that $\Sigma^n \dashv \kappa^n$ at $F$. First prove $\eta_F^{\Sigma} = id_{\Sigma^n F}$. The equation $\kappa^n(f^{\Sigma}) \circ \eta_F = f$ where $f \in \forall^n \underline{X}(F(\underline{X}) \Rightarrow C)$ implies

$$[\![\eta_F\{\underline{A}\}]\!] \xmapsto{id_{F(\underline{A})} \Rightarrow |f^{\Sigma}|} [\![f\{\underline{A}\}]\!]$$

for all $\underline{A} \in \mathcal{U}^n$. Since $\forall^n \underline{X}(F(\underline{X}) \Rightarrow C)$ is parametric,

$$[\![f\{\underline{A}\}]\!] \xmapsto{F(\underline{r}) \Rightarrow id_C} [\![f\{\underline{B}\}]\!]$$

for any $\underline{r} : \underline{A} \to \underline{B}$ in $\mathcal{R}^n_{Ar}$. Hence $F(\underline{r}) \Rightarrow |f^{\Sigma}| : [\![\eta_F\{\underline{A}\}]\!] \mapsto [\![f\{\underline{B}\}]\!]$ for any $\underline{r}$, namely $\forall^n \underline{X}(F(\underline{X}) \Rightarrow |f^{\Sigma}|) : \eta_F \mapsto f$. Then for any $a \in \Sigma^n F$, $|f^{\Sigma}| : [\![a\{\Sigma^n F\}\eta_F]\!] \mapsto [\![a\{C\}f]\!]$, since $\Sigma^n$ is parametric. From this, $\eta_F^{\Sigma} = id_{\Sigma^n F}$ is easily obtained.

Next prove $(\kappa^n(g) \circ \eta_F)^{\Sigma} = g$ for all $g \in \Sigma^n F \Rightarrow C$. For any $\underline{A} \in \mathcal{U}^n$, $id_{F(\underline{A})} \Rightarrow |g| : [\![\eta_F\{\underline{A}\}]\!] \mapsto [\![(\kappa^n(g) \circ \eta_F)\{\underline{A}\}]\!]$. As in the first half, we obtain $\forall^n \underline{X}(F(\underline{X}) \Rightarrow |g|) : \eta_F \mapsto \kappa^n(g) \circ \eta_F$. Since $\Sigma^n$ is parametric, $|g| : [\![a\{\Sigma^n F\}\eta_F]\!] \mapsto [\![a\{C\}(\kappa^n(g) \circ \eta_F)]\!]$ for any $a \in \Sigma^n F$. From this we have $g \circ \eta_F^{\Sigma} = (\kappa^n(g) \circ \eta_F)^{\Sigma}$. With the result of the first half, $g = (\kappa^n(g) \circ \eta_F)^{\Sigma}$ is obtained.

iii) Since $\Sigma^n \dashv \kappa^n$ at $F$, there is a natural bijection $\forall^n \underline{X}(F(\underline{X}) \Rightarrow A) \overset{\varphi_A}{\cong} \Sigma^n F \Rightarrow A$ where $\varphi_A = (\text{-})^{\Sigma}$. Suppose $\forall^n \underline{X}(F(\underline{X}) \Rightarrow |f|) : g \mapsto h$ where $f : A \to B$ is an arrow of $\mathbf{C}_\xi$. Then for any $\underline{C} \in \mathcal{U}^n$, $F(\underline{id_C}) \Rightarrow |f| : [\![g\{\underline{C}\}]\!] \mapsto [\![h\{\underline{C}\}]\!]$. Since $F$ is assumed to preserve identities, we have $f \circ [\![g\{\underline{C}\}]\!] = [\![h\{\underline{C}\}]\!]$ and thus $\kappa^n(f) \circ g = h$. By the naturality of $\varphi$, $f \circ g^{\Sigma} = h^{\Sigma}$. Then for any $a \in \Sigma^n F$, $|f| : [\![a\{\underline{A}\}g]\!] \mapsto [\![a\{\underline{B}\}h]\!]$, which shows $a$ is parametric w.r.t. $\mathcal{R}_{Ar}$. □

**Corollary 4.16.** *If $\xi$ is parametric w.r.t. $\mathcal{R}_{Ar}$, $\Sigma^n \dashv \kappa^n \dashv \Pi^n$ holds.* □

**Remark 4.17.** i) We write $\exists(X_1 \cdots X_n).F(\underline{X})$ for $\Sigma^n F$. Note that in general

$$\exists(X_1 \cdots X_n).F(\underline{X}) \not\cong \exists X_1 \cdots \exists X_n.F(\underline{X})$$

ii) $\kappa$ and $\Pi$ can be extended to $\kappa^{m,n} : [\mathbf{C}_\xi^m \to \mathbf{C}_\xi] \to [\mathbf{C}_\xi^n \to \mathbf{C}_\xi]$ and $\Pi^{n,m} : [\mathbf{C}_\xi^n \to \mathbf{C}_\xi] \to [\mathbf{C}_\xi^m \to \mathbf{C}_\xi]$ for any $m$ and $n$ with $m < n$, so that the adjunction $\kappa^{m,n} \dashv \Pi^{n,m}$ holds. For $\Sigma^{n,m}$, however, the situation is more delicate. The natural design for representing $\Sigma^{n,m}$ is

$$\begin{aligned} \Sigma^{n,m} F = \forall T^{Type^m \Rightarrow Type}(&\forall X_{m+1} \cdots \forall X_n \\ &(F(X_1, \ldots, X_n) \Rightarrow T(X_1, \ldots, X_m)) \Rightarrow T(X_1, \ldots, X_m)) \end{aligned}$$

where $T^{Type^m \Rightarrow Type}$ is a third order variable. Thus we need a higher order calculus even for representing a second order object with some parameters. The development of higher order parametric models are left to the future.

iii) Nonetheless, if $\xi$ is a parametric model for the system with *full comprehension*,

$$\Sigma^{n,m} F = \forall Y(\forall X_{m+1} \cdots \forall X_n(F(X_{m+1}, \ldots, X_n) \Rightarrow Y) \Rightarrow Y)$$

works as a left adjoint of $\kappa^{m,n}$. □

*4.4. Initial and terminal fixed points*

Let $\mathbf{C}$ be a category and $F : \mathbf{C} \to \mathbf{C}$ an endofunctor. $F$-**Alg** is the category whose objects are pairs $(A, f) = FA \xrightarrow{f} A$ (called *$F$-algebras*) and whose arrows $g : (A, f) \to (A', f')$ are arrows $g : A \to A'$ of $\mathbf{C}$ making the diagram

$$\begin{tikzcd}
FA \arrow[r, "Fg"] \arrow[d, "f"'] & FA' \arrow[d, "f'"] \\
A \arrow[r, "g"'] & A'
\end{tikzcd}$$

commute. If, for an $F$-algebra $(A, f)$, $f$ is an isomorphism, then $A$ is called a *fixed point* of $F$.

**Theorem 4.18.** *If $(A, f)$ is an initial $F$-algebra (i.e., an initial object of $F$-**Alg**), then $A$ is a fixed point of $F$.* $\square$

The proof is found in [Lambek 1968, Smyth and Plotkin 1982]. So we will call the object obtained from an initial $F$-algebra an *initial fixed point* of $F$.

Initial fixed points in category theory correspond to inductively defined sets in set theory. For example, the set of natural numbers is inductively defined as the least fixed point of operator $A(X) = \{0\} \dot{+} X$ where $X$ is a set variable and $\dot{+}$ is disjoint sum. Translating $\{0\}$ into a terminal object $\top$, and $\dot{+}$ into coproduct $+$, we obtain an endofunctor $F(X) = \top + X$. Then an initial fixed point of $F$ should represent a natural numbers object.

**Remark 4.19.** For the functor $FX = \top + X$, consider a *weakly initial* $F$-algebra $\top + N \xrightarrow{[z,s]} N$ (if any). We can simulate recursion in $N$. For example, $\mathtt{plus} : N \Rightarrow N \Rightarrow N$ is definable as an arrow from $\top$ to $N \Rightarrow N \Rightarrow N$. However recursive functions so defined may fail to have desirable properties. For example, the equation $\lambda x^N \lambda y^N \mathtt{plus}(x)(y) = \lambda x^N \lambda y^N \mathtt{plus}(y)(x)$ does not necessarily hold. It is because weak initiality ensures only existence of certain arrows, not uniqueness. We need therefore an *initial* $F$-algebra to obtain recursive functions with such desirable properties as the above equation (indeed initiality means the induction principle in some sense [Lehmann and Smyth 1981]). $\square$

From now on, we assume that always $\mathcal{R}_0 = \mathcal{R}_{Ar}$. $F$ a member of Obj $[\mathbf{T}^n_{Ar} \to \mathbf{T}]$ is said to be a *functor* if both i) $F$ preserves $\mathcal{R}_{Ar}$, namely, for any $\underline{r} \in \mathcal{R}^n_{Ar}$ it holds that $F(\underline{r}) \in \mathcal{R}_{Ar}$; ii) $F|\mathcal{R}^n_{Ar} : \mathcal{R}^n_{Ar} \to \mathcal{R}_{Ar}$ behaves as a functor, namely, $F(\underline{id_A}) = id_{F(\underline{A})}$ and $F(\underline{|g \circ f|}) = F(\underline{|g|}) \circ F(\underline{|f|})$. (N.B. The RHS of the last equation is a composition of relations, where, for $r : A \rightharpoonup B$ and $s : B \rightharpoonup C$, $s \circ r : A \rightharpoonup C$ is $\{a \mapsto c \mid a \in A, c \in C$, and $\exists b \in B, r : a \mapsto b \,\&\, s : b \mapsto c\}$. Note, if $r$ is $|f|$ and $s$ is $|g|$, then $|g \circ f| = |g| \circ |f|$.)

Remark that a functor in Obj $[\mathbf{T}^n_{Ar} \to \mathbf{T}]$ corresponds to an actual functor from $\mathbf{C}^n_\xi$ to $\mathbf{C}_\xi$.

**Definition 4.20.** A functor $F \in$ Obj $[\mathbf{T}^n_{Ar} \to \mathbf{T}]$ is *universally strong* iff it has a *universal* *strength*, a member of $\forall^n X_1 \cdots X_n \forall^n Y_1 \cdots Y_n((X_1 \Rightarrow Y_1) \Rightarrow \cdots \Rightarrow (X_n \Rightarrow Y_n) \Rightarrow (F(\underline{X}) \Rightarrow F(\underline{Y})))$ (the strength is also denoted by $F$), such that for any $f_i \in A_i \Rightarrow B_i$ $(i = 1 \ldots n)$ it holds that $|F(\underline{f})| = F(|\underline{f}|)$ where $F(\underline{f})$ in the LHS is an abbreviation of $[\![F\{A_1, \ldots, A_n\}\{B_1, \ldots, B_n\} f_1 \cdots f_n]\!]$ and $F$ in the RHS is an object of $[\mathbf{T}^n_{Ar} \to \mathbf{T}]$. $\Box$

The last definition is stronger than that of usual strong (or $\mathbf{C}_\xi$-) functors because the strength is in the universally quantified form.

**Theorem 4.21.** *Suppose $\mathcal{R}_0 = \mathcal{R}_{Ar}$. If $F \in \forall^n \underline{X} \forall^n \underline{Y}((\underline{X \Rightarrow Y}) \Rightarrow (F(\underline{X}) \Rightarrow F(\underline{Y})))$ is a universal strength of a functor, then the strength $F$ is self-related.*

*Proof.* Let $f_i \in A_i \Rightarrow B_i$ $(i = 1, \ldots, n)$. By Lemma 4.3, $|f_i| \Rightarrow |g_i| : h_i \mapsto h'_i$ means $g_i \circ h_i = h'_i \circ f_i$. Since $F$ is universally strong, $F(\underline{g}) \circ F(\underline{h}) = F(\underline{h'}) \circ F(\underline{f})$, namely, $|F(\underline{f})| \Rightarrow |F(\underline{g})| : F(\underline{h}) \mapsto F(\underline{h'})$. By the definition of functor, we obtain $F(\underline{|f|}) \Rightarrow F(\underline{|g|}) : F(\underline{h}) \mapsto F(\underline{h'})$. Therefore $\forall^n \underline{X} \forall^n \underline{Y}((\underline{X \Rightarrow Y}) \Rightarrow (F(\underline{X}) \Rightarrow F(\underline{Y}))) : F \mapsto F$. $\Box$

**Remark 4.22.** In this proof the assumption $\mathcal{R}_0 = \mathcal{R}_{Ar}$ is essential, since a functor determines only the operations on the graphs of functions. Therefore working in the system with basic comprehension is necessary. To make the following argument valid for the models of the full comprehension system, in which $\mathcal{R}_0 = \mathcal{R}$, we must enforce the considered universally strong functors to have an additional condition that their universal strengths are self-related. $\Box$

Representation of inductive definition in the fashion of $\forall X((F(X) \Rightarrow X) \Rightarrow X)$ is well-known. In [Böhm and Berarducci 1985], it is showed that all free heterogeneous algebras without generators (or only with parameters) are representable. In [Reynolds 1984], the case $F(X)$ is $\forall X((X \Rightarrow B) \Rightarrow B)$, where $B$ is the type of boolean values $\forall X(X \Rightarrow X \Rightarrow X)$, is used to prove that second order polymorphic lambda calculus has no full set-theoretical model. For more general formulation, we refer the reader to [Pfenning and Paulin-Mohring 1990].

**Definition 4.23.** Given an endofunctor $\langle F_1, \ldots, F_n \rangle : \mathbf{C}^n_\xi \to \mathbf{C}^n_\xi$ where $F_i : \mathbf{C}^n_\xi \to \mathbf{C}_\xi$, we denote the $i$-th component of an initial fixed point of $\langle F_1, \ldots, F_n \rangle$ by $\mu_i F_1 \cdots F_n$ or $\mu_i \underline{F}$. We use the same notation for the object part of a *weakly* initial $\langle F_1, \ldots, F_n \rangle$-algebra. $\Box$

**Theorem 4.24.** *Suppose, given a natural model, type constructors $F_1, \ldots, F_n \in \mathcal{K}$ all of arity n are interpreted as universally strong functors,*

*i) the objects represented by*

$$\mu_i \underline{F} = \forall X_1 \cdots \forall X_n((F_1(\underline{X}) \Rightarrow X_1) \Rightarrow \ldots \Rightarrow (F_n(\underline{X}) \Rightarrow X_n) \Rightarrow X_i)$$

*provide a weakly initial $\langle F_1, \ldots, F_n \rangle$-algebra.*

*ii) $\langle \mu_1 \underline{F}, \ldots, \mu_n \underline{F} \rangle$ is an initial fixed point of $\langle F_1, \ldots, F_n \rangle$ if and only if all the $\mu_i \underline{F}$ are parametric w.r.t. $\mathcal{R}_{Ar}$.*

*Proof.* To obtain a weakly initial $\langle F_1, \ldots, F_n \rangle$-algebra from $\langle \mu_1 \underline{F}, \ldots, \mu_n \underline{F} \rangle$, we must define *constructors* $\gamma_i : F_i(\mu_1 \underline{F}, \ldots, \mu_n \underline{F}) \to \mu_i \underline{F}$ $(i = 1, \ldots, n)$ and $\langle f_1, \ldots, f_n \rangle^{\mu_i} : \mu_i \underline{F} \to A_i$ $(i = 1, \ldots, n)$ for every $\langle f_1, \ldots, f_n \rangle$ where $f_i : F_i(A_1, \ldots, A_n) \to A_i$, so that a diagram

$$
\begin{tikzcd}
F_i(\underline{\mu F}) \arrow[r, "F_i(\langle \underline{f} \rangle^{\mu})"] \arrow[d, "\gamma_i"'] & F_i(\underline{A}) \arrow[d, "f_i"] \\
\mu_i \underline{F} \arrow[r, "\langle \underline{f} \rangle^{\mu_i}"'] & A_i
\end{tikzcd}
$$

commutes for each $i$. Put

$$
\begin{aligned}
\langle f_1, \ldots, f_n \rangle^{\mu_i} &= [\![\lambda x^{\mu_i \underline{F}} x\{A_1\} \cdots \{A_n\} f_1 \cdots f_n]\!] \\
\gamma_i &= [\![\lambda x^{F_i(\underline{\mu F})} \Lambda X_1 \cdots \Lambda X_n \lambda y_1^{F_1(\underline{X}) \Rightarrow X_1} \cdots \lambda y_n^{F_n(\underline{X}) \Rightarrow X_n} \\
&\qquad y_i(F_i(\langle \underline{y} \rangle^{\mu_1}, \ldots, \langle \underline{y} \rangle^{\mu_n})x)]\!]
\end{aligned}
$$

where $\langle \underline{y} \rangle^{\mu_i}$ stands for the defining term of $\langle y_1, \ldots, y_n \rangle^{\mu_i}$. The commutativity of the diagram follows immediately by reductions.

ii) A proof is more or less parallel to that of Theorem 4.5 for coproducts, in this case using Lemma 4.3. For example, the if-part in the case of $n = 1$ is proved by first showing $\gamma^{\mu} = id_{\mu F}$ and second showing $k \circ \gamma^{\mu} = f^{\mu}$ for any $f : FA \to A$ and $k : \mu F \to A$. (N.B. $\gamma : F(\mu F) \to \mu F$ and $f^{\mu} : \mu F \to A$.) $\square$

The strategy of proving ii) in the above theorem is very similar to those in [Bainbridge et al. 1990, Freyd 1989, Lambek 1989, Wadler 1990].

We can consider the dual of initial fixed points. Let $F$-**Coalg** be the category of $F$-coalgebras whose objects are pairs $(A, f) = f : A \to FA$ and whose arrows $g : (A, f) \to (A', f')$ are given by commutative diagrams

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
FA \arrow[r, "Fg"'] & FA'
\end{tikzcd}
.
$$

We call $A$ of a terminal $F$-algebra $(A, f)$ a *terminal fixed point* of $F$. Indeed a terminal object of $F$-**Coalg**, $(A, f)$, gives an isomorphism $FA \cong A$, as proved dually to the case of initial fixed points.

**Definition 4.25.** The $i$-th component of a terminal fixed point of $\langle F_1, \ldots, F_n \rangle$ (or the object part of a weakly terminal $\langle F_1, \ldots, F_n \rangle$-algebra) is denoted by $\nu_i F_1 \cdots F_n$ or $\nu_i \underline{F}$. $\square$

In the following argument, for simplicity, we assume $\mathbf{C}_{\xi}$ has products. The representation in the fashion of

$$\nu F = \exists X((X \Rightarrow F(X)) \times X)$$

was discovered independently by the author and Wraith. [Hasegawa 1989, Wraith 1989]. In general, $\nu F$ satisfies only the weak universal property as well as other representations. The next theorem gives us the condition for $\nu F$ to be a terminal fixed point.

**Theorem 4.26.** *Suppose, given a natural model, type constructors $F_1, \ldots, F_n \in \mathcal{K}$ all of arity $n$ are interpreted as universally strong functors,*

*i) the objects represented by*

$$\nu_i \underline{F} = \exists(X_1, \ldots, X_n)((X_1 \Rightarrow F_1(\underline{X})) \times \cdots \times (X_n \Rightarrow F_n(\underline{X})) \times X_i)$$

*provide a weakly terminal $\langle F_1, \ldots, F_n\rangle$-algebra.*

*ii) $\langle \nu_1 \underline{F}, \ldots, \nu_n \underline{F}\rangle$ is a terminal fixed point if and only if $\Sigma^n = \exists(X_1, \ldots, X_n)(\text{-})(X_1, \ldots, X_n)$ is a colimiting quantifier of $n$ functors $G_i$ $(i = 1, \ldots, n)$ where $G_i(X_1, \ldots, X_n)$ is put as $(X_1 \Rightarrow F_1(\underline{X})) \times \cdots \times (X_n \Rightarrow F_n(\underline{X})) \times X_i$.*

*Proof.* i) We have to define $\delta_i : \nu_i \underline{F} \to F_i(\nu_1 \underline{F}, \ldots, \nu_n \underline{F})$, and $\langle f_1, \ldots, f_n\rangle^{\nu_i} : A_i \to \nu_i \underline{F}$ for every $n$ arrows $f_i : A_i \to F_i(A_1, \ldots, A_n)$ $(i = 1, \ldots, n)$ so that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\langle \underline{f}\rangle^{\nu_i}} & \nu_i \underline{F} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ F_i(\underline{A}) & \xrightarrow[F_i(\underline{\langle \underline{f}\rangle^{\nu}})]{} & F_i(\underline{\nu \underline{F}}) \end{array}$$

commutes. Put

$$\langle f_1, \ldots, f_n\rangle^{\nu_i} = [\![\lambda x^{A_i} \eta_{G_i}\{A_1\} \cdots \{A_n\} \langle f_1, \ldots, f_n, x\rangle]\!]$$

where $\eta_{G_i} \in [\mathbf{C}^n_\xi \to \mathbf{C}](G_i, \kappa^n \Sigma^n G_i)$ is as given in the subsection 4.3. And put

$$\delta_i = ([\![\Lambda^n X \lambda \langle f_1, \ldots, f_n, z\rangle^{G_i(\underline{X})} F_i(\langle \underline{f}\rangle^{\nu_1}, \ldots, \langle \underline{f}\rangle^{\nu_n})(f_i z)]\!])^{\Sigma}$$

where $(\text{-})^{\Sigma}$ is as defined in the subsection 4.3. Then the commutativity of the diagram is proved, only using that $\Sigma^n$ is a weak left adjoint of $\kappa^n$ and that $\times$ is a weak product.

ii) For simplicity, assume $n = 1$ and write $\nu F$ and $\delta$ omitting subscripts.

$(\Rightarrow)$ We have to show two things: $\eta_G$ is a relational cone, i.e., parametric w.r.t. $\mathcal{R}_{Ar}$, and $\nu F = \exists X((X \Rightarrow F(X)) \times X) = \forall Y(\forall X(((X \Rightarrow F(X)) \times X) \Rightarrow Y) \Rightarrow Y)$ is parametric w.r.t. $\mathcal{R}_{Ar}$. For the former, suppose $g : (A, f) \to (A', f')$ is an arrow of $F$-**Coalg**. Since

$\nu F$ is terminal, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f^\nu"] \arrow[d, "g"'] & \nu F \\ A' \arrow[ur, "f'^\nu"'] & \end{tikzcd},$$

i.e., $[\![\eta_G\{A\}\langle f,a\rangle]\!] = [\![\eta_G\{A'\}\langle f',g(a)\rangle]\!]$ for all $a \in A$. From this, it is easily seen that $\eta_G$ is parametric w.r.t. $\mathcal{R}_{Ar}$.

For the latter, consider $\delta^\nu : \nu F \to \nu F$. It can be lifted to an arrow of category $F$-**Coalg** and since $\nu F$ is terminal, it must be equal to $id_{\nu F}$. Hence for any $a \in \nu F$,

$$\begin{aligned} a &= [\![\delta^\nu(a)]\!] \\ &= [\![\Lambda Y \lambda y^{\forall X(G(X)\Rightarrow Y)} y\{\nu F\}\langle \delta, a\rangle]\!] . \end{aligned}$$

Since $\xi$ is supposed to be a natural model, the Abstraction theorem shows,

$$a\{A\} \xmapsto{\forall X((G(X)\Rightarrow r)\Rightarrow r)} a\{B\}$$

for any $r : A \multimap B$ in $\mathcal{R}_{Ar}$. Therefore $\nu F$ is parametric w.r.t. $\mathcal{R}_{Ar}$.

There is disguise in this argument: the constant $a$ does not always satisfy $[\![\vdash \mathrm{Type}(a)]\!] : a \mapsto a$, so an assumption of the Abstraction Theorem fails. But if one looks at the usage of the theorem closely, he will find what is required for $a$ is only that $id_{\nu F} : a \mapsto a$.

($\Leftarrow$) Let us first prove $\delta^\nu = id_{\nu F}$. For any $f : A \to F(A)$ and any $a \in A$,

$$\langle f, a\rangle \xmapsto{G(|f^\nu|)} \langle \delta, f^\nu(a)\rangle .$$

Since $\eta_G$ is parametric w.r.t. $\mathcal{R}_{Ar}$,

$$\begin{aligned} \eta_G\{A\}\langle f,a\rangle &= \eta_G\{\nu F\}\langle \delta, f^\nu(a)\rangle \\ &= \delta^\nu(\eta_G\{A\}\langle f,a\rangle) , \end{aligned}$$

namely, $\eta_G = \kappa^n(\delta^\nu) \circ \eta_G$. Since $\eta_G$ is the unit of adjunction of $\Sigma^n \dashv \kappa^n$, we have $id_{\nu F} = \delta^\nu$.

Next, for any $g : (A, f) \to (\nu F, \delta)$ in the category $F$-**Coalg** and any $a \in A$, as in the first half, $\eta_G\{A\}\langle f,a\rangle = \delta^\nu(g(a))$, that is, $f^\nu = \delta^\nu \circ g$. Combining it with the first half, we have $f^\nu = g$. Namely $\nu F$ is a terminal fixed point. □

Recalling that $\Pi^n$ is a limiting quantifier of $F$ iff $\forall^n \underline{X}.F(\underline{X})$ is parametric, we should notice this theorem is just the dual of Theorem 4.24.

**Corollary 4.27.** *Suppose $\xi$ is a natural model which is parametric w.r.t. $\mathcal{R}_{Ar}$, and $F_1, \ldots, F_n \in \mathrm{Obj}\,[\mathbf{T}^n_{Ar} \to \mathbf{T}]$ are all universally strong functors. Then $\langle F_1, \ldots, F_n\rangle : \mathbf{C}^n_\xi \to \mathbf{C}^n_\xi$ has both initial and terminal fixed points.* □

**Remark 4.28.** i) Terminal fixed points are used to define "lazy" types. See, for example, [Wraith 1989].

ii) What do the representations

$$\mu F = \forall X((F(X) \Rightarrow X) \Rightarrow X)$$
$$\nu F = \exists X((X \Rightarrow F(X)) \times X)$$

mean as logical formulas? Logicians would read them

$$\mu F = (\forall X \text{ such that } F(X) \Rightarrow X).X$$
$$\nu F = (\exists X \text{ such that } X \Rightarrow F(X)).X\,.$$

Furthermore regarding $\forall$ as inf, $\exists$ as sup, and $\Rightarrow$ as order relation $\leq$,

$$\mu F = \bigwedge\{X \mid F(X) \leq X\}$$
$$\nu F = \bigvee\{X \mid X \leq F(X)\}\,.$$

In the context of complete lattices, these are the expressions defining the least and the greatest fixed point of a monotonic function $F$, in the Tarski fixed point theorem [Tarski 1955]. In fact, initial fixed points are introduced first by Lambek [Lambek 1968] as a categorical generalization of the Tarski fixed points. □

## 5. Examples of Parametric Models

In this section, we give two kinds of parametric models to which the theorems in the previous section are applicable. The first is complete Heyting algebras that turn out to be parametric by a trivial reason.

**Example 5.1.** Every complete Heyting algebra (cHa) $\Omega$ [Fourman and Scott 1979] is made into a parametric model of the second order lambda calculus with basic comprehension (even the calculus with full comprehension). To each $A, B \in \Omega$, we associate exactly one relation $r_{A,B} : A \twoheadrightarrow B$ non-empty only when $A = B = \top$, in which case $r_{\top,\top} : * \mapsto *$ where $*$ is a only point of $\top$. Then we collect all $r_{A,B}$ to define $\mathcal{R}$, and only those satisfying $A \leq B$ to define $\mathcal{R}_0$. The remainder of the model construction is left to the reader. See also the beginning of Section 3. $\Omega$ is parametric since all inhabited closed types are interpreted to be $\top$ and an only element of $\top$ is self-related by $r_{\top,\top}$. Therefore the theorems in Section 4 are applicable. In particular, $\mu F = \forall X((FX \Rightarrow X) \Rightarrow X)$ and $\nu F = \forall X((X \Rightarrow FX) \times X)$ recapture Tarski fixed points [Tarski 1955]. □

Another example of parametric model is obtained modifying a standard per model. Categories of partial equivalence relations (shorthand, per) are well-known to be models of not only higher order lambda calculus [Girard 1972, Mitchell 1986], but its extension including dependent types, known as the calculus of constructions [Luo 1988, Longo and Moggi 1987]. We provide a parametric per model on natural numbers $N$, which model is the same as the standard one except in interpretation of $\forall$. Our model is very similar to that in [Bainbridge et al. 1990]. In both models, the interpretation of variable types operates on binary relations as well as pers. In their model, the operation on binary relations is a map from the set of all binary relations, while the operation in our model is from the set of graphs of functions at the expense of interpreting not the full comprehension system, but only the basic comprehension fragment. If one looks for models of the full calculus, their model provides the one. The reason we choose our model of the weak fragment is first that basic comprehension suffices to prove the existence of limits and colimits, and second that the class of universally strong functors are identical with the class of internal functors that reside in Hyland's effective topos. Note that the initial fixed point construction in [Bainbridge et al. 1990] is valid only for syntactically definable functors.

A *per* $A$ on the set of natural numbers $N$ is a subset of $N \times N$ which is, regarded as a binary relation, symmetric and transitive, but not always reflexive. We write $m\ A\ n$ for $(m,n) \in A$, and $n \in \mathit{Dom}(A)$ for $n\ A\ n$, in the latter case of which we say $n$ is in the *domain* of $A$. Then $A$ is an equivalence relation on $\mathit{Dom}(A)$. The equivalence class of $m$ is denoted by $[m]_A$.

Suppose $\varphi$ is an enumeration of the partial recursive functions so that $\varphi_e$ denotes the $e$-th partial recursive function. For pers $A$ and $B$, $A \Rightarrow B$ is a per such that

$$e\ (A \Rightarrow B)\ f \quad \text{iff} \quad \varphi_e(m)\ B\ \varphi_f(n) \text{ for all } m\ A\ n.$$

Then $[e]_{A\Rightarrow B}$ determines a function from $\mathit{Dom}(A)/A$ into $\mathit{Dom}(B)/B$, written $[\varphi_e] : A \to B$, by $[m]_A \mapsto [\varphi_e(m)]_B$. Defining hom-sets as $\mathrm{Hom}(A,B) = \{[\varphi_e] \mid e \in \mathit{Dom}(A \Rightarrow B)\}$, we obtain a category $\mathbf{Per}(N)$. It is easy to show this category is a ccc, but the result is also obtained at the end of this subsection from the fact that the pers on $N$ are to be a parametric natural model.

A parametric natural model $\xi_{\mathrm{Per}(N)}$ is defined as follows;

- $\mathcal{U}$ is the set of all pers on $N$.
- For $A \in \mathcal{U}$, $D_A$ is a set of equivalence classes, $\{[n]_A \mid n \in \mathit{Dom}(A)\}$.
- An element $r : A \multimap B$ of $\mathcal{R}$ is a subset of $N \times N$ (we write $r : m \mapsto n$ for $(m,n) \in r$) which satisfies both

  1 (strictness) $r : m \mapsto n$ only when $m \in \mathit{Dom}(A)$ and $n \in \mathit{Dom}(B)$,

  2 (closedness under equivalence) $m\ A\ m'$, $n\ B\ n'$, and $r : m \mapsto n$ imply $r : m' \mapsto n'$.

  Namely $r$ is a binary relation between the equivalence classes modulo $A$ and those modulo $B$. $id_A : A \multimap A$ for $A \in \mathcal{U}$ is defined as; $id_A : m \mapsto n$ iff $m\ A\ n$.
- For $r : A \multimap B$ in $\mathcal{R}$, $D_r : D_A \multimap D_B$ is a relation defined as

$$[m]_A \xmapsto{D_r} [n]_B \quad \text{iff} \quad m \xmapsto{r} n\,.$$

  By closedness under equivalence, this definition is independent from the choice of representatives.
- $\mathcal{R}_0$ consists of those relations $r : A \multimap B$ such that $r = \{m \mapsto n \mid \varphi_e(m)\ B\ n\}$ for some $[\varphi_e] \in \mathrm{Hom}(A,B)$. Denote such a relation $r$ by $|e|$, or precisely $|[e]_{A\Rightarrow B}|$, which latter we do not use since it is a little tedious.
- $\mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$ is the set of the r-frame morphisms which preserve identities. Then $[\mathbf{T}_0^n \to \mathbf{T}]$ is the full r-subframe of $\mathbf{R}(\mathbf{T}_0^n, \mathbf{T})$, namely, if $q : F \multimap G$ is in $\mathrm{Rel}(\mathbf{R}(\mathbf{T}_0^n, \mathbf{T}))$ and $F, G \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$, then $q$ belongs to $\mathrm{Rel}\,[\mathbf{T}_0^n, \mathbf{T}]$.
- Let the signature $\Sigma = (\mathcal{K}, \mathcal{C})$ be any pair such that if $F \in \mathcal{K}$ and the arity of $F$ is $n$, then $F \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$, and if $c \in \mathcal{C}$ and $\mathrm{Type}(c) = A$, then $I(c)$ is an equivalence class $[m]_A$ for some $m \in \mathit{Dom}(A)$.

- $A \Rightarrow B$ for $A, B \in \mathcal{U}$ is a per as defined above. For $r : A \multimap A'$ and $s : B \multimap B'$ in $\mathcal{R}_0$, $r \Rightarrow s : (A \Rightarrow B) \to (A' \Rightarrow B')$ is defined as: $m \xmapsto{r \Rightarrow s} m'$, iff, $m \in Dom(A \Rightarrow B)$, $m' \in Dom(A' \Rightarrow B')$, and $\varphi_m(n) \xmapsto{s} \varphi_{m'}(n')$ for all $n$ and $n'$ such that $r : n \mapsto n'$.
- For $F \in \mathrm{Obj}\,[\mathbf{T}_0 \to \mathbf{T}]$, $\forall F$ is a per defined as $m\ (\forall F)\ n$ iff for all $r : A \multimap B$ in $\mathcal{R}_0$, it holds that $F(r) : m \mapsto n$. For $q : F \multimap F'$ in $\mathrm{Rel}\,[\mathbf{T}_0 \to \mathbf{T}]$, $\forall q : \forall F \multimap F'$ is defined as; $\forall q : m \mapsto m'$ iff $m \in Dom(\forall F)$, $m' \in Dom(\forall F')$, and $q(r) : m \mapsto m'$ for all $r \in \mathcal{R}_0$.

**Lemma 5.2.** *For $F$ an element of $\mathrm{Obj}\,[\mathbf{T}_0 \to \mathbf{T}]$, $m\ (\forall F)\ n$ iff*

$$\begin{array}{ll} m\ F(A)\ n & \textit{for all } A \in \mathcal{U}, \textit{ and} \\ m \xmapsto{F(r)} m & \textit{for all } r \in \mathcal{R}_0. \end{array}$$

*Proof.* Assume $m\ (\forall F)\ n$, i.e., $F(r) : m \mapsto n$ for all $r \in \mathcal{R}_0$. Taking $id_A$ as $r$, $F(id_A) : m \mapsto n$, and thus $m\ F(A)\ n$ since $F$ preserves identities. With $F(r) : m \mapsto n$, we obtain $F(r) : m \mapsto m$ from closedness under equivalence.

The converse direction is immediate from closedness under equivalence. □

In the ordinary per model, $\forall F$ is interpreted, written $\forall' F$ for the moment, as $m\ \forall' F\ n$ iff $m\ F(A)\ n$ for all per $A$. The lemma above shows that $\{[m]_{\forall F} \mid m \in Dom(\forall F)\}$ is a subset of $\{[m]_{\forall' F} \mid m \in Dom(\forall' F)\}$. But the author does not know whether the former is a "proper" subset. By [Freyd 1987, Bainbridge et al. 1990], at least the interpretation of natural number types $\forall X.((X \Rightarrow X) \Rightarrow X \Rightarrow X)$ and $\forall' X.((X \Rightarrow X) \Rightarrow X \Rightarrow X)$ coincide. In fact, both include only those elements representing the Church numerals.

**Proposition 5.3.** *(i) $A \Rightarrow B$ is in $\mathcal{U}$ for $A, B \in \mathcal{U}$, and $r \Rightarrow s$ is in $\mathcal{R}$ for $r, s \in \mathcal{R}$.*

*(ii) $\forall F$ is in $\mathcal{U}$ for $F \in \mathrm{Obj}\,[\mathbf{T}_0 \to \mathbf{T}]$, and $\forall q$ is in $\mathcal{R}$ for $q \in \mathrm{Rel}\,[\mathbf{T}_0 \to \mathbf{T}]$.*

*Proof.* (i) Left to the reader.

(ii) $\forall F \in \mathcal{U}$: immediate from Lemma 5.2

$\forall q \in \mathcal{R}$: let $q$ be $F \multimap F'$. Strictness is clear from definition, and in order to prove closedness under equivalence, put $m\ (\forall F)\ n$, $m'\ (\forall F')\ n'$, and $\forall q : m \mapsto m'$. Then for any $A$, $A'$ and $r : A \multimap A'$,

$$\left\{ \begin{array}{ll} m\ F(A)\ n & \text{since } F(id_A) = id_{F(A)}. \\ m \xmapsto{q(r)} n & \\ m'\ F'(A')\ n' & \text{since } F'(id'_A) = id_{F'(A')}. \end{array} \right.$$

Then $q(r) : m' \mapsto n'$, and thus $\forall q : m' \mapsto n'$ since $r$ is arbitrary. □

The definition of $\xi_{\mathrm{Per}(N)}$ is continued.

- $\Phi^{\Rightarrow}_{A,B} : D_{A \Rightarrow B} \to (D_A \to D_B)$ is defined as $\Phi^{\Rightarrow}_{A,B}([e]_{A \Rightarrow B}) = [\varphi_e]$. This definition is independent from the choice of a representative $e$, by the definition of $A \Rightarrow B$.
- $\Phi^{\forall}_F : D_{\forall F} \to \Pi X \in \mathcal{U}.D_{F(X)}$ is defined as $\Phi^{\forall}_F([m]_{\forall F})$ sending $X$ to $[m]_{F(X)}$. Well-definedness comes from the assumption that $F$ preserves identities.

**Proposition 5.4.** *$\Phi^{\Rightarrow}_{A,B}$ and $\Phi^{\forall}_F$ are all injective in $\xi_{\mathrm{Per}}(N)$.*

*Proof.* There is no problem for $\Phi^{\Rightarrow}_{A,B}$. For $\Phi^{\forall}_F$, suppose $[m]_{\forall F} \neq [n]_{\forall F}$. Then $[m]_{F(A)} \neq [n]_{F(A)}$ for some $A \in \mathcal{U}$ by the lemma 5.2, and thus $\Phi^{\forall}_F$ is one to one. □

Hence $[\![\,]\!]$ is determined uniquely if it exists, and it surely exists, constructed as follows.

It is convenient to divide the construction of $[\![\,]\!]$ into two steps. The first step, though left afterwards, is to define $p = [\Gamma; \Theta \vdash M : \tau]$ as a $|\Theta|$-ary partial recursive function such that

$$(\dagger) \qquad p(\underline{m}) \xmapsto{[\![\Gamma\vdash\tau]\!](\underline{r})} p(\underline{n})$$

for any $\underline{r} : \underline{A} \Rightarrow \underline{B}$ in $\mathcal{R}_0$, and $\underline{m}$, $\underline{n}$ such that $[\![\Gamma \vdash \mathrm{Type}\,\Theta]\!](\underline{r}) : \underline{m} \mapsto \underline{n}$. Second $t = [\![\Gamma; \Theta \vdash M : \tau]\!]$ is defined by

$$t(\underline{A})([m]_{[\![\Gamma\vdash\mathbf{Type}\,\Theta]\!](\underline{X})}) = [p(\underline{m})]_{[\![\Gamma\vdash\tau]\!](\underline{X})}$$

as an element of $\Pi X \in \mathcal{U}^{|\Gamma|}.([\![\Gamma \vdash \mathrm{Type}\,\Theta]\!](\underline{X}) \to [\![\Gamma \vdash \tau]\!](\underline{X}))$. It is easily seen that $t$ is independent from the choice of representatives, by regarding $r$ in the definition of $p$ as $A$ $(= id_A)$.

$[\Gamma; \Theta \vdash M : \tau]$ is defined by induction on the derivation of $M$. For simplicity, we write $p_M$ for $[\Gamma; \Theta \vdash M : \tau]$.

- (te proj) $p_{x_i}$ is the $i$-th projection when $x_i$ is the $i$-th element of $\Theta$.
- (te const) $p_c$ is the constant function $\lambda\underline{x}.n$ where $[n]_{\mathrm{Type}(c)} = c$.
- (te $\Rightarrow$I) $p_{\lambda x^\sigma.M}$ is defined using the S-m-n theorem [Soare 1987]. In symbols, $p_{\lambda x^\sigma.M}(\text{-}) = S_1^{|\Theta|}(e, \text{-})$ where $e$ is the code of $p_M$. Note the construction of $p_M$ is completely constructive, thus there is an effective way to obtain a code from $p_M$.
- (te $\Rightarrow$E) $p_{MN}(\text{-}) = \varphi_{p_M(\text{-})}(p_N(\text{-}))$.
- (te $\forall$I) $p_{\Lambda X.M} = p_M$.
- (te $\forall$E) $p_{M\{Y\}} = p_M$ and $p_{M\{C\}} = p_M$.

We have to check the condition $(\dagger)$. Only the case (te $\forall$E) is treated. The induction assumption says $[\![\Gamma \vdash \forall X.\tau]\!](\underline{r}) : p_M(\underline{m}) \mapsto p_M(\underline{n})$ for all $[\![\Gamma \vdash \mathrm{Type}\,\Theta]\!](\underline{r}) : \underline{m} \mapsto \underline{n}$. Thus for all $r' \in \mathcal{R}_0$, $[\![\Gamma, X \vdash \tau]\!](\underline{r}, r') : p_M(\underline{m}) \mapsto p_M(\underline{n})$. Since

$$[\![\Gamma \vdash \tau[X := C]]\!](\underline{r}) = [\![\Gamma, X \vdash \tau]\!](\underline{r}, [\![\Gamma \vdash C]\!](\underline{r})) = [\![\Gamma, X \vdash \tau]\!](\underline{r}, id_C)\,,$$

we obtain $[\![\Gamma \vdash \tau[X := C]]\!](\underline{r}) : p_{M\{C\}}(\underline{m}) \mapsto p_{M\{C\}}(\underline{n})$. The proof is similar for $p_{M\{Y\}}$, but it does not extend to full comprehension, since in $[\![\Gamma, X \vdash \tau]\!](\underline{r}, [\![\Gamma \vdash \sigma]\!](\underline{r}))$, $[\![\Gamma \vdash \sigma]\!](\underline{r})$ is not always an element of $\mathcal{R}_0$.

It is straightforward to check Condition 3.4.

**Proposition 5.5.** *i) $\Phi^{\Rightarrow}$ and $\Phi^{\forall}$ are natural.*

*ii) $\xi_{\mathrm{Per}}(N)$ is parametric.*

*Proof.* Immediate from the definition of $\Rightarrow$, $\forall$, $\Phi^{\Rightarrow}$, and $\Phi^{\forall}$. $\square$

Now we have finished showing $\xi_{\mathbf{Per}(N)}$ is a parametric natural model.

The objects and arrows of $\mathbf{C}_{\xi_{\mathbf{Per}(N)}}$ bijectively correspond to those of category $\mathbf{Per}(N)$. In fact, objects of $\mathbf{C}_{\xi_{\mathbf{Per}(N)}}$ and of $\mathbf{Per}(N)$ are both pers. $\mathbf{C}_{\xi_{\mathbf{Per}(N)}}(A, B)$ is the set of $[e]_{A\Rightarrow B}$, and $\mathbf{Per}(N)(A, B)$ is the set of $[\varphi_e] : A \to B$. They are isomorphic with correspondence $[e]_{A\Rightarrow B} \mapsto [\varphi_e]$. Therefore we can apply the results in the previous section to $\mathbf{Per}(N)$, remarking that $\mathcal{R}_0$ is just $\mathcal{R}_{Ar}$.

**Theorem 5.6.** $\mathbf{Per}(N)$ *is a ccc with finite coproducts.* $\square$

Of course this theorem is obtained more easily by directly constructing finite products, exponentiation, and finite coproducts.

An endofunctor is universally strong in $\mathbf{C}_{\mathrm{Per}(N)}$ just when it is an internal functor in the effective topos. See [Hyland 1988] for how to regard $\mathbf{Per}(N)$ as an internal category.

**Theorem 5.7.** *Every universally strong endofunctor on* $\mathbf{Per}(N)$ *has initial and terminal fixed points.* □

A similar result of this theorem, for initial fixed points, is seen in [Freyd 1989, Bainbridge et al. 1990]. In [Freyd 1989], it is proved that all realizable endofunctor has an initial fixed point, where a universally strong functor is called realizable if further $F(1) = 1$ holds.

There is an example of a functor $F$ universally strong but not realizable. $F$ sends a per on $N$ to a per on $N \cup \{-1\}$ (it is easy to modify the definition to obtain one on $N$). For a per $A$ on $N$, the per $FA$ on $N \cup \{-1\}$ is defined the same as $A$ with respect to the members of $N$. $-1$ is in the domain of $FA$ iff $0 \in Dom(A)$ and 1 is not equivalent with 2 modulo $A$, in which case $-1$ is added in the equivalence class of 0. As for functions between pers, partial recursive functions $f$ are mapped to $Ff$ that extends $f$ by $Ff(-1) = 0$. This $F$ turns out to be a universal strength of a functor. However $F$ is not realizable, since considering pers $A = \{\{0\}, \{1\}, \{2\}\}$ and $B = \{\{0\}, \{1, 2\}\}$ and $1 : A \to B$, the identity function 1 cannot code a function from $FA = \{\{-1, 0\}, \{1\}, \{2\}\}$ to $FB = \{\{0\}, \{1, 2\}\}$.

The author does not know, at the present, any example of an endofunctors that is not universally strong and has no initial or terminal fixed points.

## 6. Appendix: Conditions on $[\mathbf{T}_0^n \to \mathbf{T}]$

To interpret all types, we claim $[\mathbf{T}_0^n \to \mathbf{T}]$ $(n \geq 0)$ satisfies;

0. closed under exchanges of arguments: if $F \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$, then $F \circ \mu \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$ where $\mu : \mathbf{T}_0^n \to \mathbf{T}_0^n$ is any permutation. And for $q : F \rightarrow\!\!\!\!\!\rightarrow G$ in $\mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$, $q \circ \mu \in \mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$.
1. including projections: The projection $P^{(n)}$ defined as $P^{(n)}(A_1, \ldots, A_n) = A_n$, $P^{(n)}(r_1, \ldots, r_n) = r_n$ is an object of $[\mathbf{T}_0^n \to \mathbf{T}]$. And $id_{P^{(n)}}$ is a relation of $[\mathbf{T}_0^n \to \mathbf{T}]$.
2. closed under the addition of dummy variables: if $F \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$ $(n \geq 0)$ then $\kappa F \in \mathrm{Obj}\,[\mathbf{T}_0^{n+1} \to \mathbf{T}]$ where $\kappa F(X_1, \ldots, X_n, X_{n+1}) = F(X_1, \ldots, X_n)$, $X_i$ standing for both objects and relations. As for relations of r-frame morphisms, if $q \in \mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$ $(n \geq 0)$ then $\kappa q \in \mathrm{Rel}\,[\mathbf{T}_0^{n+1} \to \mathbf{T}]$ where $\kappa q(r_1, \ldots, r_n, r_{n+1}) = q(r_1, \ldots, r_n)$.
3. closed under argumentwise instantiations: if $F \in \mathrm{Obj}\,[\mathbf{T}_0^{n+1} \to \mathbf{T}]$, then $\lambda X_1 \cdots X_n. F(X_1, \ldots, X_n, A) \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$. And for $q : F \rightarrow\!\!\!\!\!\rightarrow G$ in $\mathrm{Rel}\,[\mathbf{T}_0^{n+1} \to \mathbf{T}]$ and $r : A \rightarrow\!\!\!\!\!\rightarrow B$ in $\mathcal{R}_0$, $\lambda r_1 \cdots r_n.q(r_1, \ldots, r_n, r) \in \mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$.
4. closed under $\Rightarrow$: if $F, G \in \mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$, then $F \Rightarrow G$ is also in $\mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$. And if $p, q \in \mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$, then $p \Rightarrow q \in \mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$.
5. closed under $\tilde{\forall}$: if $F \in \mathrm{Obj}\,[\mathbf{T}_0^{n+1} \to \mathbf{T}]$, then $\tilde{\forall} F$ is also in $\mathrm{Obj}\,[\mathbf{T}_0^n \to \mathbf{T}]$. And if $q \in \mathrm{Rel}\,[\mathbf{T}_0^{n+1} \to \mathbf{T}]$, then $\tilde{\forall} q \in \mathrm{Rel}\,[\mathbf{T}_0^n \to \mathbf{T}]$.

**Remark 6.1.** A "model-constructor" might feel so imposed, for he/she has to construct all $[\mathbf{T}_0^n \to \mathbf{T}]$ and to check many conditions. But in many cases, $[\mathbf{T}_0^n \to \mathbf{T}]$ would be $\mathbf{R}(\mathbf{T}_0^n, \mathbf{T})$ (maximum), or the collection of those which are representable as $[\![\Gamma \vdash \sigma]\!]$ (minimum). Moreover as showed in Corollary 3.17 and the following remark, the collection of all relation maps preserving identities is sometimes available. $\square$

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