

APPENDIX B

Symmetric Norms and Banach Ideals

In this appendix we present a detailed account on the theory of Calkin and Gohberg-Krein of operator ideals and operator ideals associated to symmetric norms. These ideals play an important role in noncommutative geometry. This is already illustrated by Chapter ?? and the construction of the Dixmier trace on the Dixmier ideal $\mathcal{L}^{(1,\infty)}$.

Most of the material of this appendix is taken from [GK] and [Si] (see also [Co, Chap. 4, Appendix C], [GVF, Section 7.C]).

B.1. Banach Ideals

This section is devoted to presenting the primary definitions and properties of Banach ideals. We start with basic facts about two-sided ideals in $\mathcal{L}(\mathcal{H})$.

PROPOSITION B.1. *Let \mathcal{I} be a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

(1) *For any $T \in \mathcal{L}(\mathcal{H})$,*

$$T \in \mathcal{I} \iff |T| \in \mathcal{I} \iff T^* \in \mathcal{I}.$$

(2) *Any $T \in \mathcal{I}$ can be written as*

$$T = T_1 - T_2 + i(T_3 - T_4) \quad \text{with } T_j \in \mathcal{I} \cap \mathcal{L}(\mathcal{H})_+.$$

PROOF. Once (1) is proved the proof of (2) follows along the same lines as that of the proof of Lemma ?.?. Thus, we only have to prove (1).

Let $T \in \mathcal{L}(\mathcal{H})$ have polar decomposition $T = U|T|$. If $|T|$ is in \mathcal{I} then, as \mathcal{I} is an ideal, $T = U|T|$ is in \mathcal{I} too. Since by Proposition ?? $|T| = U^*T$ we also see that if $|T|$ is in \mathcal{I} , then so is T .

It also follows from Proposition ?? that $T^* = U^*TU$, and $T = (U^*TU^*)^* = UT^*U$. Therefore T is in \mathcal{I} if and only if T^* is in \mathcal{I} . The proof is complete. \square

PROPOSITION B.2. *Let \mathcal{I} be a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*

(1) *If $\mathcal{I} \supseteq \{0\}$, then every finite-rank operator is contained in \mathcal{I} .*

(2) *If $\mathcal{I} \subsetneq \mathcal{L}(\mathcal{H})$, then every operator in \mathcal{I} is compact.*

PROOF. Assume $\mathcal{I} \supseteq \{0\}$. Since the finite-rank operators are linear combinations of rank 1 operators $\xi \otimes \eta^*$, $\xi, \eta \neq 0$, in order to prove (1) it is enough to show that any such projection is contained in \mathcal{I} .

Let $\xi, \eta \in \mathcal{H} \setminus \{0\}$ and let $T \in \mathcal{I} \setminus \{0\}$. Since $T \neq 0$ there exists $\xi' \in \mathcal{H} \setminus \{0\}$ such that $\eta' := T\xi' \neq 0$. Set $A = \xi \otimes \xi'^*$ and $B = \eta' \otimes \eta^*$. Then the operator ATB is contained in \mathcal{I} and is equal to $(\xi \otimes \eta'^*)T(\xi \otimes \eta^*) = \langle \eta', T\xi' \rangle (\xi \otimes \eta^*) = \|\eta'\|^2 (\xi \otimes \eta^*)$. Since $\eta' \neq 0$ it follows that $\xi \otimes \eta^*$ is contained in \mathcal{I} , proving (1).

Suppose now that \mathcal{I} contains a non-compact operator T . By Proposition ?? and Proposition B.1 the operator $|T|$ too is non-compact and contained in \mathcal{I} . Therefore,

possibly by replacing T by $|T|$, we may assume T positive. For $\lambda > 0$ set $\Pi_\lambda = 1_{[\lambda, \infty)}(T)$. If $g(t) := t^{-1}1_{[\lambda, \infty)}$, then $g(T)$ is a bounded operator. As $\Pi_\lambda = Tg(T)$ it follows that Π_λ is contained in \mathcal{I} .

As $\|T - T\Pi_\lambda(T)\| = \|1_{[0, \lambda)}(T)\| \leq \lambda$, we see that $T\Pi_\lambda(T)$ converges to T in norm as $\lambda \rightarrow 0^+$. Since T is non-compact, it follows there is at least one $\lambda > 0$ such that $T\Pi_\lambda$ does not have finite rank. Then Π_λ does not have finite rank.

Let $(\xi_n)_{n \geq 0}$ be an orthonormal basis of \mathcal{H} , let $(\eta_n)_{n \geq 0}$ be an orthonormal basis of $\text{im } \Pi_\lambda$, and let $V \in \mathcal{L}(\mathcal{H})$ be such that $V\xi_n = \eta_n$. As $V^*\eta_n = \xi_n$ for all $n \in \mathbb{N}$, we see that $V^*\Pi_\lambda V = 1$. Thus $1 \in \mathcal{I}$, which implies that $\mathcal{I} = \mathcal{L}(\mathcal{H})$. Therefore, if $\mathcal{I} \subsetneq \mathcal{L}(\mathcal{H})$, then \mathcal{I} cannot contain any non-compact operator, i.e., T is contained in \mathcal{K} . The proof is complete. \square

The previous proposition shows that, among non-trivial ideals of $\mathcal{L}(\mathcal{H})$, the ideal of finite-rank operators is minimal and the ideal of compact operators is maximal. Since the former is the closure of the latter in $\mathcal{L}(\mathcal{H})$ we obtain:

COROLLARY B.1. *The only closed non-trivial two-sided ideal of $\mathcal{L}(\mathcal{H})$ is \mathcal{K} .*

DEFINITION B.1. *A Banach ideal is a two-ideal \mathcal{I} of $\mathcal{L}(\mathcal{H})$ which is equipped with a norm $\|\cdot\|_{\mathcal{I}}$ such that*

- (i) \mathcal{I} is a Banach space for \mathcal{I} .
- (ii) We have

$$(B.1.1) \quad \|ATB\|_{\mathcal{I}} \leq \|A\| \|T\|_{\mathcal{I}} \|B\| \quad \forall T \in \mathcal{I} \quad \forall A, B \in \mathcal{L}(\mathcal{H}).$$

EXAMPLE B.2. $\mathcal{L}(\mathcal{H})$ and \mathcal{K} are Banach ideals for the operator norm $\|\cdot\|$.

EXAMPLE B.3. It follows from (??) and Proposition ?? that \mathcal{L}^1 is a Banach ideal for the norm $\|\cdot\|_1$. Likewise, using (??) and Proposition ??, we see that \mathcal{L}^2 is a Banach ideal for the Hilbert-Schmidt norm $\|\cdot\|_2$.

In the sequel we let \mathcal{I} be a Banach ideal with norm $\|\cdot\|_{\mathcal{I}}$. We assume \mathcal{I} non-trivial, so by Proposition B.2 all the finite-rank operators are contained in \mathcal{I} and all the elements of \mathcal{I} are compact operators.

LEMMA B.1. *Let $T \in \mathcal{I}$ and let $S \in \mathcal{K}$.*

- (i) *If $\mu_n(S) \leq \mu_n(T) \forall n \in \mathbb{N}_0$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.*
- (ii) *If $\mu_n(S) = \mu_n(T) \forall n \in \mathbb{N}_0$, then $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}$.*

PROOF. We only have to prove (i), since it implies (ii). Thus, let us assume that $\mu_n(S) \leq \mu_n(T) \forall n \in \mathbb{N}_0$, and let $T = U|T|$ and $S = V|S|$ be the respective polar decompositions of T and S . Let $(\xi_n)_{n \geq 0}$ and $(\eta_n)_{n \geq 0}$ be orthonormal families in \mathcal{H} such that $|T|\xi_n = \mu_n(T)\xi_n$ and $|S|\eta_n = \mu_n(S)\eta_n$ for all $n \in \mathbb{N}_0$. Let $C \in \mathcal{L}(\mathcal{H})$ be such that $C = 0$ on $\ker |S|$ and $C\eta_n = (\sqrt{\mu_n(S)}/\sqrt{\mu_n(T)})\xi_n$ for all $n \in \mathbb{N}_0$ such that $\mu_n(T) > 0$ (i.e., ξ_n is in $\text{im } |S| = (\ker |S|)^\perp$). This defines bounded operator of norm ≤ 1 , since by assumption $\mu_n(S) \leq \mu_n(T)$ for all $n \in \mathbb{N}_0$. As $C^*\xi_n = (\sqrt{\mu_n(S)}/\sqrt{\mu_n(T)})\eta_n$ for all $n \in \mathbb{N}_0$, we see that $C^*|T|C = |S|$.

By Proposition ?? we know that $|T| = U^*T$, so we have

$$VC^*U^*TC = VC^*|T|C = V|S| = S.$$

Therefore S is contained in \mathcal{I} and, by (B.1.1), we have $\|S\|_{\mathcal{I}} \leq \|V\| \|C^*\| \|U\| \|T\|_{\mathcal{I}} \|C\|$. Since the operator norms of U , V and C are ≤ 1 , it follows that $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$, as claimed. \square

Combining this lemma with (??) and (??) we see that, for any $T \in \mathcal{I}$,

$$\begin{aligned} \|T\|_{\mathcal{I}} &= \|\|T\|_{\mathcal{I}}\|_{\mathcal{I}} = \|T^*\|_{\mathcal{I}}, \\ \|U^*TU\|_{\mathcal{I}} &= \|T\|_{\mathcal{I}} \quad \forall U \in \mathcal{L}(\mathcal{H}), U \text{ unitary.} \end{aligned}$$

PROPOSITION B.3. *There exists a constant $c > 0$ such that*

$$(B.1.2) \quad \|T\|_{\mathcal{I}} = c\|T\| \quad \forall T \in \mathcal{R}_1.$$

Furthermore, we have

$$(B.1.3) \quad c\|T\| \leq \|T\|_{\mathcal{I}} \quad \forall T \in \mathcal{I}.$$

PROOF. Let $R \in \mathcal{R}_1$ be such that $\|R\| = 1$ and set $c = \|R\|_{\mathcal{I}}$. It follows from (??) that $\mu_0(R) = \|R\| = 1$ and $\mu_n(R) = 0$ for $n \geq 1$. Likewise, if $S \in \mathcal{R}_1$, then $\mu_0(S) = \|S\|$ and $\mu_n(S) = 0$ for $n \geq 1$, so the operators S and $\|S\|R$ have the same characteristic values. Lemma B.1 then implies that $\|S\|_{\mathcal{I}} = \|S\|\|R\|_{\mathcal{I}} = c\|S\|$.

Let $T \in \mathcal{I}$. Then $\mu_0(T) = \|T\| = \mu_0(\|T\|R)$ and $\mu_n(T) \geq 0 = \mu_n(\|T\|R)$ for $n \geq 1$, so by Lemma B.1 we have $\|T\|_{\mathcal{I}} \geq \|(\|T\|R)\|_{\mathcal{I}} = c\|T\|$, as claimed. \square

Because the norm $\|\cdot\|_{\mathcal{I}}$ on rank-one operators is constant, we sometimes require the normalization,

$$(B.1.4) \quad \|T\|_{\mathcal{I}} = \|T\| \quad \text{for any operator } T \text{ of rank 1.}$$

In this case, the inequality (B.1.3) holds with $c = 1$.

PROPOSITION B.4. *Any other Banach norm on \mathcal{I} satisfying (B.1.1) is equivalent to $\|\cdot\|_{\mathcal{I}}$.*

PROOF. Let $\|\cdot\|'_{\mathcal{I}}$ be another Banach norm on \mathcal{I} satisfying (B.1.1) and let $|\cdot|_{\mathcal{I}}$ be the norm on \mathcal{I} defined by

$$|T|_{\mathcal{I}} := \sup\{\|T\|_{\mathcal{I}}, \|T\|'_{\mathcal{I}}\} \quad \forall T \in \mathcal{I}.$$

Let $(T_n)_{n \geq 0}$ be a Cauchy sequence in $(\mathcal{I}, |\cdot|_{\mathcal{I}})$, i.e., it is a Cauchy sequence both in $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ and $(\mathcal{I}, \|\cdot\|'_{\mathcal{I}})$. It thus converges in $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ and in $(\mathcal{I}, \|\cdot\|'_{\mathcal{I}})$. The limits may be different. However, using (B.1.3) we see that $(T_n)_{n \geq 0}$ is a Cauchy sequence in $(\mathcal{L}(\mathcal{H}), \|\cdot\|)$ and its limit in $(\mathcal{L}(\mathcal{H}), \|\cdot\|)$ agrees with the limits in $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ and $(\mathcal{I}, \|\cdot\|'_{\mathcal{I}})$. Thus, the last two limits are equal and $(T_n)_{n \geq 0}$ converges in $(\mathcal{I}, |\cdot|_{\mathcal{I}})$. This shows that $(\mathcal{I}, |\cdot|_{\mathcal{I}})$ is a Banach space.

Notice that the identity map is continuous from $(\mathcal{I}, |\cdot|_{\mathcal{I}})$ to $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$. Since this is a bijection and both $(\mathcal{I}, |\cdot|_{\mathcal{I}})$ to $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ are Banach spaces, the open mapping theorem insures us that its inverse is continuous. Therefore $|\cdot|_{\mathcal{I}}$ and $\|\cdot\|_{\mathcal{I}}$ are equivalent norms. Likewise, the norms $|\cdot|_{\mathcal{I}}$ and $\|\cdot\|'_{\mathcal{I}}$ are equivalent, so $\|\cdot\|_{\mathcal{I}}$ and $\|\cdot\|'_{\mathcal{I}}$ are equivalent norms, proving the proposition. \square

As we shall now see the separability of the topology of \mathcal{I} defined by the norm $\|\cdot\|_{\mathcal{I}}$ is intimately related to the density of finite-rank operators.

DEFINITION B.4. \mathcal{I}^0 is the closure in \mathcal{I} of the the ideal \mathcal{R}_{∞} of finite-rank operators.

Since \mathcal{R}_{∞} is a two-sided ideal, \mathcal{I}^0 can easily be seen to be a Banach ideal for the norm of \mathcal{I} .

Let $T \in \mathcal{K}$ have polar decomposition $T = U|T|$ and let $(\xi_n)_{n \geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. Then, by Proposition ??,

$$(B.1.5) \quad T = \sum_{n \geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*,$$

where the series converges in \mathcal{K} . Any series of the form (B.1.5) is called a *Schmidt series* for T .

LEMMA B.2. *Let $T \in \mathcal{K}$. Then the following are equivalent:*

- (i) T is contained in \mathcal{I}^0 .
- (ii) Any Schmidt series for T converges in \mathcal{I} to T .
- (iii) There is a Schmidt series for T which converges in \mathcal{I} .

PROOF. It is clear that (ii) implies (iii). Moreover, if there is a Schmidt series for T converging in \mathcal{I} then, as it converges to T in \mathcal{K} , using (B.1.3) we see that its sum is equal to T . Thus T is contained in \mathcal{I} and is the sum of a series of finite-rank operators, hence T is an element of \mathcal{I}^0 .

Suppose now that T is in \mathcal{I} . For any $N \in \mathbb{N}$ set

$$T_N := T - \sum_{n < N} \mu_n(T)(U\xi_n) \otimes \xi_n^* = \sum_{n \geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^*.$$

As $\sum_{n < N} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ has rank $\geq N$, it is immediate that

$$(B.1.6) \quad \|T_N\|_{\mathcal{I}} \geq \inf \left\{ \|T - R\|_{\mathcal{I}}; R \in \mathcal{R}_N \right\}.$$

It follows from Proposition ?? the operator U^*U is the orthogonal projection onto $(\ker T)^\perp = (\ker |T|)^\perp = \text{im } |T|$, so $U^*U\xi_n = \xi_n$ if $\mu_n(T) \neq 0$. Thus,

$$(T_N)^*T_N = \sum_{n \geq N} \mu_n(T)^2 \xi_n \otimes \xi_n^* \quad \text{and} \quad |T_N| = \sum_{n \geq N} \mu_n(T) \xi_n \otimes \xi_n^*.$$

Using the min-max principle we then see that

$$(B.1.7) \quad \mu_n(T_N) = \mu_{n+N}(T) \quad \forall n \in \mathbb{N}.$$

Let $R \in \mathcal{R}_N$. Then (??) implies that $\mu_N(R) = 0$, so using (??) we get

$$\mu_n(T_N) = \mu_{n+N}(T) \leq \mu_n(T - R) + \mu_N(R) = \mu_n(T - R).$$

Therefore, applying Lemma B.1 we see that $\|T_N\|_{\mathcal{I}} \leq \|T - R\|_{\mathcal{I}}$ for all $R \in \mathcal{R}_N$. Combining this with (B.1.6) then shows that

$$\left\| \sum_{n \geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^* \right\|_{\mathcal{I}} = \inf \left\{ \|T - R\|_{\mathcal{I}}; R \in \mathcal{R}_N \right\}.$$

This implies that T is a limit of finite-rank operators in \mathcal{I} (i.e., T is in \mathcal{I}^0) if and only if the Schmidt series $\sum_{n \geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ converges to T in \mathcal{I} . The proof is complete. \square

LEMMA B.3. *The Banach ideal \mathcal{I}^0 is separable.*

PROOF. Without any loss of generality we may assume that in (B.1.2)–(B.1.3) the constant c is equal to 1. Let $(\zeta_k)_{k \geq 0}$ be a countable dense subset of \mathcal{H} . Let

$\xi, \eta \in \mathcal{H}$. For any $\epsilon > 0$ there exist $k, l \in \mathbb{N}_0$ such that $\|\xi - \zeta_k\| \leq \epsilon$ and $\|\eta - \zeta_l\| \leq \epsilon$. Then using (B.1.2) we get

$$(B.1.8) \quad \begin{aligned} \|\xi \otimes \eta^* - \zeta_k \otimes \zeta_l^*\|_{\mathcal{I}} &\leq \|(\xi - \zeta_k) \otimes \eta^*\|_{\mathcal{I}} + \|\zeta_k \otimes (\eta - \zeta_l)^*\|_{\mathcal{I}} \\ &\leq \|(\xi - \zeta_k) \otimes \eta^*\| + \|\zeta_k \otimes (\eta - \zeta_l)^*\| \\ &\leq \|\xi - \zeta_k\| \|\eta\| + \|\zeta_k\| \|\eta - \zeta_l\| \leq \epsilon \|\eta\| + (\epsilon + \|\xi\|)\epsilon. \end{aligned}$$

Let \mathcal{D} be the set of operators of the form,

$$\sum_{(k,l) \in K \times L} \zeta_k \otimes \zeta_l^*,$$

where K and L range over all finite subsets of \mathbb{N}_0 . Then \mathcal{D} is a countable subset of \mathcal{R}_∞ . As any operator in \mathcal{R}_∞ is a finite sum of rank one operators $\xi \otimes \eta^*$, it follows from (B.1.8) that, for any $T \in \mathcal{R}_\infty$ and for any $\epsilon > 0$, there exists $R \in \mathcal{D}$ such that $\|T - R\|_{\mathcal{I}} < \epsilon$. Combining this with the density of \mathcal{R}_∞ in \mathcal{I}^0 we deduce that \mathcal{D} is dense in \mathcal{I}^0 . Since \mathcal{D} is countable, this proves that \mathcal{I}^0 is separable. \square

PROPOSITION B.5. *The following are equivalent:*

- (1) *The finite-rank operators are dense in \mathcal{I} , i.e., $\mathcal{I} = \mathcal{I}_0$.*
- (2) *\mathcal{I} is separable.*

PROOF. It immediately follows from Lemma B.3 that if $\mathcal{I} = \mathcal{I}^0$ then \mathcal{I} is separable.

Conversely, suppose that $\mathcal{I}_0 \subsetneq \mathcal{I}$. Let $T \in \mathcal{I} \setminus \mathcal{I}_0$. Since \mathcal{I} and \mathcal{I}_0 both are ideals, using Proposition B.1 we see that $|T|$ is in \mathcal{I} , but is not in \mathcal{I} . Therefore, possibly by replacing T by $|T|$ we may assume T positive.

Since T is in $\mathcal{I} \setminus \mathcal{I}_0$ and is positive, Lemma B.2 implies that there is a Schmidt series $\sum_{n \geq 0} \mu_n(T) \xi_n \otimes \xi_n^*$ which does not converges in \mathcal{I} . As \mathcal{I} is a Banach space, this implies that the series does not satisfy Cauchy's criterion, so there exists $\delta > 0$ and an increasing sequence $(n_k)_{k \geq 0} \subset \mathbb{N}_0$ such that

$$(B.1.9) \quad \left\| \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* \right\|_{\mathcal{I}} \geq \delta \quad \forall k \in \mathbb{N}_0.$$

For any sequence $a = (a_k)_{k \geq 0} \in \{0, 1\}^{\mathbb{N}_0}$ we set

$$T_a := \sum_{k=0}^{\infty} a_k \left(\sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* \right) = \sum_{a_k \neq 0} \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^*.$$

If we let Π_a be the orthogonal projection onto the closure of the vector space spanned by $\bigcup_{a_k \neq 0} \{\xi_n; n_k \leq n < n_{k+1}\}$, then $T_a = \Pi_a T$. Therefore, the operator T_a is in \mathcal{I} .

Let $b = (b_k)_{k \geq 0} \in \{0, 1\}^{\mathbb{N}_0}$ be such that $b \neq a$, i.e., there exists $k \in \mathbb{N}_0$ such that $b_k \neq a_k$. Set $\Pi_k = \sum_{n=n_k}^{n_{k+1}-1} \xi_n \otimes \xi_n^*$. Then

$$\Pi_k(T_b - T_a) = (b_k - a_k) \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^* = \pm \sum_{n_k \leq n < n_{k+1}} \mu_n(T) \xi_n \otimes \xi_n^*.$$

Combining this with (B.1.9) we get

$$\delta \leq \|\Pi_k(T_b - T_a)\|_{\mathcal{I}} \leq \|\Pi_k\| \|T_b - T_a\|_{\mathcal{I}} = \|T_b - T_a\|_{\mathcal{I}}.$$

Since $\{0, 1\}^{\mathbb{N}}$ is not countable, it follows that no countable subset of \mathcal{I} can be dense, so \mathcal{I} is not separable if $\mathcal{I}_0 \subsetneq \mathcal{I}$. Equivalently, if \mathcal{I} is separable, then $\mathcal{I} = \mathcal{I}_0$. The proof is complete. \square

The following result shows that, among the non-trivial Banach ideals, the ideal \mathcal{L}^1 of trace-class operators is minimal.

PROPOSITION B.6. *There is a continuous inclusion,*

$$\mathcal{L}^1 \subset \mathcal{I}^0.$$

In fact, if the normalization (B.1.4) holds, then

$$(B.1.10) \quad \|T\|_{\mathcal{I}} \leq \|T\|_1 \quad \forall T \in \mathcal{I}.$$

PROOF. We may assume that the normalization (B.1.4) holds, so that we can take $c = 1$ in (B.1.3). Let $T \in \mathcal{L}^1$ and let $\sum_{n \geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ be a Schmidt series for T as in (B.1.5). Using (B.1.1) and (B.1.3) we see that, for all N and p in \mathbb{N} , we have

$$(B.1.11) \quad \left\| \sum_{N \leq n \leq N+p} \mu_n(T)(U\xi_n) \otimes \xi_n^* \right\|_{\mathcal{I}} \leq \sum_{N \leq n \leq N+p} \mu_n(T) \|U\| \|\xi_n \otimes \xi_n^*\|_{\mathcal{I}} \\ \leq \sum_{N \leq n \leq N+p} \mu_n(T) \|\xi_n \otimes \xi_n^*\| \leq \sum_{N \leq n \leq N+p} \mu_n(T).$$

Since $\sum_{n \geq 0} \mu_n(T) < \infty$ it follows that the series $\sum_{n \geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^*$ converges in \mathcal{I} . Lemma B.2 then insures us that T is contained in \mathcal{L}^0 and the Schmidt series converges to T in \mathcal{I} . Therefore, using (B.1.11), we get

$$\|T\|_{\mathcal{I}} = \left\| \sum_{n \geq 0} \mu_n(T)(U\xi_n) \otimes \xi_n^* \right\|_{\mathcal{I}} \leq \sum_{n \geq 0} \mu_n(T) = \|T\|_1.$$

This proves (B.1.10) when T is in \mathcal{L}^1 and shows there is a continuous inclusion of \mathcal{L}^1 in \mathcal{I}^0 . In addition, if $T \in \mathcal{I} \setminus \mathcal{L}^1$, then $\|T\|_1 = \infty$ and (B.1.10) holds trivially, so (B.1.10) holds for all $T \in \mathcal{I}$. The proof is complete. \square

B.2. Symmetric norms

In the sequel we denote by l_f the vector space of sequences $a = (a_n)_{n \geq 0}$ of complex numbers that have finite support (i.e., $a_n = 0$ for n large enough). We denote by l_0 the space of sequences $(a_n)_{n \geq 0}$ of complex numbers such that

$$\lim_{n \rightarrow \infty} a_n = 0.$$

For any sequence $a = (a_n)_{n \geq 0}$ in l_0 we denote by $\sigma(a) = (\sigma_N(a))_{N \geq 1}$ the sequence defined by

$$\sigma_N(a) := \sum_{n < N} a_n \quad \forall N \in \mathbb{N}.$$

In addition, for any $a \in l_0$ we denote by $a^* = (a_n^*)_{n \geq 0}$ the sequence defined by

$$a_n^* = \inf_{\substack{J \subset \mathbb{N}_0 \\ |J|=n}} \sup_{j \in J} |a_j| \quad \forall n \in \mathbb{N}_0.$$

In other word, the sequence $(a_n^*)_{n \geq 0}$ is the sequence obtained by re-ordering the sequence $(|a_n|)_{n \geq 0}$ into a non-increasing sequence. In particular, for any $N \in \mathbb{N}$, we always have

$$|\sigma_N(a_n)| \leq \sum_{n < N} |a_n| \leq \sigma_N(a^*).$$

It can also be shown (see [Si, Lem. 1.8]) that, for all $a, b \in l_f$,

$$(B.2.1) \quad \left| \sum_{n < N} a_n b_n \right| \leq \sum_{n < N} a_n^* b_n^* \quad \forall N \in \mathbb{N}.$$

DEFINITION B.5. Let Φ be a norm on l_f . We say that Φ is symmetric when

$$\Phi(a) = \Phi(a^*) \quad \forall a \in l_f.$$

REMARK B.6. It is not difficult to check that a norm Φ on l_f is symmetric if and only if it satisfies the following two conditions:

(i) For any sequence $(a_n)_{n \geq 0}$ in l_f and any bijection $\sigma : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, we have

$$\Phi((a_{\sigma(n)})_{n \geq 0}) = \Phi((a_n)_{n \geq 0}).$$

(ii) For any sequence $(a_n)_{n \geq 0}$ in l_f and any sequence $(\theta_n)_{n \geq 0} \subset [0, 2\pi)$, we have

$$\Phi((e^{i\theta_n} a_n)_{n \geq 0}) = \Phi((a_n)_{n \geq 0}).$$

EXAMPLE B.7. For $p \in [1, \infty)$ the p -norm Φ_p on l_f is defined by

$$\Phi_p(a) = \left(\sum_{n \geq 0} |a_n|^p \right)^{\frac{1}{p}} \quad \forall a = (a_n)_{n \geq 0} \in l_f.$$

For $p = \infty$ we define the Φ_∞ -norm by

$$(B.2.2) \quad \Phi_\infty(a) = \sup_{n \geq 0} |a_n| \quad \forall a = (a_n)_{n \geq 0} \in l_f.$$

All the p -norms are symmetric norms on l_f .

Let Φ be a symmetric norm on l_f .

LEMMA B.4 (Markus; see [GK, Lem. 3.1], [Si, Thm. 1.9]). Let $a, b \in l_f$. Then

$$(B.2.3) \quad \left(\sigma_N(a^*) \leq \sigma_N(b^*) \quad \forall N \in \mathbb{N} \right) \implies \Phi(a) \leq \Phi(b).$$

It follows from Markus' lemma that if $a_n^* \leq b_n^*$ for all $n \in \mathbb{N}_0$, then $\Phi(a) \leq \Phi(b)$. In particular, if $a = (a_n)_{n \geq 0}$ is a sequence in l_0 , then

$$\Phi(a_0, \dots, a_{N-1}, 0, 0, \dots) \leq \Phi(a_0, \dots, a_N, 0, 0, \dots) \quad \forall N \in \mathbb{N}.$$

This means that $(\Phi(a_0, \dots, a_N, 0, 0, \dots))_{N \geq 0}$ is a non-decreasing sequence of non-negative numbers, so it admits a limit as $N \rightarrow \infty$. We then set

$$(B.2.4) \quad \Phi(a) = \lim_{N \rightarrow \infty} \Phi(a_0, \dots, a_N, 0, 0, \dots) = \sup_{N \geq 1} \Phi(a_0, \dots, a_N, 0, 0, \dots)$$

This extends Φ to a function $\Phi : l_0 \rightarrow [0, \infty]$.

It is not hard to check that

$$(B.2.5) \quad \Phi(a) = 0 \implies a = 0,$$

$$(B.2.6) \quad \Phi(\lambda a) = |\lambda| \Phi(a) \quad \forall a \in l_0 \quad \forall \lambda \in \mathbb{C},$$

$$(B.2.7) \quad \Phi(a + b) \leq \Phi(a) + \Phi(b) \quad \forall a, b \in l_0.$$

In addition, we have

PROPOSITION B.7 (see [Si, Thm. 1.16]). *Let $a, b \in l_0$. Then*

$$\begin{aligned} \Phi(a) &= \Phi(a^*), \\ \left(\sigma_N(a^*) \leq \sigma_N(b^*) \quad \forall N \in \mathbb{N} \right) &\implies \Phi(a) \leq \Phi(b). \end{aligned}$$

It follows from Proposition B.7 that, for any $a, b \in l_0$,

$$\left(a_n^* \leq b_n^* \quad \forall n \in \mathbb{N}_0 \right) \implies \Phi(a) \leq \Phi(b).$$

In the sequel, we denote by l_f^+ the positive cone of l_f consisting of non-increasing sequences of non-negative numbers with finite supports.

LEMMA B.5 ([GK, Lem. 3.2]). *Let $\Phi : l_f^+ \rightarrow [0, \infty)$ be a function such that*

$$(B.2.8) \quad \Phi(a) = 0 \implies a = (0, 0, \dots),$$

$$(B.2.9) \quad \Phi(\lambda a) = \lambda \Phi(a) \quad \forall a \in l_f^+, \forall \lambda \geq 0,$$

$$(B.2.10) \quad \Phi(a + b) \leq \Phi(a) + \Phi(b) \quad \forall a, b \in l_f^+,$$

$$(B.2.11) \quad \left(\sigma_N(a) \leq \sigma_N(b) \quad \forall N \in \mathbb{N} \right) \implies \Phi(a) \leq \Phi(b).$$

Then Φ can be uniquely extend into a symmetric norm on l_f by letting

$$\Phi(a) := \Phi(a^*) \quad \forall a \in l_f.$$

Finally, let $\Phi' : l_f \rightarrow [0, \infty)$ be the function defined by

$$(B.2.12) \quad \Phi'(a) := \sup \left\{ \left| \sum_{n \geq 0} a_n b_n \right| ; b \in l_f, \Phi(b) \leq 1 \right\}$$

This is a norm on l_f called the *dual norm* of Φ . Using (B.2.1) and the fact that Φ is symmetric, we can check that

$$(B.2.13) \quad \Phi'(a) = \sup \left\{ \sum_{n \geq 0} a_n^* b_n ; b \in l_f^+, \Phi(b) \leq 1 \right\},$$

from which it follows that Φ' is a symmetric norm. It also implies that

$$(B.2.14) \quad \sum_{n \geq 0} a_n^* b_n^* \leq \Phi'(a) \Phi(b) \quad \forall a, b \in l_0.$$

LEMMA B.6 ([GK, Thm. 1.11]). *The dual norm of Φ' is equal to Φ , i.e., $(\Phi')' = \Phi$.*

REMARK B.8. Two norms Φ and Ψ on l_f are equivalent when there exists $c > 0$ such that

$$(B.2.15) \quad c^{-1} \Phi(a) \leq \Psi(a) \leq c \Phi(a) \quad \forall a \in l_f.$$

It is not hard to see that Φ and Ψ are equivalent if and only if their dual norms are equivalent.

EXAMPLE B.9. Let $p \in [1, \infty]$ and let $p' \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the dual norm of Φ_p is equal to $\Phi_{p'}$. This follows from the following simple facts:

- The Hölder inequality,

$$\left| \sum_{n \geq 0} a_n b_n \right| \leq \Phi_{p'}(a) \Phi_p(b) \quad \forall a, b \in l_f.$$

- If $p > 1$, the Hölder inequality is an equality if $b_n = \frac{\overline{a_n}}{|\overline{a_n}|} |a_n|^{\frac{p'}{p}} = \frac{\overline{a_n}}{|\overline{a_n}|} |a_n|^{p'-1}$ when $a_n \neq 0$ and $b_n = 0$ otherwise.
- If $p = 1$ the the Hölder inequality is an equality if $b_n = \frac{a_{n_0}}{a_{n_0}}$ for $n = n_0$ and $b_n = 0$ for $n \neq n_0$, where n_0 is such that $|a_{n_0}| = \Phi_\infty(a)$.

B.3. Banach ideals associated to symmetric norms

Let Φ be a symmetric norm on l_f . We shall also denote by Φ its extension to l_0 given by (B.2.4).

For any operator $T \in \mathcal{K}$, the sequence of characteristic values $\mu(T) := (\mu_n(T))_{n \geq 0}$ is an element of l_0 . Therefore, we can set

$$\|T\|_\Phi := \Phi(\mu(T)).$$

We then define

$$\mathcal{I}_\Phi := \left\{ T \in \mathcal{K}; \|T\|_\Phi < \infty \right\}.$$

For $T \in \mathcal{K}$ and $N \in \mathbb{N}$ we define

$$\mu^N(T) := (\mu_0(T), \dots, \mu_{N-1}(T), 0, 0, \dots) \in l_f.$$

Then by (B.2.4) we have

$$(B.3.1) \quad \|T\|_\Phi = \lim_{N \rightarrow \infty} \Phi(\mu^N(T)) = \sup_{N \geq 1} \Phi(\mu^N(T)).$$

In addition, as in Chapter ??, we also set

$$\sigma_N(T) := \sum_{n < N} \mu_n(T) = \sigma_N(\mu(T)).$$

We recall that by Proposition ??, for any $N \in \mathbb{N}$,

$$(B.3.2) \quad \sigma_N(S + T) \leq \sigma_N(S) + \sigma_N(T) \quad \forall S, T \in \mathcal{K}.$$

LEMMA B.7. *The following hold.*

(1) *Let $T \in \mathcal{K}$. Then*

$$(B.3.3) \quad \|T\|_\Phi = 0 \implies T = 0,$$

$$(B.3.4) \quad \|\lambda T\|_\Phi = |\lambda| \|T\|_\Phi \quad \forall \lambda \in \mathbb{C},$$

$$(B.3.5) \quad \|ATB\|_\Phi \leq \|A\| \|T\|_\Phi \|B\| \quad \forall A, B \in \mathcal{L}(\mathcal{H}).$$

(2) *Let $S, T \in \mathcal{K}$. Then*

$$\|S + T\|_\Phi \leq \|S\|_\Phi + \|T\|_\Phi.$$

(3) *If $\Phi(1, 0, 0, \dots) = 1$, then*

$$(B.3.6) \quad \|T\|_\Phi = \|T\| \quad \forall T \in \mathcal{R}_1,$$

$$(B.3.7) \quad \|T\| \leq \|T\|_\Phi \quad \forall T \in \mathcal{K}.$$

PROOF. The implication (B.3.3) is due to (B.2.5) and the fact that $\mu_0(T) = \|T\|$. We obtain (B.3.4) by using (??) and (B.2.6). The inequality (B.3.5) follows by combining (??) and (B.2).

Let $S, T \in \mathcal{K}$. Then (B.3.2) shows that $\sigma_N(\mu(S+T)) \leq \sigma_N(\mu(S) + \mu(T))$ for all $N \in \mathbb{N}$, so using Proposition B.7 and (B.2.7) we get

$$\|S+T\|_{\Phi} = \Phi(\mu(S+T)) \leq \Phi(\mu(S) + \mu(T)) \leq \Phi(\mu(S)) + \Phi(\mu(T)) = \|S\|_{\Phi} + \|T\|_{\Phi}.$$

Suppose now that $\Phi(1, 0, 0, \dots) = 1$ and let $T \in \mathcal{K}$. As $\mu^0(T) = \|T\|(1, 0, 0, \dots)$ we see that $\Phi(\mu^0(T)) = \|T\|\Phi(1, 0, 0, \dots) = \|T\|$. Since $\mu_n(T) \geq \mu_n^0(T)$ for all $n \in \mathbb{N}_0$, using (B.2) we see that $\|T\|_{\Phi} \geq \Phi(\mu^0(T)) = \|T\|$. Moreover, if $\text{rk } T = 1$ then $\mu(T) = \mu^0(T)$, and hence $\|T\|_{\Phi} = \Phi(\mu^0(T)) = \|T\|$. The lemma is proved. \square

PROPOSITION B.8. \mathcal{I}_{Φ} is a Banach ideal for $\|\cdot\|_{\Phi}$, i.e., \mathcal{I}_{Φ} is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|\cdot\|_{\Phi}$ is a Banach norm on \mathcal{I}_{Φ} satisfying (B.1.1).

PROOF. It follows from Lemma B.7 that \mathcal{I}_{Φ} is a two-sided ideal of $\mathcal{L}(\mathcal{H})$ and $\|\cdot\|_{\Phi}$ is a norm on \mathcal{I}_{Φ} satisfying (B.1.1). It remains to check that \mathcal{I}_{Φ} is complete for the norm $\|\cdot\|_{\Phi}$.

Let $(T_n)_{n \geq 0}$ be a Cauchy sequence in $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ and let us show that it converges in $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$. Then (B.3.7) implies that $(T_n)_{n \geq 0}$ is a Cauchy sequence in \mathcal{K} , hence converges in \mathcal{K} to some operator T .

Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\|T_p - T_q\|_{\Phi} < \epsilon$ when p and q are greater than n_0 . Let $N \in \mathbb{N}$ and denote by $e^N = (e_n^N)_{n \geq 0}$ the sequence such that $e_n^N = 1$ for $n < N$ and $e_n^N = 0$ for $n \geq N$. Let p and q be integers greater than n_0 . It follows from (??) that, for all $n \in \mathbb{N}_0$,

$$\mu_n^N(T - T_p) \leq \mu_n^N(T_p - T_q) + \|T - T_q\|_{\Phi} e_n^N.$$

Therefore, using (B.2.7) and (B.2) we see that

$$\begin{aligned} \Phi(\mu^N(T - T_p)) &\leq \Phi(\mu^N(T_p - T_q)) + \Phi(\|T - T_q\|_{\Phi} e^N) \\ &\leq \|T_p - T_q\|_{\Phi} + \|T - T_q\|_{\Phi} \Phi(e^N) \leq \epsilon + \|T - T_q\|_{\Phi} \Phi(e^N). \end{aligned}$$

Letting $q \rightarrow \infty$ shows that

$$\Phi(\mu^N(T - T_p)) \leq \epsilon + \|T - T_q\|_{\Phi} \Phi(e^N) \quad \forall N \in \mathbb{N}.$$

Combining this with (B.2.4) we get

$$(B.3.8) \quad \|T - T_p\|_{\Phi} \leq \epsilon \quad \forall p > n_0.$$

This implies that T is contained in \mathcal{I}_{Φ} and the sequence $(T_n)_{n \geq 0}$ converges to T in $(\mathcal{I}_{\Phi}, \|\cdot\|_{\Phi})$, proving that $(\mathcal{I}_{\Phi}, \|\cdot\|_{\Phi})$ is a Banach space. The proof is complete. \square

REMARK B.10. If Ψ is another symmetric norm on l_f then $\mathcal{I}_{\Phi} = \mathcal{I}_{\Psi}$ if and only if Φ and Ψ are equivalent in the sense of (B.2.15). Furthermore, it is immediate that in that case the norms $\|\cdot\|_{\Phi}$ and $\|\cdot\|_{\Psi}$ are equivalent on $\mathcal{I}_{\Phi} = \mathcal{I}_{\Psi}$.

LEMMA B.8. Let $T \in \mathcal{I}_{\Phi}$. Then T is contained in \mathcal{I}_{Φ}^0 if and only if

$$(B.3.9) \quad \lim_{N \rightarrow \infty} \Phi(\mu_N(T), \mu_{N+1}(T), \dots) = 0.$$

PROOF. Lemma B.2 says that T is contained in \mathcal{I}_{Φ}^0 if and only if any Schmidt series (B.1.5) for T converges to T in \mathcal{I}_{Φ} . Let $T = U|T|$ be the polar decomposition

of T and let $(\xi_n)_{n \geq 0}$ be an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all $n \in \mathbb{N}_0$. It immediately follows from (B.1.7) that

$$\left\| \sum_{n \geq N} \mu_n(T)(U\xi_n) \otimes \xi_n^* \right\|_{\Phi} = \Phi(\mu_N(T), \mu_{N+1}(T), \dots).$$

Thus T is contained in \mathcal{I}_{Φ}^0 if and only if (B.3.9) holds. \square

Combining this lemma with Proposition B.5 we obtain:

PROPOSITION B.9. *The following are equivalent.*

- (1) *The Banach ideal \mathcal{I}_{Φ} is separable.*
- (2) *The finite-rank operators are dense in \mathcal{I}_{Φ} , i.e., $\mathcal{I}_{\Phi}^0 = \mathcal{I}_{\Phi}$.*
- (3) *For any $a \in l_0$,*

$$(B.3.10) \quad \Phi(a) < \infty \implies \lim_{N \rightarrow \infty} \Phi(a_N, a_{N+1}, \dots) = 0.$$

PROPOSITION B.10. *Let \mathcal{I} be a Banach ideal with norm $\|\cdot\|_{\mathcal{I}}$. Then*

- (1) *There exists a unique symmetric norm on l_f such that*

$$(B.3.11) \quad \mathcal{I} \subset \mathcal{I}_{\Phi} \quad \text{and} \quad \|T\|_{\mathcal{I}} = \|T\|_{\Phi} \quad \forall T \in \mathcal{R}_{\infty}.$$

- (2) *The Banach ideals \mathcal{I}^0 and \mathcal{I}_{Φ}^0 coincide.*

PROOF. Let $(\xi_n)_{n \geq 0}$ be an orthonormal basis of \mathcal{H} . For any $a \in l_0$ set

$$T_a = \sum_{n \geq 0} a_n \xi_n \otimes \xi_n^*.$$

As $\lim_{n \rightarrow \infty} a_n = 0$ the above series converges in \mathcal{K} , i.e., T_a is a compact operator. Observe also that $\mu_n(T_a) = a_n^* \forall n \in \mathbb{N}_0$. In addition, if $T \in \mathcal{K}$, then $\mu(T_{\mu(T)}) = \mu(T)$.

Let $\Phi : l_f \rightarrow [0, \infty)$ be the function defined by

$$\Phi(a) = \|T_a\|_{\mathcal{I}} \quad \forall a \in l_f.$$

It is not hard to check that Φ is a norm on l_f . Moreover, as $\mu(T_a) = a^* = \mu(T_{a^*})$, using Lemma B.1 we see that $\|T_a\|_{\mathcal{I}} = \|T_{a^*}\|_{\mathcal{I}}$, i.e., $\Phi(a) = \Phi(a^*)$. Thus Φ is a symmetric norm on l_f . Let \mathcal{I}_{Φ} be the associated Banach ideal with norm $\|\cdot\|_{\Phi}$.

Let $T \in \mathcal{I}$ and let $N \in \mathbb{N}$. As $\mu_n(T_{\mu^N(T)}) = \mu_n^N(T) \leq \mu_n(T)$, it follows from Lemma B.1 that

$$\Phi(\mu^N(T)) = \|T_{\mu^N(T)}\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}.$$

Thus,

$$\|T\|_{\Phi} = \sup_{N \geq 1} \Phi(\mu^N(T)) \leq \|T\|_{\mathcal{I}} < \infty,$$

that is, the operator T is contained in \mathcal{I}_{Φ} . If in addition T has finite-rank then, as $\mu(T) = \mu(T_{\mu(T)})$, Lemma B.1 insures us that

$$\|T\|_{\mathcal{I}} = \|T_{\mu(T)}\|_{\mathcal{I}} = \Phi(\mu(T)) = \|T\|_{\Phi}.$$

Therefore $\|\cdot\|_{\mathcal{I}}$ and $\|\cdot\|_{\Phi}$ agree on finite-rank operators.

Let Ψ be another symmetric norm on l_f such that $\mathcal{I} \subset \mathcal{I}_{\Psi}$ and $\|\cdot\|_{\Psi} = \|\cdot\|_{\mathcal{I}}$ on \mathcal{R}_{∞} . Let $a \in l_f$. As Ψ is symmetric and $\mu(T_a) = a^*$, we have

$$\Psi(a) = \Psi(a^*) = \Psi(\mu(T_a)) = \|T_a\|_{\Psi} = \|T_a\|_{\mathcal{I}} = \Phi(a).$$

Therefore Ψ and Φ agrees, so Φ is the unique symmetric norm on l_f satisfying (B.3.11).

Let $T \in \mathcal{K}$. As a Schmidt series for T is a series of finite-rank operators, it follows from (B.3.11) that a Schmidt series for T satisfies Cauchy's criterion for $\|\cdot\|_{\mathcal{I}}$ if and only if it satisfies it for $\|\cdot\|_{\Phi}$. Using Lemma B.2 we then deduce that T is contained in \mathcal{I}^0 if and only if it is contained in \mathcal{I}_{Φ}^0 . Thus, as sets, \mathcal{I}^0 and \mathcal{I}_{Φ}^0 agree.

Let T be in $\mathcal{I}^0 = \mathcal{I}_{\Phi}^0$ and let $\sum_{n \geq 0} \mu_n(T)(U\eta_n) \otimes \eta_n^*$ be a Schmidt series for T . As this series converges to T both in \mathcal{I} and in \mathcal{I}_{Φ} , using (B.3.11) we get

$$\|T\|_{\mathcal{I}} = \lim_{N \rightarrow \infty} \left\| \sum_{n < N} \mu_n(T)(U\eta_n) \otimes \eta_n^* \right\|_{\mathcal{I}} = \lim_{N \rightarrow \infty} \left\| \sum_{n < N} \mu_n(T)(U\eta_n) \otimes \eta_n^* \right\|_{\Phi} = \|T\|_{\Phi}.$$

Thus $\|\cdot\|_{\mathcal{I}}$ and $\|\cdot\|_{\Phi}$ agrees on \mathcal{I}^0 . This proves that the Banach ideals \mathcal{I}^0 and \mathcal{I}_{Φ}^0 coincide. The proof is complete. \square

Combining Proposition B.9 and Proposition B.10 we obtain:

PROPOSITION B.11. *A Banach ideal \mathcal{I} is separable if and only if there exists a symmetric norm Φ on l_f such that \mathcal{I} coincides with the Banach ideal \mathcal{I}_{Φ} .*

Next, let us denote by Φ' the dual symmetric norm of Φ as defined in (B.2.12). We can relate the Banach ideal $\mathcal{I}_{\Phi'}$ to the dual of \mathcal{I}_{Φ} as follows.

LEMMA B.9 (Horn Inequality; see [Si, Thm. 1.15]). *Let $S, T \in \mathcal{K}$. Then*

$$\sum_{n < N} \mu_n(ST) \leq \sum_{n < N} \mu_n(S)\mu_n(T) \quad \forall N \in \mathbb{N}.$$

PROPOSITION B.12. *The following hold.*

(1) *For all $S, T \in \mathcal{K}$, we have*

$$\|ST\|_1 \leq \|S\|_{\Phi'} \|T\|_{\Phi}.$$

(2) *Let $S \in \mathcal{I}_{\Phi'}$ and $T \in \mathcal{I}_{\Phi}$. Then the operator ST is trace-class and*

$$(B.3.12) \quad |\text{Trace}(ST)| \leq \|S\|_{\Phi'} \|T\|_{\Phi}.$$

(3) *For all $S \in \mathcal{K}$, we have*

$$(B.3.13) \quad \|S\|_{\Phi'} = \sup_{\substack{\|T\|_{\Phi}=1 \\ T \in \mathcal{R}_{\infty}}} |\text{Trace}(ST)|.$$

PROOF. Let $S, T \in \mathcal{K}$. Then using Horn's inequality and (B.2.14) we see that, for any $N \in \mathbb{N}$, we have

$$\sum_{n < N} \mu_n(ST) \leq \sum_{n \geq 0} \mu_n^N(S)\mu_n^N(T) \leq \Phi'(\mu^N(S))\Phi(\mu^N(T)) \leq \|S\|_{\Phi'} \|T\|_{\Phi}.$$

Thus,

$$\|ST\|_1 = \sum_{n \geq 0} \mu_n(ST) \leq \|S\|_{\Phi'} \|T\|_{\Phi}.$$

Therefore, if $S \in \mathcal{I}_{\Phi'}$ and $T \in \mathcal{I}_{\Phi}$, then ST is trace-class and, using (??), we get

$$(B.3.14) \quad |\text{Trace}(ST)| \leq \|ST\|_1 \leq \|S\|_{\Phi'} \|T\|_{\Phi}.$$

Let $S \in \mathcal{K}$. Then (B.3.14) implies that

$$(B.3.15) \quad \|S\|_{\Phi'} \geq \sup_{\substack{\|T\|_{\Phi}=1 \\ T \in \mathcal{R}_{\infty}}} |\text{Trace}(ST)|.$$

Let $A \in (0, \|S\|_\Phi)$. In view of (B.3.1) we can find $N \in \mathbb{N}$ large enough such that $A < \Phi'(\mu^N(S))$. Using (B.2.14) we see that there exists a sequence $b = (b_n)_{n \geq 0}$ in l_f^+ with same support as μ^N such that $\Phi'(b) = 1$ and

$$(B.3.16) \quad A < \sum_{n \geq 0} \mu_n^N(S) b_n \leq \sum_{n \geq 0} \mu_n(S) b_n.$$

Let $S = U|S|$ be the polar decomposition of S and let $(\xi_n)_{n \geq 0} \subset \mathcal{H}$ be an orthonormal family such that $|S|\xi_n = \mu_n(S)\xi_n \forall n \in \mathbb{N}_0$. Set

$$T = \sum_{n \geq 0} b_n (\xi_n \otimes \xi_n^*) U^*.$$

The operator T has finite rank, since the support of b is finite.

By Proposition ??, the operator U^*U is the orthogonal projection onto $(\ker S)^\perp = (\ker |S|)^\perp$, so $U^*U\xi_n = \xi_n$ whenever $\mu_n(S) \neq 0$. Therefore, we can check that

$$T^*T = \sum_{n \geq 0} b_n^2 \xi_n \otimes \xi_n^* \quad \text{and} \quad |T| = \sum_{n \geq 0} b_n \xi_n \otimes \xi_n^*$$

Using the min-max principle we then deduce that $\mu_n(T) = b_n$ for all $n \in \mathbb{N}_0$. Thus,

$$\|T\|_\Phi = \Phi(\mu(T)) = \Phi(b) = 1.$$

We also have

$$ST = \left(\sum_{n \geq 0} \mu_n(S) U(\xi_n \otimes \xi_n^*) \right) \left(\sum_{n \geq 0} b_n (\xi_n \otimes \xi_n^*) U^* \right) = \sum_{n \geq 0} \mu_n(S) b_n U(\xi_n \otimes \xi_n^*) U^*,$$

Thus $\text{Trace}(ST)$ is equal to

$$\sum_{n \geq 0} \mu_n(S) b_n \text{Trace}(U(\xi_n \otimes \xi_n^*) U^*) = \sum_{n \geq 0} \mu_n(S) b_n \text{Trace}(U^*U(\xi_n \otimes \xi_n^*)) = \sum_{n \geq 0} \mu_n(S) b_n.$$

In view of (B.3.16) this implies that $\text{Trace}(ST) > A$. Since $\|T\|_\Phi = 1$ we deduce that

$$A < \sup_{\substack{\|T\|_\Phi=1 \\ T \in \mathcal{R}_\infty}} |\text{Trace}(ST)| \quad \forall A \in (0, \|S\|_\Phi).$$

Combining this with (B.3.15) yields (B.3.13). The proof is complete. \square

Recall that by Proposition ?? we have an isometric isomorphism from $\mathcal{L}(\mathcal{H})$ onto $(\mathcal{L}^1)'$ given by

$$(B.3.17) \quad \mathcal{L}(\mathcal{H}) \ni S \longrightarrow (S, \cdot) \in (\mathcal{L}^1)', \quad (S, \cdot) : \mathcal{L}(\mathcal{H}) \ni T \longrightarrow (S, T) := \text{Trace}(ST).$$

It follows from Proposition B.12 that we also have linear map,

$$(B.3.18) \quad \mathcal{I}_{\Phi'} \ni S \longrightarrow (S, \cdot) \in (\mathcal{I}_{\Phi}^0)', \quad (S, \cdot) : \mathcal{L}(\mathcal{H}) \ni T \longrightarrow (S, T) := \text{Trace}(ST).$$

and the density of \mathcal{L}^1 in \mathcal{I}_{Φ}^0 show that if $S \in \mathcal{I}_{\Phi}'$ then (S, \cdot) uniquely extends to a continuous linear map on \mathcal{I}_{Φ}^0 , which we shall continue to denote (S, \cdot) .

PROPOSITION B.13. *If $\mathcal{I}_{\Phi} \supsetneq \mathcal{L}^1$, then (B.3.18) yields an isometric isomorphism,*

$$\mathcal{I}_{\Phi}' \simeq (\mathcal{I}_{\Phi}^0)'.$$

PROOF. Let us denote by \mathcal{T}_Φ the linear map (B.3.18). It follows from (B.3.12) and (B.3.13) that, for all $S \in \mathcal{I}_{\Phi'}$,

$$\|S\|_{\Phi'} = \sup_{T \in \mathcal{R}_\infty} \|T\|_{\Phi=1} |\text{Trace}(ST)| \leq \sup_{\substack{\|T\|_{\Phi}=1 \\ T \in \mathcal{R}_\infty}} |\text{Trace}(ST)| \leq \|S\|_{\Phi'}.$$

Therefore \mathcal{T}_Φ is an isometry, so it follows from Lemma ?? that for proving that \mathcal{T}_Φ is an isomorphism it is enough to show that it is onto.

Consider the following subspace of $\mathcal{L}(\mathcal{H})$,

$$\mathcal{I} := \mathcal{T}^{-1}((\mathcal{I}_\Phi^0)') = \left\{ S \in \mathcal{L}(\mathcal{H}); \sup_{T \in \mathcal{R}_\infty \setminus 0} \frac{|\text{Trace}(ST)|}{\|T\|_\Phi} < \infty \right\}.$$

Let $S \in \mathcal{I}$, let $A, B \in \mathcal{L}(\mathcal{H})$ and $C := \sup_{T \in \mathcal{R}_\infty \setminus 0} \frac{|\text{Trace}(ST)|}{\|T\|_\Phi}$. Then, for all $T \in \mathcal{R}_\infty$,

$$|\text{Trace}(ASBT)| = |\text{Trace}(SBTA)| \leq C \|BTA\|_\Phi \leq C \|B\| \|T\|_\Phi \|A\|,$$

hence ASB is contained in \mathcal{I} . This shows that \mathcal{I} is a two-sided ideal.

Suppose that \mathcal{I} is not contained in \mathcal{K} . As \mathcal{I} is a two-sided ideal, Proposition B.2 then insures us that $\mathcal{I} = \mathcal{L}(\mathcal{H})$ and $\mathcal{K} = \mathcal{I} \cap \mathcal{K}$. Since (B.3.13) shows that $\mathcal{I} \cap \mathcal{K} = \mathcal{I}_{\Phi'}$, we see that $\mathcal{I}_{\Phi'} = \mathcal{K}$. Observe that \mathcal{K} is the Banach ideal $\mathcal{I}_{\Phi_\infty}$ associated to the norm Φ_∞ in (B.2.2). Using Remark B.10 we see that Φ' and Φ_∞ are equivalent norms on l_f , so by Remark B.8 their dual norms. By Lemma B.6 the dual norm of Φ' is Φ and, as shown in Example B.9, the dual norm of Φ_∞ is the norm Φ_1 . Combining this with Remark B.10, we see that $\mathcal{I}_\Phi = \mathcal{I}_{\Phi_1} = \mathcal{L}^1$. This contradicts the assumption that \mathcal{I}_Φ does not coincide with \mathcal{L}^1 , so \mathcal{I} must be contained in \mathcal{K} . As $\mathcal{I} \cap \mathcal{K} = \mathcal{I}_{\Phi'}$, this proves that $\mathcal{I} = \mathcal{I}_{\Phi'}$.

Let $\varphi \in (\mathcal{I}_\Phi^0)'$. By Proposition B.6 the inclusion $\mathcal{L}^1 \subset \mathcal{L}_\Phi^0$ is continuous, so φ induces a continuous linear map on \mathcal{L}^1 . The isomorphism (B.3.17) insures us that there exists $S \in \mathcal{L}(\mathcal{H})$ such that

$$(B.3.19) \quad \langle \varphi, T \rangle = \text{Trace}(ST) \quad \forall T \in \mathcal{L}^1.$$

Then, for all $T \in \mathcal{R}_\infty$, we have

$$|\text{Trace}(ST)| = |\langle \varphi, T \rangle| \leq \|\varphi\|_{\mathcal{I}_\Phi'} \|T\|_\Phi.$$

This shows that S is contained in $\mathcal{I} = \mathcal{I}_{\Phi'}$. Observe that (B.3.19) shows that φ and \mathcal{T}_Φ agree on \mathcal{L}^1 . In particular, they agree on finite-rank operators. As finite-rank operators are dense in \mathcal{I}_Φ^0 , it follows that φ and \mathcal{T}_Φ agrees on all \mathcal{I}_Φ^0 , i.e., $\mathcal{T}_\Phi(S) = \varphi$. This proves that \mathcal{T}_Φ is onto, completing the proof. \square

B.4. Schatten Ideals

Let $p \in [0, \infty]$. The Schatten ideal \mathcal{L}^p is the Banach ideal \mathcal{L}_{Φ_p} associated to the p -norm Φ_p . Thus, if for any $T \in \mathcal{K}$, we set

$$\|T\|_p := \Phi_p((\mu_n(T))_{n \geq 0}) = \left(\sum_{n \geq 0} \mu_n(T)^p \right)^{\frac{1}{p}},$$

then

$$\mathcal{L}^p = \{T \in \mathcal{K}; \|T\|_p < \infty\}.$$

For $p = 1$ (resp. $p = 2$) we recover the Banach ideal of trace-class operators (resp. Hilbert-Schmidt operators). As alluded to in the proof of Proposition B.13,

for $p = \infty$ the Banach ideal $\mathcal{L}^\infty = \mathcal{I}_{\Phi_\infty}$ is the whole Banach ideal of compact operators.

As shown in Example ?? we have $\mu_n(|T|^p) = \mu_n(T)^p \forall n \in \mathbb{N}$, so we have

$$\sum_{n \geq 0} \mu_n(T)^p = \text{Trace } |T|^p \quad \forall T \in \mathcal{L}(\mathcal{H}).$$

Thus,

$$T \in \mathcal{L}^p \iff \text{Trace } |T|^p < \infty \iff |T|^p \in \mathcal{L}^1.$$

PROPOSITION B.14. *Let $q \in (p, \infty)$. Then*

$$\|T\|_q \leq \|T\|_p \quad \forall T \in \mathcal{K}.$$

In particular, there is a continuous inclusion,

$$\mathcal{L}^q \subset \mathcal{L}^p.$$

PROOF. Let $T \in \mathcal{K}$. Observe that, for all $n \in \mathbb{N}_0$,

$$\mu_n(T)^q = \mu_n(T)^{q-p} \cdot \mu_n(T)^p \leq (\|T\|_p)^{q-p} \cdot \mu_n(T)^p.$$

Therefore, we get

$$\|T\|_q = \left(\sum_{n \geq 0} \mu_n(T)^q \right)^{\frac{1}{q}} \leq (\|T\|_p)^{\frac{q-p}{q}} \left(\sum_{n \geq 0} \mu_n(T)^p \right)^{\frac{1}{q}} = (\|T\|_p)^{\frac{q-p}{q}} (\|T\|_p)^{\frac{p}{q}} = \|T\|_p,$$

proving the lemma. \square

It can be easily seen that, for $p < \infty$, the symmetric norm Φ_p satisfies the condition (B.3.10). Therefore, Proposition B.9 gives

PROPOSITION B.15. *The Schatten ideal \mathcal{L}^p is separable and the finite-rank operators are dense in \mathcal{L}^p .*

Assume $p > 1$ and let $p' \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the dual norm of Φ_p is the p' -norm $\Phi_{p'}$. This follows from the Hölder inequality,

$$\left| \sum_{n \geq 0} a_n b_n \right| \leq \left(\sum_{n \geq 0} |a_n|^{p'} \right)^{\frac{1}{p'}} \left(\sum_{n \geq 0} |b_n|^p \right)^{\frac{1}{p}} \quad \forall a, b \in l_f,$$

and the fact that we actually have an equality when $b_n = \frac{\overline{a_n}}{|a_n|} |a_n|^{\frac{p'}{p}}$. Therefore, using Proposition B.12 and Proposition B.13 we obtain:

PROPOSITION B.16. *The following hold.*

(1) *For all $S, T \in \mathcal{K}$, we have*

$$\|ST\|_1 \leq \|S\|_{p'} \|T\|_p.$$

(2) *If $S \in \mathcal{L}^{p'}$ and $T \in \mathcal{L}^p$, then the operator ST is trace-class and*

$$|\text{Trace}(ST)| \leq \|S\|_q \|T\|_p.$$

(3) *The linear map (B.3.18) gives rise to an isometric isomorphism,*

$$\mathcal{L}^{p'} \simeq (\mathcal{L}^p)'$$

The Horn inequality admits the following generalization (see [Si, Thm. 1.15]). Let $r \in [1, \infty)$ and $N \in \mathbb{N}$. Then

$$\sum_{n < N} \mu_n(ST)^r \leq \sum_{n < N} \mu_n(S)^r \mu_n(T)^r \quad \forall S, T \in \mathcal{K}.$$

Using this generalization we can show that, if $\frac{1}{p_1} + \dots + \frac{1}{p_k} = \frac{1}{r}$, then

$$\|T_1 T_2 \cdots T_k\|_1 \leq \|T_1\|_{p_1} \|T_2\|_{p_2} \cdots \|T_k\|_{p_k} \quad \forall T_j \in \mathcal{K}.$$

In particular, if for every $j = 1, \dots, k$ the operator T_j is in \mathcal{L}^{p_j} , then $T_1 T_2 \cdots T_k$ is a trace-class operator and we have

$$|\text{Trace}(T_1 T_2 \cdots T_k)| \leq \|T_1\|_{p_1} \|T_2\|_{p_2} \cdots \|T_k\|_{p_k}.$$

B.5. Banach ideals associated to divergent series

Following [GK] we can produce a large class of non-separable ideals as follows.

Let $\pi = (\pi_n)_{n \geq 0}$ be a non-increasing sequence of positive real numbers satisfying the following two conditions:

$$(B.5.1) \quad \lim_{n \rightarrow \infty} \pi_n = 0 \quad \text{and} \quad \sum_{n \geq 0} \pi_n = \infty.$$

Examples of sequences satisfying all these conditions are provided by the sequences

$$(B.5.2) \quad \pi^{(p)} := ((n+1)^{\frac{1}{p}})_{n \geq 0}, \quad p \geq 1.$$

Using Lemma B.5 it is not difficult to check that we define a symmetric norm on l_f by letting

$$\Phi_\pi(a) := \sup_{N \geq 1} \frac{\sigma_N(a^*)}{\sigma_N(\pi)}.$$

We denote by \mathcal{I}_{Φ_π} the associated Banach ideal. In particular,

$$(B.5.3) \quad \mathcal{I}_{\Phi_\pi} = \left\{ T \in \mathcal{K}; \sigma_N(T) = O(\sigma_N(\pi)) \right\}.$$

LEMMA B.10. *Let $a = (a_n)_{n \geq 0}$ be a non-increasing sequence in l_0 such that $\Phi_\pi(a) < \infty$. Then*

$$\lim_{N \rightarrow \infty} \Phi(a_N, a_{N+1}, \dots) = \limsup_{N \rightarrow \infty} \frac{\sigma_N(a)}{\sigma_N(\pi)}.$$

PROOF. For $N \in \mathbb{N}$ set $a^N = (a_N, a_{N+1}, \dots) \in l_0$. Let $n \in \mathbb{N}_0$. As the sequence $(a_n)_{n \geq 0}$ is non-increasing, we have

$$\sigma_n(a^N) = \sum_{j < N} a_{j+N} = \sigma_{n+N}(a) - \sigma_n \geq \sigma_n(a) - \sigma_N(a).$$

Thus,

$$\frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \frac{\sigma_n(a^N)}{\sigma_n(\pi)} + \frac{\sigma_N(a)}{\sigma_n(\pi)} \leq \Phi_\pi(a^N) + \frac{\sigma_N(a)}{\sigma_n(\pi)}.$$

Since (B.5.1) implies that $\lim_{n \rightarrow \infty} \sigma_n(\pi) = \sum_{j \geq 0} a_j = \infty$, we deduce that

$$\limsup_{n \rightarrow \infty} \frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \Phi_\pi(a^N) \quad \forall N \in \mathbb{N}.$$

Thus,

$$(B.5.4) \quad \limsup_{n \rightarrow \infty} \frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \lim_{N \rightarrow \infty} \Phi_\pi(a^N).$$

Let N and m be positive integers. As the sequence $(a_n)_{n \geq 0}$ is non-decreasing, for any $n \in \mathbb{N}$, we have

$$\sigma_n(a^N) = \sum_{j < n} a_{N+j} \leq \sum_{j < n} a_j = \sigma_n(a).$$

Therefore, for all $n \geq m$, we have

$$\frac{\sigma_n(a^N)}{\sigma_n(\pi)} \leq \frac{\sigma_n(a)}{\sigma_n(\pi)} \leq \sup_{p \geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)}.$$

Notice also that, for all $n \leq m - 1$,

$$\frac{\sigma_n(a^N)}{\sigma_n(\pi)} \leq \frac{na_0^N}{\pi_0} = \frac{m}{\pi_0} a_N.$$

Therefore, we have

$$\Phi_\pi(a^N) = \sup_{n \geq 1} \frac{\sigma_n(a^N)}{\sigma_n(\pi)} \leq \sup \left\{ \sup_{p \geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)}, \frac{m}{\pi_0} a_N \right\}.$$

Since $\lim_{N \rightarrow \infty} a_N = 0$, it follows that ,

$$\lim_{N \rightarrow \infty} \Phi_\pi(a^N) \leq \sup_{p \geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)} \quad \forall m \in \mathbb{N}.$$

Thus,

$$\lim_{N \rightarrow \infty} \Phi_\pi(a^N) \leq \lim_{m \rightarrow \infty} \sup_{p \geq m} \frac{\sigma_p(a)}{\sigma_p(\pi)} = \limsup_{n \rightarrow \infty} \frac{\sigma_n(a)}{\sigma_n(\pi)}.$$

Combining this with (B.5.4) yields the lemma. \square

PROPOSITION B.17. *The Banach ideal \mathcal{I}_{Φ_π} is not separable and*

$$(B.5.5) \quad \mathcal{I}_{\Phi_\pi}^0 = \left\{ T \in \mathcal{K}; \sigma_N(T) = o(\sigma_N(\pi)) \right\}.$$

PROOF. It follows from Lemma B.8 and Lemma B.10 that an operator $T \in \mathcal{K}$ is in $\mathcal{I}_{\Phi_\pi}^0$ if and only if $\sigma_N(T) = o(\sigma_N(\pi))$.

Thanks to (B.5.1) the sequence $\pi = (\pi_n)_{n \geq 0}$ is contained in l_0 . It is immediate that $\Phi_\pi(\pi) = 1$. Moreover, using Lemma B.10, we see that

$$\lim_{N \rightarrow \infty} \Phi_\pi(\pi_N, \pi_{N+1}, \dots) = \limsup_{N \rightarrow \infty} \frac{\sigma_N(\pi)}{\sigma_N(\pi)} = 1.$$

Therefore (B.3.10) does hold, so using Proposition B.9 we see that the Banach ideal \mathcal{I}_{Φ_π} is not separable. The proof is complete. \square

PROPOSITION B.18. *Suppose that $\sigma_n(\pi) = O(n\pi_n)$. Then*

$$\mathcal{I}_{\Phi_\pi} = \left\{ T \in \mathcal{K}; \sigma_N(T) = O(\pi_n) \right\} \quad \text{and} \quad \mathcal{I}_{\Phi_\pi}^0 = \left\{ T \in \mathcal{K}; \sigma_N(T) = o(\pi_n) \right\}.$$

PROOF. Let $T \in \mathcal{K}$. Let $m \in \mathbb{N}$ and set $a_m = \sup_{n \geq m} \frac{\mu_n(T)}{\pi_n}$. For $n \geq m$ we have $\mu_n(T) = \frac{\mu_n(T)}{\pi_n} \pi_n \leq a_m \pi_n$, so for all $N > m$ we get

$$\sigma_N(T) = \sigma_m(T) + \sum_{m \leq n < N} \mu_n(T) \leq \sigma_m(T) + a_m \sum_{m \leq n < N} \pi_n \leq \sigma_m(T) + a_m \sigma_N(T).$$

Since $\lim_{N \rightarrow \infty} \sigma_N(\pi) = \infty$, we deduce that

$$\limsup_{N \geq \infty} \frac{\sigma_N(T)}{\sigma_N(\pi)} \leq a_m \quad \forall m \in \mathbb{N}.$$

Thus,

$$(B.5.6) \quad \limsup_{N \rightarrow \infty} \frac{\sigma_N(T)}{\sigma_N(\pi)} \leq \lim_{m \rightarrow \infty} a_m = \limsup_{n \rightarrow \infty} \frac{\mu_n(T)}{\pi_n}.$$

Set $C := \sup_{n \geq 1} \frac{\sigma_n(\pi)}{n\pi_n}$. Since the sequence $(\mu_n(T))_{n \geq 0}$ is non-decreasing, for any $n \in \mathbb{N}$, we have $\sigma_n(T) = \sum_{j < n} \mu_j(T) \geq n\mu_n(T)$, and hence

$$\frac{\mu_n(T)}{\pi_n} \leq \frac{\sigma_n(T)}{n\pi_n} = \frac{\sigma_n(\pi)}{n\pi_n} \cdot \frac{\sigma_n(T)}{\sigma_n(\pi)} \leq C \frac{\sigma_n(T)}{\sigma_n(\pi)}.$$

Thus,

$$(B.5.7) \quad \limsup_{n \rightarrow \infty} \frac{\mu_n(T)}{\pi_n} \leq C \limsup_{n \rightarrow \infty} \frac{\sigma_n(T)}{\sigma_n(\pi)}.$$

Combining (B.5.6)–(B.5.7) with (B.5.3) and (B.5.5) yields the proposition. \square

Let Φ'_π be the dual norm of Φ_π as defined in (B.2.12). This is a symmetric norm on l_f .

LEMMA B.11. *We have*

$$(B.5.8) \quad \Phi'_\pi(a) = \sum_{n \geq 0} \pi_n a_n^* \quad \forall a \in l_f.$$

PROOF. Let $\tilde{\Phi}$ be the function on l_f^+ defined by

$$\tilde{\Phi}(a) := \sum_{n \geq 0} \pi_n a_n \quad \forall a \in l_f^+.$$

In view of Lemma B.5, in order to prove that $\tilde{\Phi}$ agrees with Φ'_π it is enough to show that $\tilde{\Phi}$ satisfies the conditions (B.2.8)–(B.2.11) and agrees with Φ'_π on l_f^+ .

Clearly, $\tilde{\Phi}$ satisfies (B.2.8)–(B.2.10). As, for all $a \in l_f$, we have

$$(B.5.9) \quad \sum_{n \geq 0} \pi_n a_n = \pi_0 \sigma_0 + \sum_{n \geq 1} \pi_n (\sigma_{n+1}(a) - \sigma_n(a)) = \sum_{N \geq 1} (\pi_{N-1} - \pi_N) \sigma_N(a),$$

we see that $\tilde{\Phi}$ satisfies (B.2.11) as well.

It remains to prove that $\tilde{\Phi}$ agrees with Φ'_π on l_f^+ . Let $a \in l_f^+$. Then by (B.2.13) we have

$$\Phi'_\pi(a) = \sup \left\{ \sum_{n \geq 0} a_n b_n; b \in l_f^+, \Phi_\pi(b) = 1 \right\}.$$

Notice that, for any $N \in \mathbb{N}$, the sequence $(\pi_0, \dots, \pi_N, 0, 0, \dots)$ belongs to l_f^+ and we can check that $\Phi_\pi(\pi_0, \dots, \pi_N, 0, 0, \dots) = 1$. Therefore, if N is large enough so that $a_n = 0$ for $n \geq N$, then

$$(B.5.10) \quad \Phi'_\pi(a) \geq \sum_{n \leq N} a_n \pi_n = \sum_{n \geq 0} a_n \pi_n = \tilde{\Phi}(a).$$

Let $b \in l_f^+$ be such that $\Phi_\pi(b) = 1$. Then $\sigma_N(b) \leq \sigma_N(\pi)$ for all $N \in \mathbb{N}$. Therefore, arguing as in (B.5.9), we get

$$\sum_{n \geq 0} a_n b_n = \sum_{N \geq 1} (a_N - a_{N-1}) \sigma_N(b) \leq \sum_{N \geq 1} (a_N - a_{N-1}) \sigma_N(\pi) = \sum_{n \geq 0} a_n \pi_n.$$

It then follows that $\Phi'_\pi(a) \leq \tilde{\Phi}(a)$. Combining this with (B.5.10) proves that $\tilde{\Phi}$ and Φ'_π agree on l_f^+ . The proof is complete. \square

It follows from Lemma B.11 that

$$\mathcal{I}_{\Phi'_\pi} = \left\{ T \in \mathcal{K}; \sum_{n \geq 0} \pi_n \mu_n(T) < \infty \right\} \quad \text{and} \quad \|T\|_{\Phi'_\pi} = \sum_{n \geq 0} \pi_n \mu_n(T) \quad \forall T \in \mathcal{K}.$$

Using (B.5.8) it is not hard to check that the symmetric norm Φ'_π satisfies (B.3.10). Therefore, from Proposition B.9 we get

PROPOSITION B.19. *The Banach ideal $\mathcal{I}_{\Phi'_\pi}$ is separable and the finite-rank operators are dense in it, i.e., $\mathcal{I}_{\Phi'_\pi}^0 = \mathcal{I}_{\Phi'_\pi}$.*

Using Proposition B.13, Lemma B.6 and the fact that $\mathcal{I}_{\Phi'_\pi}^0 = \mathcal{I}_{\Phi'_\pi}$ we get

PROPOSITION B.20. *The linear map (B.3.18) gives rise to isometric isomorphisms,*

$$\mathcal{I}_{\Phi'_\pi} \simeq (\mathcal{I}_{\Phi_\pi}^0)' \quad \text{and} \quad \mathcal{I}_{\Phi_\pi} \simeq (\mathcal{I}_{\Phi'_\pi})'$$

B.6. The Banach ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{L}^{(p,1)}$

Let $p \in (1, \infty)$. We denote by $\mathcal{L}^{(p,\infty)}$ the Banach ideal $\mathcal{I}_{\Phi_{(p,\infty)}}$ associated to the symmetric norm $\Phi_{(p,\infty)}$ on l_f defined by

$$\Phi_{(p,\infty)}(a) := \sup_{N \geq 1} \frac{\sigma_N(a)}{N^{1-\frac{1}{p}}} \quad \forall a \in l_f.$$

Thus,

$$\mathcal{L}^{(p,\infty)} = \left\{ T \in \mathcal{K}; \sigma_N(T) = O(N^{1-\frac{1}{p}}) \right\},$$

and $\mathcal{L}^{(p,\infty)}$ is a Banach ideal for the norm,

$$\|T\|_{(p,\infty)} := \|T\|_{\Phi_{(p,\infty)}} = \sup_{N \geq 1} \frac{\sigma_N(T)}{N^{1-\frac{1}{p}}}.$$

Since $\sum_{n < N} (n+1)^{-\frac{1}{p}} \sim \frac{1}{1-\frac{1}{p}} N^{1-\frac{1}{p}}$ as $N \rightarrow \infty$, we see that $\Phi_{(p,\infty)}$ is equivalent to the symmetric norm $\Phi_{\pi^{(p)}}$ associated to the sequence $\pi^{(p)}$ in (B.5.2). Therefore, the Banach ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{I}_{\Phi_{\pi^{(p)}}}$ have same underlying sets and their norms are equivalent. Since for $p > 1$ we have $\sigma_n(\pi^{(p)}) = O(n\pi_n^{(p)})$, using Proposition B.17 and Proposition B.18 we obtain:

PROPOSITION B.21. *Let $p \in (1, \infty)$. Then*

(1) We have

$$\mathcal{L}^{(p,\infty)} = \left\{ T \in \mathcal{K}; \mu_n(T) = O(n^{-\frac{1}{p}}) \right\}.$$

(2) The Banach ideal $\mathcal{L}^{(p,\infty)}$ is not separable and the closure of finite-rank operators in $\mathcal{L}^{(p,\infty)}$ is

$$\begin{aligned} \mathcal{L}_0^{(p,\infty)} &= \left\{ T \in \mathcal{K}; \sigma_N(T) = o(N^{1-\frac{1}{p}}) \right\} \\ &= \left\{ T \in \mathcal{K}; \mu_n(T) = o(n^{-\frac{1}{p}}) \right\}. \end{aligned}$$

For $p = 1$ we let $\mathcal{L}^{(1,\infty)}$ be the Banach ideal $\mathcal{I}_{\Phi_{(1,\infty)}}$ associated to the symmetric norm $\Phi_{(p,\infty)}$ on l_f defined by

$$\Phi_{(1,\infty)}(a) := \sup_{N \geq 2} \frac{\sigma_N(a)}{\log N} \quad \forall a \in l_f.$$

Thus,

$$\mathcal{L}^{(1,\infty)} = \left\{ T \in \mathcal{K}; \sigma_N(T) = O(\log N) \right\},$$

and the norm of $\mathcal{L}^{(p,\infty)}$ is

$$\|T\|_{(1,\infty)} := \|T\|_{\Phi_{(1,\infty)}} = \sup_{N \geq 2} \frac{\sigma_N(T)}{\log N}.$$

This Banach ideal is sometimes called the *Dixmier ideal*, since this is the natural domain of the Dixmier trace (cf. Chapter ??).

As in the case $p > 1$, the symmetric norms $\Phi_{(1,\infty)}$ and $\Phi_{\pi(1)}$ are equivalent, so the Banach ideals $\mathcal{L}^{(1,\infty)}$ and $\mathcal{I}_{\Phi_{\pi(1)}}$ have same underlying sets and their norms are equivalent. Therefore, using Proposition B.17 we get:

PROPOSITION B.22. *The Banach ideal $\mathcal{L}^{(1,\infty)}$ is not separable and the closure of finite-rank operators in $\mathcal{L}^{(1,\infty)}$ is*

$$\mathcal{L}_0^{(1,\infty)} = \left\{ T \in \mathcal{K}; \sigma_N(T) = o(\log N) \right\}.$$

REMARK B.11. Unlike in the case $p > 1$, we have a strict inclusion,

$$\mathcal{L}^{(1,\infty)} \supsetneq \left\{ T \in \mathcal{K}; \mu_n(T) = O(n^{-1}) \right\}.$$

Clearly, if $T \in \mathcal{K}$ is such that $\mu_n(T) = O(n^{-1})$, then $\sigma_N(T) = O(\log N)$, and hence T is contained in $\mathcal{L}^{(1,\infty)}$.

To show that the inclusion is strict we only have to exhibit a non-increasing sequence of positive numbers $(a_n)_{n \geq 0}$ such that $\sigma_N(a) = O(\log N)$ and na_n is not bounded. An example of such a sequence is obtained as follows.

For any $k \in \mathbb{N}$ set $n_k = k^k$ and let $(a_n)_{n \geq 0}$ be the sequence defined by

$$a_0 = a_1 = 1 \quad \text{and} \quad a_n = \frac{1 + \log k}{n_k} \quad \text{for } n_{k-1} < n \leq n_k.$$

As $n_k a_{n_k} = 1 + \log k \rightarrow \infty$ as $k \rightarrow \infty$, we see that a_n is not a $O(n^{-1})$.

Furthermore, for $k \geq 3$, we have

$$\begin{aligned} \sum_{2 \leq n \leq n_k} a_n &= \sum_{2 \leq j \leq k} \frac{1 + \log j}{n_j} (n_j - n_{j-1}) \leq \sum_{2 \leq j \leq k} (1 + \log j) \\ &\leq \int_1^k (1 + \log x) dx = k \log k = \log n_k. \end{aligned}$$

Therefore, if $n_{k-1} \leq N \leq n_k$, then

$$\sum_{2 \leq n < N} a_n \leq \sum_{2 \leq n \leq n_k} a_n \leq \frac{\log n_k}{\log n_{k-1}} \log n_{k-1} \leq \frac{k \log k}{(k-1) \log(k-1)} \log N \leq C \log N,$$

where we have set $C := \sup_{k \geq 3} \frac{k \log k}{(k-1) \log(k-1)}$. This shows that $\sigma_N(a) = O(\log N)$, concluding the remark.

Next, let $p \in (1, \infty]$ and denote by $\mathcal{L}^{(p,1)}$ the Banach ideal associated to the symmetric norm $\Phi_{(1,\infty)}$ on l_f defined by

$$\Phi_{(p,1)}(a) := \sum_{n \geq 0} (n+1)^{\frac{1}{p}-1} a_n \quad \forall a \in l_f.$$

In other words,

$$\mathcal{L}^{(p,1)} = \left\{ T \in \mathcal{K}; \sum_{n \geq 0} (n+1)^{\frac{1}{p}-1} \mu_n(T) < \infty \right\},$$

and the norm of $\mathcal{L}^{(p,1)}$ is given by

$$\|T\|_{(p,1)} := \sum_{n \geq 0} (n+1)^{\frac{1}{p}-1} \mu_n(T).$$

When $p = \infty$ the Banach ideal $\mathcal{L}^{(\infty,1)}$ is called the *Macaev ideal*.

Let $p' \in [1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Since Lemma B.11 shows that $\Phi_{(p,1)}$ is the dual norm $\Phi'_{\pi^{(p')}}$, we see that $\mathcal{L}^{(p,1)}$ is the Banach ideal $\mathcal{I}_{\Phi'_{\pi^{(p')}}}$. Therefore, Proposition B.19 and Proposition B.20 yields:

PROPOSITION B.23. *Let $p \in (1, \infty]$ and $p' \in [1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$.*

- (1) *The Banach ideal $\mathcal{L}^{(p,1)}$ is separable and agrees with the closure of finite-rank operators.*
- (2) *The linear map (B.3.18) yields isomorphisms,*

$$\mathcal{L}^{(p,1)} \simeq (\mathcal{L}_0^{(p',\infty)})' \quad \text{and} \quad \mathcal{L}^{(p',\infty)} \simeq (\mathcal{L}^{(p,1)})'$$

Finally, the ideals $\mathcal{L}^{(p,\infty)}$ and $\mathcal{L}^{(p,1)}$ can be compared to the Schatten ideals.

PROPOSITION B.24. *We have continuous inclusions,*

$$(B.6.1) \quad \mathcal{L}^p \subset \mathcal{L}_0^{(p,\infty)} \quad \text{and} \quad \mathcal{L}^{(p,\infty)} \subset \mathcal{L}^q, \quad 1 \leq p < q < \infty,$$

$$(B.6.2) \quad \mathcal{L}^q \subset \mathcal{L}^{(p,1)}, \quad 1 \leq q < p \leq \infty,$$

$$(B.6.3) \quad \mathcal{L}^{(p,1)} \subset \mathcal{L}^p, \quad 1 \leq p < \infty.$$

PROOF. In view of Proposition B.23 it is enough to prove that we have continuous inclusions,

$$\begin{aligned}\mathcal{L}^p &\subset \mathcal{L}_0^{(p,\infty)}, & 1 \leq p < \infty, \\ \mathcal{L}^q &\subset \mathcal{L}^{(p,1)}, & 1 \leq q < p \leq \infty,\end{aligned}$$

since the other continuous inclusions would follow by duality.

Let $p \in (1, \infty]$ and let $q \in [1, p)$. Let $T \in \mathcal{L}^q$. Let p' and q' be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. The Hölder inequality gives

$$\|T\|_{(p,1)} = \sum_{n \geq 0} (n+1)^{-\frac{1}{p'}} \mu_n(T) \leq \left(\sum_{n \geq 0} (n+1)^{-\frac{q'}{p'}} \right)^{\frac{1}{q'}} \|T\|_q.$$

The fact that $q < p$ insures us that $q' > p'$, so the series $\sum_{n \geq 0} (n+1)^{-\frac{q'}{p'}}$ is convergent, so we see that \mathcal{L}^q is contained in $\mathcal{L}^{(p,1)}$ and the inclusion is continuous.

Let $p \in [1, \infty)$ and let $T \in \mathcal{L}^p$. Using the Hölder inequality we see that, for any $N \in \mathbb{N}$,

$$\sigma_N(T) = \sum_{n < N} \mu_n(T) \leq \left(\sum_{n < N} 1^{p'} \right)^{\frac{1}{p'}} \left(\sum_{n < N} \mu_n(T)^p \right)^{\frac{1}{p}} \leq N^{1-\frac{1}{p}} \|T\|_p.$$

In view of the definition of the norm of $\mathcal{L}^{(p,\infty)}$ this implies that

$$\|T\|_{(p,\infty)} \leq \|T\|_p \quad \forall T \in \mathcal{L}^p.$$

Thus \mathcal{L}^p is contained in $\mathcal{L}^{(p,\infty)}$ and the inclusion is continuous. Since by Proposition B.15 the finite-rank operators are dense in \mathcal{L}^p , it follows that \mathcal{L}^p is contained the closure of finite-rank operators in $\mathcal{L}^{(p,\infty)}$, that is, the ideal $\mathcal{L}_0^{(p,\infty)}$. Therefore, we actually have a continuous inclusion of \mathcal{L}^p in $\mathcal{L}_0^{(p,\infty)}$. The proof is complete. \square

REMARK B.12. Let $T \in \mathcal{K}$ be such that

$$\mu_n(T) = (n+1)^{-\frac{1}{p}} (\log(n+2))^{-\alpha} \quad \forall n \in \mathbb{N}_0.$$

The following observations hold:

- If $\alpha = \frac{1}{p}$ and $p \in (1, \infty)$, then T is not in \mathcal{L}^p and Proposition B.21 and Proposition B.22 insure us that T is in $\mathcal{L}_0^{(p,\infty)}$.
- If $\alpha = -1$ and $p \in [1, \infty)$, then T is contained in every ideal \mathcal{L}^q with $q > p$. If $p > 1$, then using Proposition B.21 we see that T is not in $\mathcal{L}^{(p,\infty)}$. Likewise, when $p = 1$ the operator T is not in $\mathcal{L}^{(1,\infty)}$.
- If $\alpha = 1$ and $p \in (1, \infty)$, then T is in $\mathcal{L}^{(p,1)}$, but it is not in any ideal \mathcal{L}^q with $q > p$.
- If $p \in (1, \infty]$ and $\alpha \in (p^{-1}, 1)$, then T is in \mathcal{L}^p , but not in $\mathcal{L}^{(p,1)}$.

This shows that all the inclusions in (B.6.1)–(B.6.3) are *strict* inclusions.

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