TRACES ON PSEUDODIFFERENTIAL OPERATORS AND SUMS OF COMMUTATORS

RAPHAËL PONGE

Abstract. The aim of this paper is to show that various known characterizations of traces on classical pseudodifferential operators can actually be obtained by very elementary considerations on pseudodifferential operators, using only basic properties of these operators. Thereby, we give a unified treatment of the determinations of the space of traces (i) on ΨDOs of non-integer order or of regular parity-class, (ii) on integer order ΨDOs, (iii) on ΨDOs of non-positive orders in dimension ≥ 2, and (iv) on ΨDOs of non-positive orders in dimension 1.

Introduction

This paper deals with the description of traces and sum of commutators of classical pseudodifferential operators (ΨDOs) acting on the sections of a vector bundle $\mathcal{E}$ over a compact manifold M^n. The results depend on the class of operators under consideration.

First, if we consider integer order ΨDOs then an important result of Wodzicki [Wo2] (see also [Gu3], [Pa]) states that when M is connected every trace is proportional to the noncommutative residue trace. The latter was discovered independently by Wodzicki ([Wo1], [Wo3]) and Guillemin [Gu1]. Recall that the noncommutative residue of an integer order ΨDO is given by the integral of a density which in local coordinates can be expressed in terms of the symbol of degree $-n$ of the ΨDO. Alternatively, the noncommutative residue appears as the residual trace induced on integer ΨDOs by the analytic continuation of the usual trace to the class ΨDOs of noninteger complex orders. Since its discovery it has found numerous generalizations and applications (see, e.g., [CM], [FGLS], [Gu3], [Le], [MMS], [PR1], [PR2], [Po], [Sc], [Ug], [Va]).

Following the terminology of [KV] the analytic extension of the usual trace to noninteger order ΨDOs is called the *canonical trace*. This is a trace in the sense that it vanishes on commutators $[P_1, P_2]$ such that $\operatorname{ord} P_1 + \operatorname{ord} P_2$ is not an integer. Furthermore, it makes sense on integer order ΨDOs such that their symbols have parity properties which make their noncommutative residue densities vanish. In this paper these ΨDOs are said to be of regular parity-class (see Section 1 for the precise definition). It has been shown recently by Maniccia-Schrohe-Seiler [MSS] and Paycha [Pa] that the tracial properties of the canonical trace characterize it among the linear forms on noninteger order ΨDOs and on regular parity-class ΨDOs.

2000 *Mathematics Subject Classification.* Primary 58J40; Secondary 58J42.

Key words and phrases. Pseudodifferential operators, noncommutative residue.

Research partially supported by NSERC grant 341328-07 and by a New Staff Matching grant from the Connaught Fund of the University of Toronto.

Next, when we consider the algebra of zeroth order ΨDOs we get other traces by composing any linear form on $C^\infty(S^*M)$ with the fiberwise trace of the zeroth order symbol of the ΨDO. Such traces are called *leading-symbol traces.* It has been shown by Wodzicki [Wo2] that when M is connected and has dimension ≥ 2 every trace on zeroth order ΨDOs is the sum of a leading symbol trace and of a constant-multiple of the noncommutative residue. This result was rediscovered by Lescure-Paycha [LP] via the computation of the Hochschild homology of the algebra of zeroth order ΨDOs (which, at least in the continuous case, can also be found in [Wo4]).

Notice that in [LP] there is no distinction between the cases $n \geq 2$ and $n = 1$. However, as noticed by Wodzicki [Wo2], as well as by the author, there is a specificity to the one dimensional case since in dimension 1 we get other traces beside the sums of leading-symbol traces and constant-multiples of the noncommutative residue (see below).

The aim of this paper is to show that the aforementioned characterizations of traces on ΨDO algebras can all be obtained from elementary considerations on ΨDOs, using only very basic properties of these operators. Furthermore, this includes a characterization of the traces on zeroth order ΨDOs in dimension 1.

In his Steklov Institute thesis [Wo2] Wodzicki determined all the traces on integer order ΨDOs and on zeroth order ΨDOs, both in dimension ≥ 2 and in dimension 1. Unfortunately, the proofs of Wodzicki did not appear elsewhere, so it is difficult to have access to them. According to Wodzicki [Wo5] the proofs follow from the determination of the commutator spaces $\{\mathcal{P}_j, \mathcal{P}_k\}$, where $\mathcal{P}_j$ denotes the space of functions on $T^*M \setminus 0$ that are homogeneous of degree j.

The approach of this paper differs from that of [Wo2] and can be briefly described as follows.

The uniqueness of the canonical trace is an immediate consequence of the fact that any non-integer order (resp. parity class) ΨDO is a sum of commutators of functions with non-integer order (resp. parity class) ΨDOs up to a smoothing operator (see Proposition 3.3).

The uniqueness of the noncommutative residue follows from the fact that an integer order ΨDO supported on a local chart is a sum of commutators of compactly supported ΨDOs of given specific types, modulo a constant-multiple of a fixed given ΨDO with non-vanishing noncommutative residue (see Proposition 4.8).

A difference with the approach of [Wo2] is that we work with ΨDOs supported on a local chart, rather than with symbols defined on the whole cosphere bundle. Thus, our arguments are very much related to that [FGLS] and [MSS], except that we make use of the characterization of the ΨDOs in terms of their Schwartz kernels and of the interpretation due to [CM] of the noncommutative residue of a ΨDO in terms of the logarithmic singularity of its Schwartz kernel near the diagonal. This leads us to a simple argument showing that any smoothing operator on $\mathbb{R}^n$ can be written as a sum of commutators of coordinate functions with ΨDOs of order $-n+1$ (see Lemma 4.1). In particular, this allows us to prove the uniqueness of the noncommutative directly for ΨDOs, rather than for symbols (compare [FGLS]).

To deal with traces on zeroth order ΨDOs we observe that to a large extent in dimension ≥ 2 the sums of ΨDO commutators involved in the proof of the uniqueness of the noncommutative residue can be replaced by sum of commutators involving ΨDOs of order ≤ 0 (see Proposition 5.3 for the precise statement). This allows us to characterize traces on zeroth order ΨDOs in dimension ≥ 2. In particular, this provides us with an alternative to the spectral sequence arguments of [LP].

The main ingredient in the determination of traces on zeroth order ΨDOs in dimension 1 is the observation that in dimension 1 the symbol of degree -1 of a zeroth order ΨDO makes sense intrinsically as a section over the cosphere bundle S^*M (Proposition 6.1). As a consequence we can define *subleading-symbol traces* in the same way as leading-symbol traces are defined. The noncommutative residue is an example of subleading-symbol trace and we show that in dimension 1 any trace on zeroth order ΨDOs can be uniquely written as the sum of a leading-symbol trace and of a subleading-symbol trace (Theorem 6.3). An interesting consequence is that for scalar ΨDOs the commutator space of zeroth order ΨDOs agrees with the space of ΨDOs of order ≤ -2.

This paper is organized as follows. In Section 1, we recall some basic facts on ΨDOs and their Schwartz kernels. In Section 2, we collect some key definitions and properties of the noncommutative residue and of the canonical trace. In Section 3, we show the uniqueness of the canonical trace. In Section 4, we prove that of the noncommutative residue. In Section 5, we characterize traces on zeroth order ΨDOs in dimension ≥ 2. In Section 6, we deal with the one dimensional case.

Notation. Throughout the paper M^n denotes a manifold of dimension n and $\mathcal{E}$ denotes a (complex) vector bundle of rank r over M.

Acknowledgements. I am very grateful to Sylvie Paycha and Mariusz Wodzicki for stimulating discussions related to the subject matter of this paper, and to Maciej Zworksi for his pointer to the reference [Ka]. I also wish to thank for its hospitality the University of California at Berkeley where this paper was written.

1. Pseudodifferential operators and their Schwartz kernels

In this section we recall some notation and results about (classical) ΨDOs and their Schwartz kernels.

Let U be an open subset of $\mathbb{R}^n$. The symbols on $U \times \mathbb{R}^n$ and the ΨDOs on U are defined as follows.

Definition 1.1. *1) $S_m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, consists of smooth functions $p(x,\xi)$ on $U \times (\mathbb{R}^n \backslash 0)$ such that $p(x,\lambda\xi) = \lambda^m p(x,\xi)$ for any $\lambda > 0$.*

2) $S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, consists of smooth functions $p(x,\xi)$ on $U \times \mathbb{R}^n$ admitting an asymptotic expansion $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$, $p_{m-j} \in S_{m-j}(U \times \mathbb{R}^n)$, in the sense that, for any integer N and any compact $K \subset U$, there exists $C_{NK\alpha\beta} > 0$ such that, for all $x \in U$ and for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$, we have

$$|\partial_x^\alpha \partial_\xi^\beta (p - \sum_{j<N} p_{m-j})(x,\xi)| \leq C_{NK\alpha\beta} |\xi|^{\Re m - N - |\beta|}. \tag{1.1}$$

If $p(x,\xi)$ is a symbol in $S^m(U \times \mathbb{R}^n)$, then we denote by $p(x,D)$ the linear operator from $C_c^\infty(U)$ to $C^\infty(U)$ given by

$$p(x,D)u(x) = (2\pi)^{-n} \int e^{ix.\xi} p(x,\xi)\hat{u}(\xi)d\xi, \quad u \in C_c^\infty(U). \tag{1.2}$$

Definition 1.2. *$\Psi^m(U)$, $m \in \mathbb{C}$, consists of linear operators $P : C_c^\infty(U) \to C^\infty(U)$ of the form*

$$P = p(x,D) + R, \tag{1.3}$$

where $p(x,\xi)$ is a symbol in $S^m(U\times\mathbb{R}^n)$ (called the symbol of P) and R is a smoothing operator (i.e. R has a smooth Schwartz kernel).

Any ΨDO on U is a continuous operator from $C_c^\infty(U)$ to $C^\infty(U)$ and its Schwartz kernel is smooth off the diagonal. We then define ΨDOs on M acting on sections of $\mathcal{E}$ as follows.

Definition 1.3. *$\Psi^m(M,\mathcal{E})$, $m\in\mathbb{C}$, consists of continuous operators P from $C_c^\infty(M,\mathcal{E})$ to $C^\infty(M,\mathcal{E})$ such that the Schwartz kernel of P is smooth off the diagonal and, in any open of trivializing local coordinates $U\subset\mathbb{R}^n$, we can write P in the form*

$$P=p(x,D)+R, \tag{1.4}$$

for some symbol $p\in S^m(U\times\mathbb{R}^n)\otimes\operatorname{End}\mathbb{C}^r$ and some smoothing operator R.

In addition, we denote by $\Psi^{-\infty}(M,\mathcal{E})$ the space of smoothing operators on M acting on sections of $\mathcal{E}$.

Let us now recall the description of ΨDOs in terms of their Schwartz kernels. This description is well-known to experts (see, e.g., [BG], [Hö2], [Me], [Ta]). The exposition here follows that of [BG] and [Ta].

First, for $\tau\in\mathcal{S}'(\mathbb{R}^n)$ and $\lambda\in\mathbb{R}\setminus 0$ we denote by τ_λ, or $\tau(\lambda\xi)$ when convenient, the distribution in $\mathcal{S}'(\mathbb{R}^n)$ defined by

$$\langle\tau_\lambda(\xi),u(\xi)\rangle=|\lambda|^{-n}\langle\tau(\xi),u(\lambda^{-1}\xi)\rangle\qquad\forall u\in\mathcal{S}(\mathbb{R}^n). \tag{1.5}$$

We say that τ is *homogeneous* of degree m, $m\in\mathbb{C}$, when $\tau_\lambda=\lambda^m\tau$ for all $\lambda>0$.

It is natural to ask whether a homogeneous functions on $\mathbb{R}^n\backslash 0$ can be extended into a homogeneous distribution on $\mathbb{R}^n$. This problem is completely solved by:

Lemma 1.4 ([Hö1, Thm. 3.2.3, Thm. 3.2.4])**.** *Let $p(\xi)\in C^\infty(\mathbb{R}^n\backslash 0)$ be homogeneous of degree m, $m\in\mathbb{C}$.*

1) If m is not an integer $\leq -n$, then $p(\xi)$ can be uniquely extended into a homogeneous distribution $\tau(\xi)$ in $\mathcal{S}'(\mathbb{R}^n)$.

2) If m is an integer $\leq -n$, then at best we can extend $p(\xi)$ into a distribution $\tau(\xi)$ in $\mathcal{S}'(\mathbb{R}^n)$ such that, for any $\lambda>0$, we have

$$\tau(\lambda\xi)=\lambda^m\tau(\xi)+\lambda^m\log\lambda\sum_{|\alpha|=-(m+n)}c_\alpha(p)\delta^{(\alpha)}, \tag{1.6}$$

where we have let $c_\alpha(p)=\int_{S^{n-1}}\frac{(-\xi)^\alpha}{\alpha!}p(\xi)d^{n-1}\xi$. In particular $p(\xi)$ admits a homogeneous extension if and only if all the coefficients $c_\alpha(p)$ vanish.

In the sequel $\tau\rightarrow\hat{\tau}$ is the Fourier transform on $\mathbb{R}^n$ and $\tau\rightarrow\check{\tau}$ is its inverse transform. In addition, if $m\in\mathbb{Z}$ then we set $\hat{m}=-(m+n)$.

Let $\lambda>0$. For any $u\in\mathcal{S}(\mathbb{R}^n)$ we have

$$\langle(\check{\tau})_\lambda,u\rangle=|\lambda|^{-n}\langle\tau,(u_{\lambda^{-1}})^\vee\rangle=\langle\tau,(\check{u})_\lambda\rangle=|\lambda|^{-n}\langle(\tau_{\lambda^{-1}})^\vee,u\rangle, \tag{1.7}$$

that is, $\check{\tau}(\lambda\xi)=|\lambda|^{-n}(\tau(\lambda^{-1}\xi))^\vee$. From this we deduce that:

- τ is homogeneous of degree m if and only if $\check{\tau}$ is homogeneous of degree $\hat{m}$.
- τ satisfies (1.6) if and only if

$$\check{\tau}(\lambda.y)=\lambda^{\hat{m}}\check{\tau}(y)-\lambda^{\hat{m}}\log\lambda\sum_{|\alpha|=\hat{m}}(2\pi)^{-n}c_\alpha(p)(-iy)^\alpha\qquad\forall\lambda\in\mathbb{R}\setminus 0. \tag{1.8}$$

In the sequel we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and we denote by $\mathbb{Z}_{\geq -n}$ the set of integers that are greater than or equal to $-n$. In addition, we denote by $\mathcal{S}'_{\text{reg}}(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$ that are smooth outside the origin. We equip $\mathcal{S}'_{\text{reg}}(\mathbb{R}^n)$ with the locally convex topology induced by that of $\mathcal{S}'(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n \backslash 0)$.

Definition 1.5. *The space $\mathcal{K}_m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, consists of distributions $K(x, y)$ in $C^\infty(U)\hat{\otimes}\mathcal{S}'_{reg}(\mathbb{R}^n)$ such that, for all $\lambda > 0$, we have*

$$K(x, \lambda y) = \begin{cases} \lambda^m K(x, y) & \text{if } m \notin \mathbb{N}_0, \\ \lambda^m K(x, y) + \lambda^m \log \lambda \sum_{|\alpha|=m} c_{K,\alpha}(x) y^\alpha & \text{if } m \in \mathbb{N}_0, \end{cases} \tag{1.9}$$

where the functions $c_{K,\alpha}(x)$, $|\alpha| = m$, are in $C^\infty(U)$ when $m \in \mathbb{N}_0$.

Definition 1.6. *$\mathcal{K}^m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, consists of distributions K in $\mathcal{D}'(U \times \mathbb{R}^n)$ with an asymptotic expansion $K \sim \sum_{j \geq 0} K_{m+j}$, $K_l \in \mathcal{K}_l(U \times \mathbb{R}^n)$, in the sense that, for any integer N, provided J is large enough we have*

$$K - \sum_{j \leq J} K_{m+j} \in C^N(U \times \mathbb{R}^n). \tag{1.10}$$

Using Lemma 1.4 and the discussion that follows we get:

Lemma 1.7. 1) *For any $K(x, y) \in \mathcal{K}_{\hat{m}}(U \times \mathbb{R}^n)$ the restriction of $\hat{K}_{y \to \xi}(x, \xi)$ to $U \times (\mathbb{R}^n \backslash 0)$ is contained in $S_m(U \times \mathbb{R}^n)$.*

2) *Any $p(x, \xi) \in S_m(U \times \mathbb{R}^n)$ can be extended into a distribution $\tau(x, \xi)$ in $C^\infty(U)\hat{\otimes}\mathcal{S}'_{reg}(\mathbb{R}^n)$ such that $K(x, y) := \check{\tau}_{\xi \to y}(x, y)$ belongs to $\mathcal{K}_{\hat{m}}(U \times \mathbb{R}^n)$. Furthermore, if $m \in \mathbb{Z}_{\geq -n}$ then, in the notation of (1.9),*

$$c_{K,\alpha}(x) = (2\pi)^{-n} \int_{S^{n-1}} \frac{(i\xi)^\alpha}{\alpha!} p(x, \xi) d^{n-1}\xi. \tag{1.11}$$

This lemma is a key ingredient in the characterization of ΨDOs below.

Proposition 1.8. *Let $P : C_c^\infty(U) \to C^\infty(U)$ be a continuous operator with Schwartz kernel $k_P(x, y)$. Then the following are equivalent:*

(i) P is a ΨDO of of order m, $m \in \mathbb{C}$.

(ii) We can write $k_P(x, y)$ in the form

$$k_P(x, y) = K(x, x - y) + R(x, y), \tag{1.12}$$

with $K \in \mathcal{K}^{\hat{m}}(U \times \mathbb{R}^n)$ and $R \in C^\infty(U \times U)$.

Moreover, if (ii) holds and, in the sense of (1.10), we have $K \sim \sum_{j \geq 0} K_{\hat{m}+j}$ with $K_l \in \mathcal{K}_l(U \times \mathbb{R}^n)$, then P has symbol $p \sim \sum_{j \geq 0} p_{m-j}$, where $p_l \in S_l(U \times \mathbb{R}^n)$ is the restriction to $U \times (\mathbb{R}^n \backslash 0)$ of $(K_{m+j})^\wedge_{y \to \xi}$.

The above description of ΨDOs allows us to determine the singularities near the diagonal of the Schwartz kernels of ΨDOs. In particular, we have:

Proposition 1.9. *Let $P \in \Psi^m(M, \mathcal{E})$, $m \in \mathbb{Z}$. Then, in any trivializing local coordinates, the Schwartz kernel $k_P(x, y)$ of P has a behavior near $y = x$ of the form*

$$k_P(x, y) = \sum_{-(m+n) \leq j \leq -1} a_j(x, x - y) - c_P(x) \log |y - x| + \mathrm{O}(1), \tag{1.13}$$

where $a_j(x,y) \in C^\infty(U \times (\mathbb{R}^n \backslash 0))$ *is homogeneous of degree* j *with respect to* y *and* $c_P(x) \in C^\infty(U)$ *is given by*

$$c_P(x) = (2\pi)^{-n} \int_{S^{n-1}} p_{-n}(x,\xi) d^{n-1}\xi. \tag{1.14}$$

The description (1.13) of the behavior of $k_P(x,y)$ depends on the choice of the local coordinates, but the coefficient $c_P(x)$ makes sense intrinsically, for we have:

Proposition 1.10 ([CM])**.** *The coefficient* $c_P(x)$ *in (1.13) makes sense globally on* M *as an* $\operatorname{End}\mathcal{E}$*-valued 1-density.*

The point is that if $\phi : U' \rightarrow U$ is a change of local coordinates, then

$$c_{\phi^* P}(x) = |\phi'(x)| c_P(\phi(x)) \qquad \forall P \in \Psi^m(U), \tag{1.15}$$

showing that $c_P(x)$ behaves like a 1-density (detailed proofs of this result can be found in [GVF]).

Finally, we recall some definitions and properties regarding odd-class and even-class ΨDOs.

Definition 1.11. *An operator* $P \in \Psi^m(M,\mathcal{E})$, $m \in \mathbb{Z}$, *is odd-class (resp. even-class) if in any trivializing local coordinates its symbol* $p \sim \sum_{j \geq 0} p_{m-j}$ *satisfies*

$$p_{m-j}(x,-\xi) = \varepsilon(-1)^{m-j} p_{m-j}(x,\xi) \qquad \forall j \geq 0, \tag{1.16}$$

with $\varepsilon = 1$ *(resp.* $\varepsilon = -1$*).*

We denote by $\Psi^{\mathbb{Z}}_{\text{odd}}(M,\mathcal{E})$ (resp. $\Psi^{\mathbb{Z}}_{\text{ev}}(M,\mathcal{E})$) the space of odd-class (resp. even-class) ΨDOs.

Proposition 1.12. *The following hold.*

1) *Any differential operator is odd-class.*
2) *Any parametrix of an odd-class (resp. even-class) elliptic* Ψ*DO is odd-class (resp. even-class).*
3) *If* P *and* Q *are in* $\Psi^{\mathbb{Z}}_{odd}(M,\mathcal{E}) \cup \Psi^{\mathbb{Z}}_{ev}(M,\mathcal{E})$*, then* PQ *is an odd-class (resp. even-class)* Ψ*DO if the parity classes of* P *and* Q *agree (resp. don't agree).*

Definition 1.13. *Let* $P \in \Psi^m(M,\mathcal{E})$, $m \in \mathbb{Z}$. *Then:*

1) P *has regular parity-class if, either* P *is odd-class and* n *is odd, or* P *is even-class and* n *is even.*
2) P *has singular parity-class if, either* P *is odd-class and* n *is even, or* P *is even-class and* n *is odd.*

We denote by $\Psi^{\mathbb{Z}}_{\text{reg}}(M,\mathcal{E})$ (resp. $\Psi^{\mathbb{Z}}_{\text{sing}}(M,\mathcal{E})$) the space of ΨDOs of regular (resp. singular) parity-class.

Proposition 1.14. *The following hold.*

1) *For all* $P \in \Psi^{\mathbb{Z}}_{reg}(M,\mathcal{E})$ *the density* $c_P(x)$ *vanishes everywhere.*
2) *The product of an odd-class* Ψ*DO and of a regular parity-class* Ψ*DO has regular parity-class.*
3) *The product of an even-class* Ψ*DO and of a singular parity-class* Ψ*DO has regular parity-class.*

2. THE NONCOMMUTATIVE RESIDUE AND THE CANONICAL TRACE

In this section we recall the main definitions and properties of the noncommutative residue and of the canonical trace (see [Gu1]–[Gu3], [KV] and [Wo1]–[Wo3] for more details.)

Let $\Psi^{\text{int}}(M, \mathcal{E}) = \cup_{\Re m < -n} \Psi^m(M, \mathcal{E})$ denote the class of ΨDOs whose symbols are integrable with respect to the ξ-variable. If $P \in \Psi^{\text{int}}(M, \mathcal{E})$, then its Schwartz kernel $k_P(x, y)$ is continuous on $M \times M$ and its restriction to the diagonal to the diagonal of $M \times M$ defines a smooth $\operatorname{End} \mathcal{E}$-valued 1-density $k_P(x, x)$. If we further assume M is compact, then we see that P is a trace-class operator on $L^2(M, \mathcal{E})$ and we have

$$\operatorname{Trace}(P) = \int_M \operatorname{tr}_{\mathcal{E}} k_P(x, x). \tag{2.1}$$

In fact, the map $P \rightarrow k_P(x, x)$ can be analytically extended to a map $P \rightarrow t_P(x)$ defined on the class $\Psi^{\mathbb{C} \setminus \mathbb{Z}}(M, \mathcal{E})$ of non-integer order ΨDOs and on regular parity-class ΨDOs. More precisely, if $(P(z))_{z \in \mathbb{C}}$ is a holomorphic family of ΨDOs of non-integer orders as in [Gu3] and [KV], then $(t_P(z))_{z \in \mathbb{C}}$ is a holomorphic family of $\operatorname{End} \mathcal{E}$-values densities.

Let $P \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$ and let $(P(z))_{z \in \mathbb{C}}$ be a family of ΨDOs such that $P(0) = P$ and $\operatorname{ord} P(z) = z + \operatorname{ord} P$. Then the map $z \rightarrow t_{P(z)}(x)$ has at worst a simple pole singularity near $z = 0$ and we have

$$\operatorname{Res}_{z=0} t_{P(z)}(x) = -c_P(x), \tag{2.2}$$

where $c_P(x)$ is the density defined by the logarithmic singularity of the Schwartz kernel of P (cf. Proposition 1.10). Furthermore, if P is of regular parity-class then, under suitable conditions on the family $(P(z))$ (see, e.g., [Pa]), we have

$$\lim_{z \rightarrow 0} t_{P(z)}(x) = t_P(x). \tag{2.3}$$

From now on, and throughout the rest of the paper, we shall assume that M is compact. Then the *canonical trace* is the functional on $\Psi^{\mathbb{C} \setminus \mathbb{Z}}(M, \mathcal{E}) \cup \Psi^{\mathbb{Z}}_{\text{reg}}(M, \mathcal{E})$ defined by

$$\operatorname{TR} P = \int_M \operatorname{tr}_{\mathcal{E}} t_P(x) \qquad \forall P \in \Psi^{\mathbb{C} \setminus \mathbb{Z}}(M, \mathcal{E}) \cup \Psi^{\mathbb{Z}}_{\text{reg}}(M, \mathcal{E}). \tag{2.4}$$

This is the unique analytic continuation to $\Psi^{\mathbb{C} \setminus \mathbb{Z}}(M, \mathcal{E}) \cup \Psi^{\mathbb{Z}}_{\text{reg}}(M, \mathcal{E})$ of the operator trace. Moreover, we have:

Proposition 2.1. *1) We have* $\operatorname{TR}[P_1, P_2] = 0$ *whenever* $\operatorname{ord} P_1 + \operatorname{ord} P_2 \notin \mathbb{Z}$.

2) TR *vanishes on* $[\Psi^{\mathbb{Z}}_{odd}(M, \mathcal{E}), \Psi^{\mathbb{Z}}_{reg}(M, \mathcal{E})]$ *and* $[\Psi^{\mathbb{Z}}_{ev}(M, \mathcal{E}), \Psi^{\mathbb{Z}}_{sing}(M, \mathcal{E})]$.

Next, the noncommutative residue of an operator $P \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$ is

$$\operatorname{Res} P = \int_M \operatorname{tr}_{\mathcal{E}} c_P(x). \tag{2.5}$$

Because of (1.14) we recover the usual definition of the noncommutative residue. Moreover, if $(P(z))_{z \in \mathbb{C}}$ is a holomorphic family of ΨDOs such that $P(0) = P$ and $\operatorname{ord} P(z) = z + \operatorname{ord} P$, then by (2.2) we have

$$\operatorname{Res}_{z=0} \operatorname{TR} P(z) = -\operatorname{Res} P \tag{2.6}$$

Notice that the noncommutative residue vanishes on ΨDOs of integer order $\leq -(n+1)$, including smoothing operators, and it also vanishes on ΨDOs of regular parity-class. In addition, using (2.6) we get:

Proposition 2.2. *The noncommutative residue is a trace on the algebra $\Psi^{\mathbb{Z}}(M,\mathcal{E})$.*

Next, the trace properties of the canonical trace and of the noncommutative residue mentioned in Proposition 2.1 and Proposition 2.2 characterize these functionals. First, we have:

Theorem 2.3 ([Wo2], [Gu3])**.** *If M connected, then any trace on $\Psi^{\mathbb{Z}}(M,\mathcal{E})$ is a constant-multiple of the noncommutative residue.*

As for the canonical trace, we have:

Theorem 2.4 ([MSS], [Pa])**.** *1) Any linear map $\tau : \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E}) \rightarrow \mathbb{C}$ vanishing on $[C^\infty(M), \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})]$ and $[\Psi^{-\infty}(M,\mathcal{E}), \Psi^{-\infty}(M,\mathcal{E})]$ is a constant-multiple of the canonical trace.*

2) Any linear map $\tau : \Psi^{\mathbb{Z}}_{reg}(M,\mathcal{E}) \rightarrow \mathbb{C}$ vanishing on $[C^\infty(M), \Psi^{\mathbb{Z}}_{reg}(M,\mathcal{E})]$ and $[\Psi^{-\infty}(M,\mathcal{E}), \Psi^{-\infty}(M,\mathcal{E})]$ is a constant-multiple of the canonical trace.

In addition to the noncommutative residue, on zeroth order ΨDOs we have many other traces. For an operator $P \in \Psi^0(M,\mathcal{E})$ the zeroth order symbol uniquely defines a section $\sigma_0(P) \in C^\infty(S^*M, \operatorname{End}\mathcal{E})$ (for the sake of brevity we also denote by $\mathcal{E}$ its pullback by the canonical projection of S^*M onto M). Then *any* linear form L on $C^\infty(S^*M)$ gives rise to a trace τ_L on $\Psi^0(M,\mathcal{E})$ defined by the formula

$$\tau_L(P) := L[\operatorname{tr}_{\mathcal{E}} \sigma_0(P)] \qquad \forall P \in \Psi^0(M,\mathcal{E}). \tag{2.7}$$

Such a trace is called a *leading-symbol trace.*

Theorem 2.5 ([Wo2], [LP])**.** *Suppose that M is connected and has dimension ≥ 2. Then any trace on $\Psi^0(M,\mathcal{E})$ can be uniquely written as the sum of a leading-symbol trace and of a constant-multiple of the noncommutative residue.*

We refer to Section 6 for the description of the traces on $\Psi^0(M,\mathcal{E})$ when $n=1$.

3. Uniqueness of the canonical trace

In this section we shall give a proof of Theorem 2.4 about the uniqueness of the canonical trace. First, we have:

Lemma 3.1. *Let $K(x,y) \in \mathcal{K}_0(\mathbb{R}^n \times \mathbb{R}^n)$ be homogeneous of degree 0 with respect to y and let $P \in \Psi^{-n}(\mathbb{R}^n)$ be the ΨDO with Schwartz kernel $k_P(x,y) = K(x,x-y)$. Then P can be written in the form*

$$P = [x_1, P_1] + \ldots + [x_n, P_n], \tag{3.1}$$

with $P_1, \ldots, P_n$ in $\Psi^{-n+1}(\mathbb{R}^n)$. Moreover, if $K_0(x,-y) = -K_0(x,y)$ then $P_1, \ldots, P_n$ can be chosen to be of regular parity-class.

Proof. For $j=1,\ldots,n$ and for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \backslash 0$ set $K^{(j)}(x,y) = y_j |y|^{-2} K(x,y)$. As $K^{(j)}(x,y)$ is smooth for $y \neq 0$ and is homogeneous with respect to y of degree -1 we see that $K^{(j)}(x,y)$ is an element of $\mathcal{K}_{-1}(\mathbb{R} \times \mathbb{R})$. Therefore, by Proposition 1.8

the operator P_j with kernel $k_{P_j}(x,y) = K^{(j)}(x,x-y)$ is a ΨDO of order $-n+1$. Moreover, observe that the Schwartz kernel of $\sum_{j=1}^n [x_j, P_j]$ is

$$\sum_{1\leq j\leq n} (x_j - y_j)^2 |x-y|^{-2} K(x,x-y) = K(x,x-y) = k_P(x,y). \tag{3.2}$$

Hence $P = [x_1, P_1] + \ldots + [x_n, P_n]$.

Assume now that $K(x,-y) = -K(x,y)$. Then we have $K^{(j)}(x,-y) = K^{(j)}(x,y)$. By Proposition 1.8 the symbol of P_j is $p^{(j)}(x,\xi) \sim p_{-n+1}^{(j)}(x,\xi)$ with $p_{-n+1}^{(j)}(x,\xi) = (K^{(j)})^\wedge_{y\rightarrow\xi}(x,\xi)$, so we have

$$p_{-n+1}^{(j)}(x,-\xi) = (K^{(j)}(x,-y))^\wedge_{y\rightarrow\xi}(x,\xi) = (K^{(j)})^\wedge_{y\rightarrow\xi}(x,\xi) = p_{-n+1}(x,\xi). \tag{3.3}$$

Since $1 = (-1)^{-n+1}$ when n is odd and $1 = (-1)^{-n+1+1}$ when n is even, this shows that P_j is odd-class when n is odd and is even-class when n is even. In any case P_j is of regular parity-class. The lemma is proved. ☐

Lemma 3.2 ([FGLS], [Gu2])**.** *Let $P \in \Psi^m(\mathbb{R}^n)$, $m \in \mathbb{C}$, and assume that, either $m \notin \mathbb{Z}_{\geq -n}$, or $m \in \mathbb{Z}_{\geq -n}$ and $c_P(x)$ vanishes everywhere. Then we can write*

$$P = [x_1, P_1] + \ldots + [x_n, P_n] + R, \tag{3.4}$$

with $P_1, \ldots, P_n$ in $\Psi^{m+1}(\mathbb{R}^n)$ and $R \in \Psi^{-\infty}(\mathbb{R}^n)$. Furthermore, if P is in $\Psi^{\mathbb{Z}}_{reg}(\mathbb{R}^n)$ then $P_1, \ldots, P_n$ can be chosen to be in $\Psi^{\mathbb{Z}}_{reg}(\mathbb{R}^n)$ as well.

Proof. Let us first assume that, either $m \notin \mathbb{Z}_{\geq -n}$, or $m \in \mathbb{Z}_{\geq -n}$ and the symbol of degree $-n$ of P is zero. Let $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$ be the symbol of P. By the Euler identity $\sum_{k=1}^n \xi_k \partial_{\xi_k} p_{m-j} = (m-j)p_{m-j}$, so we have

$$\sum_{k=1}^n \partial_{\xi_k}[\xi_k p_{m-j}] = n p_{m-j} + \sum_{k=1}^n \xi_k \partial_{\xi_k} p_{m-j} = (m-j+n)p_{m-j}. \tag{3.5}$$

By assumption, either $m \notin \mathbb{Z}_{\geq -n}$, or $m \in \mathbb{Z}_{\geq -n}$ and $p_{-n} = 0$, so for $k = 1, \ldots, n$ there always exists $P_k \in \Psi^{m+1}(\mathbb{R}^n)$ with symbol $p^{(k)} \sim \frac{1}{i}\sum_{j\geq 0} \frac{1}{m-j+n}\xi_k p_{m-j}$. Then using (3.5) we see that $\sum_{k=1}^n [x_k, P_k]$ has symbol

$$\sum_{k=1}^n \partial_{\xi_k} p^{(k)} \sim \sum_{j\geq 0} \frac{1}{m-j+n} \sum_{k=1}^n \partial_{\xi_k}[\xi_k p_{m-j}] \sim \sum_{j\geq 0} p_{m-j} \sim p. \tag{3.6}$$

It follows that P agrees with $\sum_{k=1}^n [x_k, P_k]$ up to a smoothing operator, proving that P can be put in the form (3.4). Furthermore, if P has regular parity-class, then so does each operator P_k.

Suppose now that $m \in \mathbb{Z}_{\geq -n}$ and $c_P = 0$. Let $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$ be the symbol of P. By (1.14), for all $x \in \mathbb{R}^n$, we have

$$\int_{S^{n-1}} p_{-n}(x,\xi) d^{n-1}\xi = (2\pi)^n c_P(x) = 0. \tag{3.7}$$

It then follows from Lemma 1.7 that $p_{-n}(x,\xi)$ can be extended into a distribution $\tau(x,\xi)$ in $C^\infty(U)\hat{\otimes}\mathcal{S}'_{\text{reg}}(\mathbb{R}^n)$ such that $K_0(x,y) := \check{\tau}_{\xi\rightarrow y}(x,y)$ is in $\mathcal{K}_0(U\times\mathbb{R}^n)$ and is homogeneous of degree 0 with respect to y.

Let $Q \in \Psi^{-n}(\mathbb{R}^n)$ have Schwartz kernel $k_Q(x,y) = K_0(x,x-y)$. By Lemma 3.1 we can write Q in the form

$$Q = [x_1, Q_1] + \ldots + [x_n, Q_n], \tag{3.8}$$

with $Q_1,\ldots,Q_n$ in $\Psi^{-n+1}(\mathbb{R}^n)$. The symbol of Q is $q=(K_0)^\vee_{y\to\xi}\sim p_{-n}$, so the operator $\tilde{P}=P-Q$ has symbol $\tilde{p}\sim\sum_{m-j\neq -n}p_{m-j}$. The first part of the proof then shows that we can write

$$\tilde{P}=[x_1,\tilde{P}_1]+\ldots+[x_n,\tilde{P}_n]+R, \tag{3.9}$$

with $\tilde{P}_1,\ldots,\tilde{P}_n$ in $\Psi^{m+1}(\mathbb{R}^n)$ and $R\in\Psi^{-\infty}(\mathbb{R}^n)$. Since $P=\tilde{P}+Q$ we see that

$$P=[x_1,P_1]+\ldots+[x_n,P_n]+R, \tag{3.10}$$

where we have set $P_j=\tilde{P}_j+Q_j$. In other words P can be put in the form (3.4).

Finally, let us further assume that P is of regular parity-class. Then $\tilde{P}$ is of regular parity-class and, as its symbol of $\tilde{P}$ of degree $-n$ is zero, it follows from the first part of the proof that the operators $\tilde{P}_1,\ldots,\tilde{P}_n$ in (3.9) can be chosen to be of regular parity-class.

The operator Q is of regular parity-class too. Indeed, as $p_{-n}(x,-\xi)=-p_{-n}(x,\xi)$, we can choose τ so that $\tau(x,-\xi)=-\tau(x,\xi)$. Then

$$K_0(x,-y)=[\tau(x,-\xi)]^\vee_{\xi\to y}=-\check{\tau}_{\xi\to y}(x,y)=-K_0(x,y). \tag{3.11}$$

Therefore, by Lemma 3.1 the operators $Q_1,\ldots,Q_n$ in (3.8) can be chosen to be of regular parity-class. Since in (3.9) we have $P_j=\tilde{P}_j+Q_j$, it follows that $P_1,\ldots,P_n$ can be chosen to be of regular parity-class, completing the proof. ☐

In the sequel we shall use the symbol Ψ_c to denote classes of ΨDOs with compactly supported Schwartz kernels, e.g., $\Psi^{\mathbb{Z}}_{\mathrm{c}}(\mathbb{R}^n)$ is the space of all integer order ΨDOs on $\mathbb{R}^n$ whose Schwartz kernels have compact supports.

Proposition 3.3. *Let $P\in\Psi^m(M,\mathcal{E})$, $m\in\mathbb{C}$, and assume that, either $m\notin\mathbb{Z}_{\geq -n}$, or $m\in\mathbb{Z}_{\geq -n}$ and the density $c_P(x)$ vanishes everywhere. Then we can write*

$$P=[a_1,P_1]+\ldots+[a_N,P_N]+R, \tag{3.12}$$

where $a_1,\ldots,a_N$ are smooth functions on M, the operators $P_1,\ldots,P_N$ are in $\Psi^{m+1}(M,\mathcal{E})$, and R is a smoothing operator. Furthermore, if P has regular parity-class, then $P_1,\ldots,P_N$ can be chosen to have regular parity-class as well.

Proof. Let us first assume that P is a scalar ΨDO on $\mathbb{R}^n$ whose Schwartz kernel has compact support. By Lemma 3.2 there exist $P_1,\ldots,P_n$ in $\Psi^{m+1}(\mathbb{R}^n)$ and $R\in\Psi^{-\infty}(\mathbb{R}^n)$ such that

$$P=[x_1,P_1]+\ldots+[x_n,P_n]+R. \tag{3.13}$$

Let χ and ψ in $C^\infty_c(\mathbb{R}^n)$ be such that $\psi(x)\psi(y)=1$ near the support of the kernel of P, so that $\psi P\psi=P$, and let $\chi\in C^\infty_c(\mathbb{R}^n)$ be such that $\chi=1$ near $\operatorname{supp}\psi$. As $\psi[x_j,P_j]\psi=x_j\chi\psi P_j\psi-\psi P\psi x_j\chi=[x_j\chi,\psi P_j\psi]$ we get

$$P=\psi P\psi=\sum_{j=1}^n\psi[x_j,P_j]\psi+\psi R\psi=\sum_{j=1}^n[\chi x_j,\psi P_j\psi]+\psi R\psi. \tag{3.14}$$

This shows that P can be put in the form (3.12) with functions $a_1,\ldots,a_n$ of compact support and operators $P_1,\ldots,P_n$ and R with compactly supported Schwartz kernels. Furthermore, if P is of regular parity-class, then Lemma 3.2 insures us that $P_1,\ldots,P_n$ can be chosen to be of regular parity-class, so that $\psi P_1\psi,\ldots,\psi P_n\psi$ are of regular parity-class.

These results immediately extends to ΨDOs in $\Psi^*_{\mathrm{c}}(U,\mathcal{E})$, where $U\subset M$ is the domain of any local chart over which $\mathcal{E}$ is trivializable.

Now, let $P \in \Psi^m(M,\mathcal{E})$ and assume that, either $m \notin \mathbb{Z}_{\geq -n}$, or $m \in \mathbb{Z}_{\geq -n}$ and the density $c_P(x)$ vanishes everywhere. Let $(\varphi_i) \subset C^\infty(M)$ be a finite partition of unity subordinate to an open covering (U_i) by trivializing local charts. For each index i let $\psi_i \in C^\infty_c(U_i)$ be such that $\psi_i = 1$ near $\operatorname{supp}\varphi_i$. Then there exists $R \in \Psi^{-\infty}(M,\mathcal{E})$ such that

$$P = \sum \varphi_i P \psi_i + R. \tag{3.15}$$

Each operator $P_i := \varphi_i P \psi_i$ is contained in $\Psi^m_c(U_i,\mathcal{E})$. Moreover, it agrees with $\varphi_i P$ agrees up to a smoothing operator, so if $m \in \mathbb{Z}_{\geq -n}$ and the density $c_P(x)$ vanishes everywhere, then $c_{P_i}(x) = c_{\varphi_i P}(x) = \varphi_i(x)c_P(x) = 0$. In addition, if P has regular parity class, then so does P_i. In any case, the first part of the proof insures us that each operator P_i can be put in the form (3.12) and, if P has regular parity-class, then this can be done using regular parity-class ΨDOs. Combining this with (3.15) yields the proposition. ☐

For future reference we quote the following well-known result.

Lemma 3.4 ([Wo2], [Gu3])**.** *1) Any $R \in \Psi^{-\infty}(M,\mathcal{E})$ with vanishing operator trace is the sum of two commutators in $\Psi^{-\infty}(M,\mathcal{E})$.*

2) Any trace on $\Psi^{-\infty}(M,\mathcal{E})$ is a constant-multiple of the operator trace.

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Let $\tau : \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E}) \rightarrow \mathbb{C}$ be a linear map vanishing on $[C^\infty(M), \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})]$ and $[\Psi^{-\infty}(M,\mathcal{E}), \Psi^{-\infty}(M,\mathcal{E})]$. Then τ induces a trace on $\Psi^{-\infty}(M,\mathcal{E})$, so by Lemma 3.4 there exists a constant $c \in \mathbb{C}$ such that, for any $R \in \Psi^{-\infty}(M,\mathcal{E})$, we have $\tau(R) = c\,\mathrm{Trace}(R)$.

Let $P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$. Then by Proposition 3.3 there exists $R \in \Psi^{-\infty}(M,\mathcal{E})$ such that $P = R \mod [C^\infty(M), \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})]$. As τ vanishes on $[C^\infty(M), \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})]$ we have $\tau(P) = \tau(R) = c\,\mathrm{Trace}(R)$. Similarly, we have $\mathrm{TR}\,P = \mathrm{TR}\,R = \mathrm{Trace}(R)$, so we see that $\tau(P) = c\,\mathrm{TR}\,P$. Thus τ is a constant-multiple of TR, proving the first part of Theorem 2.4. The second part can be proved along similar lines. ☐

We close this section with the following.

Proposition 3.5. *1) Let $P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$. Then $\mathrm{TR}\,P = 0$ if and only if P is contained in the subspace $[C^\infty(M), \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})] + [\Psi^{-\infty}(M,\mathcal{E}), \Psi^{-\infty}(M,\mathcal{E})]$.*

2) Let $P \in \Psi^{\mathbb{Z}}_{reg}(M,\mathcal{E})$. Then $\mathrm{TR}\,P = 0$ if and only if P is contained in the subspace $[C^\infty(M), \Psi^{\mathbb{Z}}_{reg}(M,\mathcal{E})] + [\Psi^{-\infty}(M,\mathcal{E}), \Psi^{-\infty}(M,\mathcal{E})]$.

Proof. Let $R_0 \in \Psi^{-\infty}(M,\mathcal{E})$ be such that $\mathrm{Trace}\,R_0 \neq 0$ (e.g. $R_0 = e^{-\Delta_{\mathcal{E}}}$ where $\Delta_{\mathcal{E}}$ is a Laplace type operator acting on the sections of $\mathcal{E}$). Since by Lemma 3.4 the commutator space of $\Psi^{-\infty}(M,\mathcal{E})$ agrees with the null space of $\mathrm{Trace}_{|\Psi^{-\infty}(M,\mathcal{E})}$, for any $R \in \Psi^{-\infty}(M,\mathcal{E})$ we have

$$R = \frac{\mathrm{Trace}\,R}{\mathrm{Trace}\,R_0} R_0 \mod [\Psi^{-\infty}(M,\mathcal{E}), \Psi^{-\infty}(M,\mathcal{E})]. \tag{3.16}$$

Let $P \in \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})$. By Proposition 3.3 there exists $R \in \Psi^{-\infty}(M,\mathcal{E})$ such that $P = R \mod [C^\infty(M), \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})]$. Then $\mathrm{TR}\,P = \mathrm{TR}\,R = \mathrm{Trace}\,R$, so using (3.16) we get

$$P = \frac{\mathrm{TR}\,P}{\mathrm{Trace}\,R_0} R_0 \mod [C^\infty(M), \Psi^{\mathbb{C}\setminus\mathbb{Z}}(M,\mathcal{E})] + [\Psi^{-\infty}(M,\mathcal{E}), \Psi^{-\infty}(M,\mathcal{E})]. \tag{3.17}$$

It then follows that P belongs to $[C^\infty(M),\Psi^{\mathbb{C}\mathbb{Z}}(M,\mathcal{E})]+[\Psi^{-\infty}(M,\mathcal{E}),\Psi^{-\infty}(M,\mathcal{E})]$ if and only if $\operatorname{TR} P$ vanishes, proving the first part of the proposition. The second part can be shown along similar lines. ☐

4. Uniqueness of the noncommutative residue

In this section, we prove Theorem 2.3 on the uniqueness of the noncommutative residue. First, we have:

Lemma 4.1. *Any operator $R\in\Psi^{-\infty}(\mathbb{R}^n)$ can be written in the form*

$$R=[x_1,P_1]+\ldots+[x_n,P_n], \tag{4.1}$$

with $P_1,\ldots,P_n$ in $\Psi^{-n+1}(\mathbb{R}^n)$.

Proof. Let $k_R(x,y)$ denote the Schwartz kernel of R. Since $k_R(x,y)$ is smooth, we can write

$$k_R(x,y)=k_R(x,x)+(x_1-y_1)k_{R_1}(x,y)+\ldots+(x_n-y_n)k_{R_n}(x,y), \tag{4.2}$$

where $k_1(x,y),\ldots,k_n(x,y)$ are smooth functions on $\mathbb{R}^n\times\mathbb{R}^n$. Let Q be the operator with Schwartz kernel $k_Q(x,y)=k_R(x,x)$ and, for $j=1,\ldots,n$, let R_j be the smoothing operator with Schwartz kernel $k_j(x,y)$. Then (4.2) shows that

$$R=Q+[x_1,R_1]+\ldots+[x_n,R_n]. \tag{4.3}$$

Next, the Schwartz kernel of Q is of the form $k_Q(x,y)=K_0(x,x-y)$ with $K_0(x,y)=k_R(x,x)$. Obviously $K_0(x,y)$ belongs to $\mathcal{K}_0(\mathbb{R}^n\times\mathbb{R}^n)$ and is homogeneous of degree 0 with respect to y, so it follows from Lemma 3.1 that Q can be written as a sum of commutators of the form (4.1). Combining this with (4.3) proves the lemma. ☐

Using the previous lemma we shall prove:

Proposition 4.2. *Any $R\in\Psi^{-\infty}(M,\mathcal{E})$ can be written in the form*

$$R=[a_1,P_1]+\ldots+[a_N,P_N]+[R_1,R_2]+[R_3,R_4], \tag{4.4}$$

where the functions a_j are in $C^\infty(M)$, the operators P_j are in $\Psi^{-n+1}(M,\mathcal{E})$ and the operators R_j are in $\Psi^{-\infty}(M,\mathcal{E})$.

Proof. If $R\in\Psi_c^{-\infty}(\mathbb{R}^n)$ then Lemma 4.1 tells us that $R=\sum_{j=1}^n[x_j,P_j]$ with $P_1,\ldots,P_n$ in $\Psi^{-n+1}(\mathbb{R}^n)$. Let $\psi\in C_c^\infty(\mathbb{R}^n)$ be such that $\psi(x)\psi(y)=1$ near the support of the kernel of R and let $\chi\in C_c^\infty(\mathbb{R}^n)$ be such that $\chi=1$ near $\operatorname{supp}\psi$. As in (3.14) we have $\psi[x_j,P_j]\psi=x_j\chi\psi P_j\psi-\psi P\psi x_j\chi=[x_j\chi,\psi P_j\psi]$, so we get

$$R=[\chi x_1,\psi P_1\psi]+\ldots+[\chi x_n,\psi P_n\psi]. \tag{4.5}$$

More generally, if $U\subset M$ is the domain of a chart over which $\mathcal{E}$ is trivializable, then any $R\in\Psi_c^{-\infty}(U,\mathcal{E})$ can be written in the form

$$R=[a_1,P_1]+\ldots+[a_n,P_n], \tag{4.6}$$

with $a_1,\ldots,a_n$ in $C_c^\infty(U)$ and $P_1,\ldots,P_n$ in $\Psi_{\mathrm{c}}^{-n+1}(U,\mathcal{E})$.

Now, let $(\varphi_i)\subset C^\infty(M)$ be a partition of unity subordinate to an open covering (U_i) of M by domains of local charts over which $\mathcal{E}$ is trivializable. For each index i let $\psi_i\in C_c^\infty(U_i)$ be such that $\psi_i=1$ near $\operatorname{supp}\varphi_i$. Then for any $R\in\Psi^{-\infty}(M,\mathcal{E})$ we have

$$R=\sum\varphi_iR\psi_i+\sum\varphi_iR(1-\psi_i). \tag{4.7}$$

For each index i the operator $\varphi_i R \psi_i$ belongs to $\Psi_c^{-\infty}(U_i, \mathcal{E})$, so by the first part of the proof it can be written as a sum of commutators of the form (4.6). Moreover, the operator $S := \sum \varphi_i R(1-\psi_i)$ is smoothing and has a Schwartz kernel that vanishes on the diagonal. This implies that its operator trace is zero, so by Lemma 3.4 it can be written a the sum of two commutators in $\Psi^{-\infty}(M, \mathcal{E})$. Putting all this together and using (4.7) shows that R can be put in the form (4.4), proving the proposition. ☐

Remark 4.3. Wodzicki [Wo2] also proved that any smoothing operator is a sum of ΨDO commutators.

Remark 4.4. It follows from the proof of Lemma 4.1 that the operators $P_1, \ldots, P_n$ in (4.1) can be chosen to be of singular parity-class. Therefore any $R \in \Psi^{-\infty}(M, \mathcal{E})$ can be written in the form (4.4) with operators $P_1, \ldots, P_N$ in $\Psi^{-n+1}(M, \mathcal{E})$ of singular parity-class. Notice that they cannot be chosen to be of regular parity-class, since this would imply the vanishing of the canonical trace on smoothing operators, which is obviously wrong.

Combining Proposition 3.3 and Proposition 4.2 we immediately get:

Proposition 4.5. *Let $m \in \mathbb{Z}$ and set $\tilde{m} = \max(m, -n)$. Then any $P \in \Psi^m(M, \mathcal{E})$ such that $c_P(x)$ vanishes everywhere can be written in the form*

$$P = [a_1, P_1] + \ldots + [a_N, P_N] + [R_1, R_2] + [R_3, R_4], \tag{4.8}$$

where the functions a_j are in $C^\infty(M)$, the operators P_j are in $\Psi^{\tilde{m}+1}(M, \mathcal{E})$ and the operators R_j are in $\Psi^{-\infty}(M, \mathcal{E})$.

Next, in the sequel we denote by Γ_0 the operator in $\Psi^{-n}(\mathbb{R}^n)$ with Schwartz kernel $k_{\Gamma_0}(x,y) = -\log|x-y|$. Notice that $c_{\Gamma_0}(x) = 1$ for all $x \in \mathbb{R}^n$.

Lemma 4.6. *Let $c \in C_c^\infty(\mathbb{R}^n)$ be such that $\int c(x)dx = 0$. Then there exist functions $c_1, \ldots, c_n$ in $C_c^\infty(\mathbb{R}^n)$ so that we have*

$$c\Gamma_0 = [\partial_{x_1}, c_1\Gamma_0] + \ldots + [\partial_{x_n}, c_n\Gamma_0] + Q, \tag{4.9}$$

for some $Q \in \Psi^{-n}(\mathbb{R}^n)$ such that $c_Q(x)$ vanishes everywhere.

Proof. Since $c(x)$ has compact support and we have $\int c(x)dx = 0$ there exist functions $c_1, \ldots, c_n$ in $C_c^\infty(\mathbb{R}^n)$ such that $c = \sum_{j=1}^n \partial_{x_j} c_j$ (see, e.g., [Po, pp. 24-25]). Set $P = \sum_{j=1}^n [\partial_{x_j}, c_j\Gamma_0]$. Then P is a ΨDO of order $-n$. Moreover, since Γ_0 has Schwartz kernel $k_{\Gamma_0}(x,y) = -\log|x-y|$, the Schwartz kernel of P is

$$\begin{aligned} k_P(x,y) &= \sum_{j=1}^n -(\partial_{x_j} - \partial_{y_j})(c_j(x)\log|x-y|) \\ &= -\sum_{j=1}^n \partial_{x_j} c_j(x) \log|x-y| - \sum_{j=1}^n c_j(x)(x_j - y_j)|x-y|^{-2}. \end{aligned} \tag{4.10}$$

From this we see that $c_P(x) = \sum_{j=1}^n \partial_{x_j} c_j(x) = c(x) = c_{c\Gamma_0}(x)$. Therefore $c\Gamma_0$ and $P = \sum_{j=1}^n [\partial_{x_j}, c_j\Gamma_0]$ differ by an operator $Q \in \Psi^{-n}(\mathbb{R}^n)$ such that $c_Q(x) = 0$. This proves the lemma. ☐

Let ρ and χ be functions in $C_c^\infty(\mathbb{R}^n)$ such that $\int \rho(x)dx = 1$ and $\chi(x) = 1$ near $\operatorname{supp}\rho$.

Lemma 4.7. *Let $P \in \Psi_{\mathrm{c}}^{m}(\mathbb{R}^n)$, $m \in \mathbb{Z}_{\geq -n}$. Then*

$$P = (\operatorname{Res} P)\rho\Gamma_0\chi + [\psi\partial_{x_1}\psi, c_1\Gamma_0\psi] + \ldots + [\psi\partial_{x_n}\psi, c_n\Gamma_0\psi] + Q, \tag{4.11}$$

where ψ and $c_1, \ldots, c_n$ are functions in $C_c^\infty(\mathbb{R}^n)$, and Q is an operator in $\Psi_{\mathrm{c}}^{m}(\mathbb{R}^n)$ such that $c_Q(x)$ vanishes everywhere.

Proof. Let $P \in \Psi_{\mathrm{c}}^{m}(\mathbb{R}^n)$ and set $c(x) = c_P(x) - (\operatorname{Res} P)\rho(x)$. Then $c(x)$ is a function in $C_c^\infty(\mathbb{R}^n)$ such that $\int c(x)dx = \int c_P(x)dx - \operatorname{Res} P = 0$. Therefore, by Lemma 4.6 there exist functions $c_1, \ldots, c_n$ in $C_c^\infty(\mathbb{R}^n)$ such that

$$c\Gamma_0 = [\partial_{x_1}, c_1\Gamma_0] + \ldots + [\partial_{x_n}, c_n\Gamma_0] + Q, \tag{4.12}$$

with $Q \in \Psi^{-n}(\mathbb{R}^n)$ such that $c_Q(x) = 0$ for all $x \in \mathbb{R}^n$.

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ near $\operatorname{supp} c \cup \operatorname{supp} c_1 \cup \ldots \cup \operatorname{supp} c_n$. Then for $j = 1, \ldots, n$ the operator $\psi[\partial_{x_j}, c_j\Gamma_0]\psi$ is equal to

$$\psi(\partial_{x_j})\psi c_j\Gamma_0\psi - c_j\Gamma_0(\partial_{x_j})\psi = [\psi(\partial_{x_j})\psi, c_j\Gamma_0\psi] - c_j\Gamma_0(1-\psi^2)(\partial_{x_j})\psi. \tag{4.13}$$

Each operator $c_j\Gamma_0(1-\psi^2)(\partial_{x_j})\psi$ is smoothing and has a compactly supported Schwartz kernel. Therefore, using (4.12) and (4.13) we see that

$$c\Gamma_0\psi = \psi c\Gamma_0\psi = \sum_{j=1}^{n}\psi[\partial_{x_j}, c_j\Gamma_0]\psi + \psi Q\psi = \sum_{j=1}^{n}[\psi(\partial_{x_j})\psi, c_j\Gamma_0\psi] + Q', \tag{4.14}$$

with $Q' \in \Psi_{\mathrm{c}}^{-n}(\mathbb{R}^n)$ such that $c_{Q'}(x)$ vanishes everywhere.

Set $\tilde{P} = (\operatorname{Res} P)\rho\Gamma_0\chi + c\Gamma_0\psi$. As $\tilde{P}$ agrees with $(\operatorname{Res} P)\rho\Gamma_0 + c\Gamma_0$ up to a smoothing operator, we have $c_{\tilde{P}}(x) = (\operatorname{Res} P)\rho(x)c_{\Gamma_0}(x) + c(x)c_{\Gamma_0}(x) = c_P(x)$. Therefore, we can write

$$P = \tilde{P} + \tilde{Q} = (\operatorname{Res} P)\rho\Gamma_0\chi + c\Gamma_0\psi + \tilde{Q}, \tag{4.15}$$

with $\tilde{Q} \in \Psi_{\mathrm{c}}^{-m}(\mathbb{R}^n)$ such that $c_{\tilde{Q}}(x) = 0$ for all $x \in \mathbb{R}^n$. Combining this with (4.14) proves the lemma. □

Proposition 4.8. *Let $U \subset M$ be the domain of a local chart over which $\mathcal{E}$ is trivializable. Then there exists $P_0 \in \Psi_{\mathrm{c}}(U, \mathcal{E})$ such that any $P \in \Psi_{\mathrm{c}}^{m}(U, \mathcal{E})$, $m \in \mathbb{Z}$, can be written in the form*

$$\begin{aligned} P = (\operatorname{Res} P)P_0 + \sum_{j=1}^{n}[a_j, P_j] + \sum_{j=1}^{n}[L_j, Q_j] + [Q_{n+1}, Q_{n+2}] \\ + [R_1, R_2] + [R_3, R_4], \end{aligned} \tag{4.16}$$

where the functions a_j are in $C_c^\infty(U)$, the operators P_j are in $\Psi_{\mathrm{c}}^{\tilde{m}+1}(U, \mathcal{E})$ with $\tilde{m} = \sup(m, -n)$, the L_j are compactly supported first order differential operators, the R_j are in $\Psi_{\mathrm{c}}^{-\infty}(U, \mathcal{E})$ and the operators Q_j are in $\Psi_{\mathrm{c}}^{-n}(U, \mathcal{E})$ and can be chosen to be zero when $c_P(x)$ vanishing everywhere.

Proof. First, since U is diffeomorphic to an open subset of $\mathbb{R}^n$ and $\mathcal{E}$ is trivializable over U, we may as well assume that $U = \mathbb{R}^n$ and $\mathcal{E}$ is trivial. Thus we only have to prove the result for operators in $\Psi_{\mathrm{c}}^{\mathbb{Z}}(\mathbb{R}^n, \mathbb{C}^r) = \Psi_{\mathrm{c}}^{\mathbb{Z}}(\mathbb{R}^n) \otimes M_r(\mathbb{C})$.

Second, it follows from (3.14) and (4.6) that if $Q \in \Psi_{\mathrm{c}}^{m}(\mathbb{R}^n, \mathbb{C}^r)$ is such that $c_Q(x)$ vanishes everywhere, then

$$Q = [a_1, P_1] + \ldots + [a_n, P_n], \tag{4.17}$$

with $a_1, \ldots, a_n$ in $C_c^\infty(\mathbb{R}^n)$ and $P_1, \ldots, P_n$ in $\Psi_{\mathrm{c}}^{m}(\mathbb{R}^n, \mathbb{C}^r)$.

Let $P = (P_{k,l})_{1\leq k,l\leq n} \in \Psi^{m}_{\text{c}}(\mathbb{R}^n, \mathbb{C}^r)$, $m \in \mathbb{Z}_{\geq -n}$, and set $A = (\operatorname{Res} P_{k,l})_{1\leq k,l\leq n}$. Applying Lemma 4.7 to each operator $P_{k,l}$ shows that there exist compactly supported first order differential operators $L_1, \ldots, L_n$ and operators $Q_1, \ldots, Q_n$ in $\Psi^{-n}_{\text{c}}(\mathbb{R}^n, \mathbb{C}^r)$ such that

$$P = (\rho\Gamma_0\chi) \otimes A + [L_1, Q_1] + \ldots + [L_1, Q_n] + Q, \tag{4.18}$$

with $Q \in \Psi^{m}_{\text{c}}(\mathbb{R}^n, \mathbb{C}^r)$ such that $c_Q(x) = 0$ for all $x \in \mathbb{R}^n$.

As $\operatorname{tr} A = \sum \operatorname{Res} P_{k,k} = \int \operatorname{tr} c_P(x)dx = \operatorname{Res} P$, the matrix $A - \frac{1}{n}(\operatorname{Res} P)I_n$ has a zero trace, and hence is a commutator (see [Sh], [AM]). Therefore, we can write $A = \frac{1}{n}(\operatorname{Res} P)I_n + [A_1, A_2]$, with A_1 and A_2 in $M_r(\mathbb{C})$. Set $P_0 = (\rho\Gamma_0\chi) \otimes (\frac{1}{n}I_n)$ and $Q_{n+j} = (\rho\Gamma_0\chi) \otimes A_j$. Then

$$P = (\operatorname{Res} P)P_0 + [L_1, Q_1] + \ldots + [L_1, Q_n] + [Q_{n+1}, Q_{n+2}] + Q. \tag{4.19}$$

Combining this with (4.17) shows that P can be put in the form (4.16), proving the proposition. □

We are now in position to prove Theorem 2.3.

Proof of Theorem 2.3. We need to show that any trace τ on $\Psi^{\mathbb{Z}}(M, \mathcal{E})$ is a constant-multiple of the noncommutative residue. Let $U \subset M$ be domain of a local chart over which $\mathcal{E}$ is trivializable, and let τ_U denote the restriction of τ to $\Psi^{\mathbb{Z}}_{\text{c}}(U, \mathcal{E})$. By Proposition 4.8 there exists $P_0 \in \Psi^{-n}_{\text{c}}(U, \mathcal{E})$ such that for any $P \in \Psi^{\mathbb{Z}}_{\text{c}}(U, \mathcal{E})$ we have $P = (\operatorname{Res} P)P_0$ modulo $[\Psi^{\mathbb{Z}}_{\text{c}}(U, \mathcal{E}), \Psi^{\mathbb{Z}}_{\text{c}}(U, \mathcal{E})]$. Therefore, setting $c_U := \tau(P_0)$, we have

$$\tau(P) = \tau[(\operatorname{Res} P)P_0] = c_U \operatorname{Res} P \qquad \forall P \in \Psi^{\mathbb{Z}}_{\text{c}}(U, \mathcal{E}). \tag{4.20}$$

Let Λ be the set of points $x \in M$ near which there is a domain $V \subset M$ of a local chart over which $\mathcal{E}$ is trivializable in such way that $c_V = c_U$. This is a non-empty open subset of M. Let us show that Λ is closed as well. Let $x \in \overline{\Lambda}$ and let $V \subset M$ be a domain of a local chart near x over which $\mathcal{E}$ is trivializable. Let $y \in \Lambda \cap V$ and let W be a be a domain of a local chart near y over which $\mathcal{E}$ is trivializable and in such way that $c_W = c_U$. We always can find $P \in \Psi^{\mathbb{Z}}_{\text{c}}(V \cap W, \mathcal{E})$ such that $\operatorname{Res} P \neq 0$. Then $\tau(P) = c_V \operatorname{Res} P = c_W \operatorname{Res} P = c_U \operatorname{Res} P$, so we see that $c_V = c_U$. Therefore x is contained in Λ, proving that Λ is a non-empty subset of M which is both open and closed. Since M is connected it follows that Λ agrees with M. Thus there exists $c \in \mathbb{C}$ such that, for any domain $U \subset M$ of a local chart over which $\mathcal{E}$ is trivializable, we have

$$\tau(P) = c \operatorname{Res} P \qquad \forall P \in \Psi^{\mathbb{Z}}_{\text{c}}(U, \mathcal{E}). \tag{4.21}$$

Let (φ_i) be a finite partition of the unity subordinate to an open covering (U_i) of M by domains of charts over which $\mathcal{E}$ is trivializable. For each index i let $\psi_i \in C^\infty_c(U_i)$ be such that $\psi_i = 1$ near $\operatorname{supp} \varphi_i$. Then any $P \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$ can be written as $P = \sum \varphi_i P \psi_i + R$, where R is a smoothing operator. By Proposition 4.5 the operator R is a sum of commutators in $\Psi^{\mathbb{Z}}(M, \mathcal{E})$, and hence $\tau(R) = 0$. Moreover, for each index i the operator $\varphi_i P \psi_i$ belongs to $\Psi^{\mathbb{Z}}_{\text{c}}(U_i, \mathcal{E})$, so using (4.21) we get

$$\tau(P) = \sum \tau(\varphi_i P \psi_i) = \sum c \operatorname{Res}(\varphi_i P \psi_i) = c \operatorname{Res}(\sum \varphi_i P \psi_i) = c \operatorname{Res} P, \tag{4.22}$$

showing that τ is a constant-multiple of the noncommutative residue. The proof of Theorem 2.3 is complete. □

Finally, as a corollary of Theorem 2.3 we have:

Corollary 4.9 ([Wo2], [Gu3])**.** *Suppose that M is connected. Then an operator $P \in \Psi^{\mathbb{Z}}(M,\mathcal{E})$ belongs to the commutator space $[\Psi^{\mathbb{Z}}(M,\mathcal{E}),\Psi^{\mathbb{Z}}(M,\mathcal{E})]$ if and only if* $\operatorname{Res} P$ *vanishes.*

Proof. Using the isomorphism of the space of traces on $\Psi^{\mathbb{Z}}(M,\mathcal{E})$ with the dual of $\Psi^{\mathbb{Z}}(M,\mathcal{E})/[\Psi^{\mathbb{Z}}(M,\mathcal{E}),\Psi^{\mathbb{Z}}(M,\mathcal{E})]$, we deduce from Theorem 2.3 that the latter space has dimension 1.

Let $P_0 \in \Psi^{\mathbb{Z}}(M,\mathcal{E})$ be such that $\operatorname{Res} P_0 \neq 0$. Then $P_0 \notin [\Psi^{\mathbb{Z}}(M,\mathcal{E}),\Psi^{\mathbb{Z}}(M,\mathcal{E})]$. As $\Psi^{\mathbb{Z}}(M,\mathcal{E})/[\Psi^{\mathbb{Z}}(M,\mathcal{E}),\Psi^{\mathbb{Z}}(M,\mathcal{E})]$ has codimension 1, we see that, for all P in $\Psi^{\mathbb{Z}}(M,\mathcal{E})$, there exists $\lambda \in \mathbb{C}$ such that

$$P = \lambda P_0 \mod [\Psi^{\mathbb{Z}}(M,\mathcal{E}),\Psi^{\mathbb{Z}}(M,\mathcal{E})]. \tag{4.23}$$

Observe that $\operatorname{Res} P = \lambda \operatorname{Res} P_0$, so we have

$$P = \frac{\operatorname{Res} P}{\operatorname{Res} P_0} P_0 \mod [\Psi^{\mathbb{Z}}(M,\mathcal{E}),\Psi^{\mathbb{Z}}(M,\mathcal{E})]. \tag{4.24}$$

Thus P is in $[\Psi^{\mathbb{Z}}(M,\mathcal{E}),\Psi^{\mathbb{Z}}(M,\mathcal{E})]$ if and only if $\operatorname{Res} P = 0$, giving the corollary. ☐

5. TRACES ON ZEROTH ORDER ΨDOs ($n \geq 2$)

The aim of this section is to prove Theorem 2.5 om the characterization of traces on zeroth order ΨDOs in dimension greater than or equal to 2.

Recall that for any $P \in \Psi^0(M,\mathcal{E})$ the zeroth order symbol of P uniquely defines a section $\sigma_0(P) \in C^\infty(S^*M, \operatorname{End}\mathcal{E})$, where $S^*M = T^*M/\mathbb{R}_+$ denotes the cosphere bundle of M. In addition, if L is a linear form on $C^\infty(S^*M)$ then its associated leading-symbol trace τ_L is the trace on $\Psi^0(M,\mathcal{E})$ given by

$$\tau_L(P) = L[\operatorname{tr}_{\mathcal{E}} \sigma_0(P)] \qquad \forall P \in \Psi^0(M,\mathcal{E}). \tag{5.1}$$

Let $(\varphi_i)_{1\leq i\leq N}$ be a partition of the unity subordinate to an open covering of M by domains U_i of local chart maps $\kappa_i : U_i \rightarrow V_i \subset \mathbb{R}^n$ over which there are trivialization maps $\tau_i : \mathcal{E}_{|U_i} \rightarrow U_i \times \mathbb{C}^r$. For each index i let $\psi_i \in C_c^\infty(U_i)$ be such that $\psi_i = 1$ near $\operatorname{supp}\varphi_i$. In addition let $\chi \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ near $\xi = 0$.

For $\sigma \in C^\infty(S^*M, \operatorname{End}\mathcal{E})$ we let $P_\sigma \in \Psi^0(M,\mathcal{E})$ be the ΨDO given by

$$P_\sigma = \sum \varphi_i[\tau_i^*\kappa_i^* p_i(x,D)]\psi_i, \quad p_i(x,\xi) = (1-\chi(\xi))(\kappa_{i*}\tau_{i*}\sigma)(x,|\xi|^{-1}\xi). \tag{5.2}$$

On each open subset U_i the operator $\varphi_i\tau_i^*\kappa_i^* p_i(x,D)\psi_i$ has principal symbol $\varphi_i\sigma$, so we see that $\sigma_0(P) = \sum \varphi_i\sigma = \sigma$. Furthermore, the following holds.

Lemma 5.1. *Let $\sigma \in C^\infty(S^*M, \operatorname{End}\mathcal{E})$. Then:*

1) The density $c_{P_\sigma}(x)$ vanishes everywhere.

2) There exist $\sigma_1,\ldots,\sigma_{2N}$ in $C^\infty(S^*M, \operatorname{End}\mathcal{E})$ and $Q \in \Psi^{-1}(M,\mathcal{E})$ such that

$$\sigma = \frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}} + [\sigma_1,\sigma_2] + \ldots + [\sigma_{2N-1},\sigma_{2N}], \tag{5.3}$$

$$P_\sigma = P_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}}} + [P_{\sigma_1},P_{\sigma_2}] + \ldots + [P_{\sigma_{2N-1}},P_{\sigma_{2N}}] + Q. \tag{5.4}$$

Proof. In (5.2) each symbol $p_i(x,\xi)$ has no homogeneous component of degree $-n$, so $c_{p_i(x,D)}(x) = 0$ for all $x \in U_i$. As $\varphi_i[\tau_i^*\kappa_i^* p_i(x,D)]\psi_i$ and $\varphi_i[\tau_i^*\kappa_i^* p_i(x,D)]$ agree on U_i up to a smoothing operator, we see that, for all $x \in M$, we have $c_{\varphi_i[\tau_i^*\kappa_i^* p_i(x,D)]\psi_i}(x) = \varphi_i(x) c_{\tau_i^*\kappa_i^* p_i(x,D)}(x) = \varphi_i(x)\tau_i^*\kappa_i^* c_{p_i(x,D)}(x) = 0$. It then follows that, for all $x \in M$, we have $c_{P_\sigma}(x) = \sum c_{\varphi_i[\tau_i^*\kappa_i^* p_i(x,D)]\psi_i}(x) = 0$.

Next, any matrix with vanishing trace is a commutator (see [Ka], [Sh]). It can be seen from the proof in [Ka] that this result continues to hold for smooth families of matrices. This implies that, for each index i, there exist $\sigma_i^{(1)}(x,\xi)$ and $\sigma_i^{(2)}(x,\xi)$ in $S^0(S^*U_i, \operatorname{End}\mathcal{E})$ such that on S^*U_i we have

$$\sigma(x,\xi) = \frac{1}{r}(\operatorname{tr}_{\mathcal{E}_x}\sigma(x,\xi))\operatorname{id}_{\mathcal{E}_x} + [\sigma_i^{(1)}(x,\xi), \sigma_i^{(2)}(x,\xi)]. \tag{5.5}$$

Therefore, on S^*M we have

$$\begin{aligned}\sigma(x,\xi) &= \sum_{1\leq i\leq N} \varphi_i(x)\psi_i(x)\sigma(x,\xi) \\ &= \sum_{1\leq i\leq N}\varphi_i(x)\psi_i(x)\left\{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}_x}\sigma(x,\xi))\operatorname{id}_{\mathcal{E}_x} + [\sigma_i^{(1)}(x,\xi),\sigma_i^{(2)}(x,\xi)]\right\} \\ &= \frac{1}{r}(\operatorname{tr}_{\mathcal{E}_x}\sigma(x,\xi))\operatorname{id}_{\mathcal{E}_x} + \sum_{1\leq i\leq N}[\sigma_{2i-1}(x,\xi),\sigma_{2i}(x,\xi)],\end{aligned} \tag{5.6}$$

where we have set $\sigma_{2i-1}(x,\xi) = \varphi_i(x)\sigma_i^{(1)}(x,\xi)$ and $\sigma_{2i}(x,\xi) = \psi_i(x)\sigma_i^{(2)}(x,\xi)$. This proves (5.3). Combining this with the fact that $P_{[\sigma_{2i-1},\sigma_{2i}]} = [P_{\sigma_{2i-1}}, P_{\sigma_{2i}}]$ modulo $\Psi^{-1}(M,\mathcal{E})$ then gives (5.4). The proof is complete. ☐

We show next that, when the dimension is greater than or equal to 2, in Proposition 4.8 we can replace the first order differential operators L_j by zeroth order ΨDOs. The key fact is the following alternative version of Lemma 4.6.

Lemma 5.2. *Assume $n\geq 2$ and let $c\in C_c^\infty(\mathbb{R}^n)$ be such that $\int c(x)dx = 0$. Then there exist functions $c_1,\ldots,c_n$ in $C_c^\infty(\mathbb{R}^n)$ such that*

$$c(1+\Delta)^{-\frac{n}{2}} = \sum_{j=1}^n [\partial_{x_j}(1+\Delta)^{-\frac{1}{2}}, c_j(1+\Delta)^{\frac{1-n}{2}}] + Q, \tag{5.7}$$

with $Q\in\Psi^{-n}(\mathbb{R}^n)$ such that $c_Q(x)$ vanishes everywhere.

Proof. First, since $c(1+\Delta)^{-\frac{n}{2}}$ is a ΨDO of order $-n$ with principal symbol $c(x)|\xi|^{-n}$ we have

$$c_{c(1+\Delta)^{-\frac{n}{2}}}(x) = (2\pi)^{-n}c(x)\int_{S^{n-1}} d^{n-1}\xi = \frac{|S^{n-1}|}{(2\pi)^n}c(x). \tag{5.8}$$

Second, as $c(x)$ has compact support and $\int c(x)dx = 0$ there exist functions $c_1,\ldots,c_n$ in $C_c^\infty(\mathbb{R}^n)$ such that $c = \sum_{j=1}^n \partial_{x_j}c_j$. Define

$$P = \sum_{j=1}^n [\partial_{x_j}(1+\Delta)^{-\frac{1}{2}}, c_j(1+\Delta)^{\frac{1-n}{2}}]. \tag{5.9}$$

Then P is a ΨDO of order $-n$ whose principal symbol $p_{-n}(x,\xi)$ is equal to

$$\begin{aligned}
(5.10)\qquad \sum_{j,k=1}^{n} \frac{1}{i}\left[\partial_{\xi_k}(i\xi_j|\xi|^{-1})\partial_{x_k}(c_j(x)|\xi|^{1-n}) - \partial_{\xi_k}(c_j(x)|\xi|^{1-n})\partial_{x_k}(i\xi_j|\xi|^{-1})\right] \\
= \sum_{j=1}^{n}\partial_{x_j}c_j(x)|\xi|^{-n} - \sum_{j,k=1}^{n}\xi_j\xi_k\partial_{x_k}c_j(x)|\xi|^{-(n+2)} \\
= c(x)|\xi|^{-n} - \sum_{j,k=1}^{n}\xi_j\xi_k\partial_{x_k}c_j(x)|\xi|^{-(n+2)}.
\end{aligned}$$

Therefore, from (1.14) we obtain

$$(5.11)\qquad (2\pi)^n c_P(x) = c(x)\int_{S^{n-1}} d^{n-1}\xi - \sum_{k=1}^{n}\partial_{x_k}c_j(x)\int_{S^{n-1}}\xi_j\xi_k d^{n-1}\xi.$$

If $k \neq j$ then the change of variable $\xi_k \to -\xi_k$ shows that $\int_{S^{n-1}}\xi_j\xi_k d^{n-1}\xi = -\int_{S^{n-1}}\xi_j\xi_k d^{n-1}\xi = 0$, while for $k=j$ we have

$$(5.12)\qquad \int_{S^{n-1}}\xi_j^2 d^{n-1}\xi = \frac{1}{n}\sum_{l=1}^{n}\int_{S^{n-1}}\xi_l^2 d^{n-1}\xi = \frac{1}{n}\int_{S^{n-1}}d^{n-1}\xi = \frac{|S^{n-1}|}{n}.$$

Thus,

$$(5.13)\qquad (2\pi)^n c_P(x) = c(x)|S^{n-1}| - \sum_{j=1}^{n}\partial_{x_j}c_j(x)\frac{|S^{n-1}|}{n} = \frac{n-1}{n}|S^{n-1}|c(x).$$

Set $Q = c(1+\Delta)^{-\frac{n}{2}} - \frac{n}{n-1}P$. Then $c_Q(x) = \frac{|S^{n-1}|}{(2\pi)^n}c(x) - \frac{n}{n-1}c_P(x) = 0$ for all $x \in \mathbb{R}^n$. Moreover, we have

$$(5.14)\quad c(1+\Delta)^{-\frac{n}{2}} = \frac{n}{n-1}P + Q = \sum_{j=1}^{n}[\partial_{x_j}(1+\Delta)^{-\frac{1}{2}}, \frac{n}{n-1}c_j(1+\Delta)^{\frac{1-n}{2}}] + Q,$$

proving the lemma. ☐

Thanks to Lemma 5.2 we may argue as in the proofs of Lemma 4.7 and Proposition 4.8 to get:

Proposition 5.3. *Assume $n \geq 2$ and let $U \subset M$ be a local open chart over which $\mathcal{E}$ is trivializable. Then there exists $P_0 \in \Psi_{\mathrm{c}}(U,\mathcal{E})$ such that any operator $P \in \Psi_{\mathrm{c}}^m(U,\mathcal{E})$, $m \in \mathbb{Z}$, can be written in the form*

$$(5.15)\qquad P = (\operatorname{Res} P)P_0 + \sum_{j=1}^{2n}[A_j, P_j] + [Q_1,Q_2] + [R_1,R_2] + [R_3,R_4],$$

where the A_j are in $\Psi_{\mathrm{c}}^0(U,\mathcal{E})$, the P_j are in $\Psi_{\mathrm{c}}^{m+1}(U,\mathcal{E})$, the Q_j are in $\Psi_{\mathrm{c}}^{-n}(U,\mathcal{E})$, and the R_j are in $\Psi_{\mathrm{c}}^{-\infty}(U,\mathcal{E})$.

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. Let τ be a trace on the algebra $\Psi^0(M,\mathcal{E})$. Let $U \subset M$ be a local open chart over which $\mathcal{E}$ is trivializable. By Proposition 5.3 there exists $P_0 \in \Psi_{\mathrm{c}}(U,\mathcal{E})$ such that for any $P \in \Psi_{\mathrm{c}}^{-1}(U,\mathcal{E})$ we have

$$(5.16)\qquad P = (\operatorname{Res} P)P_0 \mod [\Psi_{\mathrm{c}}^0(U,\mathcal{E}), \Psi_{\mathrm{c}}^0(U,\mathcal{E})].$$

It follows that $\tau(P) = \tau(P_0) \operatorname{Res} P$ for all $P \in \Psi_c^{-1}(U, \mathcal{E})$. As in the proof of Theorem 2.3 we then can show that there exists $c \in \mathbb{C}$ such that

$$\tau(P) = c \operatorname{Res} P \qquad \forall P \in \Psi^{-1}(M, \mathcal{E}). \tag{5.17}$$

Set $\tilde{\tau}(P) = \tau(P) - c \operatorname{Res}(P)$. Then $\tilde{\tau}$ is a trace on $\Psi^0(M, \mathcal{E})$ vanishing on $\Psi^{-1}(M, \mathcal{E})$. In addition, let L be the linear form on $C^\infty(S^*M)$ defined by

$$L(\sigma) = \tilde{\tau}(P_{\frac{1}{r}\sigma \operatorname{id}_{\mathcal{E}}}) \qquad \forall \sigma \in C^\infty(S^*M). \tag{5.18}$$

Let $P \in \Psi^0(M, \mathcal{E})$. Since $P - P_{\sigma_0(P)}$ has order ≤ -1, by Lemma 5.1 there exist $\sigma_1, \ldots, \sigma_{2N}$ in $C^\infty(S^*M, \operatorname{End} \mathcal{E})$ and $Q \in \Psi^{-1}(M, \mathcal{E})$ such that

$$P = P_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}} \sigma_0(P)) \operatorname{id}_{\mathcal{E}}} + [P_{\sigma_1}, P_{\sigma_2}] + \ldots + [P_{\sigma_{2N-1}}, P_{\sigma_{2N}}] + Q. \tag{5.19}$$

Since $\tilde{\tau}$ is a trace on $\Psi^0(M, \mathcal{E})$ vanishing on $\Psi^{-1}(M, \mathcal{E})$ we get

$$\tilde{\tau}(P) = \tilde{\tau}(P_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}} \sigma_0(P)) \operatorname{id}_{\mathcal{E}}}) = L(\operatorname{tr}_{\mathcal{E}} \sigma_0(P)) = \tau_L(P). \tag{5.20}$$

Thus $\tau = \tilde{\tau} + c \operatorname{Res} = \tau_L + c \operatorname{Res}$, showing that τ is a sum of a leading-symbol trace and of a constant-multiple of the noncommutative residue.

Let us now show that the decomposition $\tau = \tau_L + c \operatorname{Res}$ is unique. Suppose that we have another decomposition $\tau = \tau_{L_1} + c_1 \operatorname{Res} P$, with $L_1 \in C^\infty(S^*M)^*$ and $c_1 \in \mathbb{C}$. Let $P_0 \in \Psi^{-1}(M, \mathcal{E})$ be such that $\operatorname{Res} P_0 \neq 0$. As τ_L and τ_{L_1} vanish on $\Psi^{-1}(M, \mathcal{E})$ we have $\tau(P_0) = c \operatorname{Res} P_0$ and $\tau(P_0) = c_1 \operatorname{Res} P_0$, showing that $c_1 = c$.

On the other hand, if $\sigma \in C^\infty(S^*M)$ then it follows from Lemma 5.1 that $\operatorname{Res}(P_{\frac{1}{r}\sigma \operatorname{id}_{\mathcal{E}}}) = 0$, so $\tau(P_{\frac{1}{r}\sigma \operatorname{id}_{\mathcal{E}}}) = \tau_L(P_{\frac{1}{r}\sigma \operatorname{id}_{\mathcal{E}}}) = L(\sigma)$. Similarly, $\tau(P_{\frac{1}{r}\sigma \operatorname{id}_{\mathcal{E}}}) = L_1(\sigma)$, so we see that $L_1 = L$. This proves that L and c are uniquely determined by τ. The proof is therefore complete. □

Finally, as a corollary to Theorem 2.5 we get:

Corollary 5.4. *Suppose that M is connected and has dimension ≥ 2. Then for any operator $P \in \Psi^0(M, \mathcal{E})$ the following are equivalent:*

(i) P belongs to the commutator space $[\Psi^0(M, \mathcal{E}), \Psi^0(M, \mathcal{E})]$.

(ii) $\operatorname{Res} P$ is zero and $\operatorname{tr}_{\mathcal{E}} \sigma_0(P)(x, \xi) = 0$ vanishes everywhere.

Proof. If $P \in [\Psi^0(M, \mathcal{E}), \Psi^0(M, \mathcal{E})]$, then $\operatorname{Res} P$ is zero and $\operatorname{tr}_{\mathcal{E}} \sigma_0(P)(x, \xi)$ vanishes everywhere.

Conversely, it follows from the first part of the proof of Theorem 2.5 that any linear form on $\Psi^{-1}(M, \mathcal{E})$ vanishing on $\Psi^{-1}(M, \mathcal{E}) \cap [\Psi^0(M, \mathcal{E}), \Psi^0(M, \mathcal{E})]$ is a constant-multiple of the noncommutative residue. Therefore, arguing as in the proof of Corollary 4.9 shows that an operator $Q \in \Psi^{-1}(M, \mathcal{E})$ is contained in $[\Psi^0(M, \mathcal{E}), \Psi^0(M, \mathcal{E})]$ if and only if $\operatorname{Res} Q = 0$.

Let $P \in \Psi^0(M, \mathcal{E})$ be such that $\operatorname{Res} P$ is zero and $\operatorname{tr}_{\mathcal{E}} \sigma_0(P)(x, \xi) = 0$ vanishes everywhere. By (5.19) there exist $\sigma_1, \ldots, \sigma_{2N}$ in $C^\infty(S^*M, \operatorname{End} \mathcal{E})$ and there exists $Q \in \Psi^{-1}(M, \mathcal{E})$ such that $P = [P_{\sigma_1}, P_{\sigma_2}] + \ldots + [P_{\sigma_{2N-1}}, P_{\sigma_{2N}}] + Q$. Observe that $\operatorname{Res} Q = \operatorname{Res} P = 0$. As Q has order ≤ -1 it then follows from the discussion above that Q is contained in $[\Psi^0(M, \mathcal{E}), \Psi^0(M, \mathcal{E})]$, and hence P is itself a sum of commutators in $\Psi^0(M, \mathcal{E})$. The proof is complete. □

6. TRACES ON ZEROTH ORDER ΨDOS ($n = 1$)

In this section we shall determine all the traces on $\Psi^0(M, \mathcal{E})$ when $n = 1$. The key observation is the following.

Proposition 6.1. *1) For any $P \in \Psi^0(M, \mathcal{E})$ there exists a unique section $\sigma_{-1}(P)$ in $C^\infty(S^*M, \operatorname{End}\mathcal{E})$ such that, for any local chart map $\kappa : U \to V$ for M and any trivialization map $\tau : \mathcal{E}_{|U} \to U \times \mathbb{C}^r$ of $\mathcal{E}$ over U, we have*

$$[\kappa_* \tau_* \sigma_{-1}(P)](x, \xi) = (x, p_{-1}(x, \xi)) \qquad \forall (x, \xi) \in S^*V, \tag{6.1}$$

where $p_{-1}(x, \xi)$ is the symbol of degree -1 of P in the local coordinates defined by κ and τ.

2) For all P_1 and P_2 in $\Psi^0(M, \mathcal{E})$ we have

$$\sigma_{-1}(P_1 P_2) = \sigma_0(P_1)\sigma_{-1}(P_2) + \sigma_{-1}(P_1)\sigma_0(P_2), \tag{6.2}$$

$$\sigma_{-1}([P_1, P_2]) = [\sigma_0(P_1), \sigma_{-1}(P_2)] + [\sigma_{-1}(P_1), \sigma_0(P_2)]. \tag{6.3}$$

Proof. First, let $\phi : U' \to U$ be a diffeomorphism between open subsets of $\mathbb{R}$, and let $P \in \Psi^0(U)$ have symbol $p(x, \xi) \sim \sum p_{-j}(x, \xi)$. Then the operator $P' = \phi^* P$ is a zeroth order ΨDO on U' whose symbol $p^\phi(x, \xi) \sim \sum p^\phi_{-j}(x, \xi)$ is such that

$$p^\phi(x, \xi) \sim \sum_k a_k(x, \xi)(\partial_\xi^k p)\left(\phi(x), \phi'(x)^{-1}\xi\right), \tag{6.4}$$

where $a_k(x, \xi) = \frac{1}{k!}\frac{\partial}{\partial y}\, e^{i\rho_x(y)\xi}\big|_{y=x}$ and $\rho_x(y) = \phi(y) - \phi(y) - \phi'(x)(x - y)$ (see, e.g., [Hö2]). As $d_y \rho_x|_{y=x} = 0$ we see that $a_k(x, \xi)$ is polynomial in ξ of degree $\leq \frac{k}{2}$, so that we can write $a_k(x, \xi) = \sum_{2l \leq k} a_{kl}(x)\xi^l$ with $a_{kl}(x) \in C^\infty(U')$. Thus,

$$p^\phi_{-j}(x, \xi) = \sum_{\substack{j'+k-l=j \\ 2l \leq l}} a_{kl}(x)\xi^l(\partial_\xi^k p_{-j'})\left(\phi(x), \phi'(x)^{-1}\xi\right). \tag{6.5}$$

Observe that in dimension 1 the degree-zero homogeneity implies that we have $p_0(x, \xi) = p_0(x, \pm 1)$ depending on the sign of ξ. In any case $\partial_\xi p_0 = 0$ everywhere, so for $j = -1$ the equation (6.5) reduces to

$$p^\phi_{-1}(x, \xi) = p_{-1}(\phi(x), \phi'(x)^{-1}\xi). \tag{6.6}$$

Next, let $Q \in \Psi^0(U)$ have symbol $q \sim \sum q_{-j}$ and suppose that P or Q is properly supported. Then PQ belongs to $\Psi^0(U)$ and has symbol $p\#q \sim \sum \frac{(-i)^k}{k!}\partial_\xi^k p \partial_x^k q$. Thus its symbol $(p\#q)_{-1}(x, \xi)$ of degree -1 is

$$p_0 q_{-1} + \frac{1}{i}\partial_\xi p_0 \partial_x q_0 + p_{-1}q_0 = p_0 q_{-1} + p_{-1} q_0. \tag{6.7}$$

The formulas (6.5) and (6.6) extend *verbatim* to vector-valued ΨDOs and matrix-valued symbols. Thus if $P \in \Psi^0(U, \mathbb{C}^r)$ has symbol $p(x, \xi) \sim \sum p_{-j}(x, \xi)$ and if and A and B are in $C^\infty(U, M_r(\mathbb{C}))$, then the symbol of degree -1 of APB is

$$(A\#p\#B)_{-1} = A(x)p_{-1}(x, \xi)B(x). \tag{6.8}$$

Together with (6.6) this implies that, for all $P \in \Psi^0(M, \mathcal{E})$, there is a unique section $\sigma_{-1}(P) \in C^\infty(S^*M, \operatorname{End}\mathcal{E})$ satisfying (6.1). Then (6.2) and (6.3) follow from (6.7). The lemma is proved. ☐

If L is linear form on $C^\infty(S^*M)$, then we denote by ρ_L the linear form on $\Psi^0(M,\mathcal{E})$ defined by

$$\rho_L(P)=L[\operatorname{tr}_{\mathcal{E}}\sigma_{-1}(P)] \qquad \forall P\in\Psi^0(M,\mathcal{E}). \tag{6.9}$$

If P_1 and P_2 are operators in $\Psi^0(M,\mathcal{E})$, then using (6.3) we get

$$\rho_L([P_1,P_2])=L\left(\operatorname{tr}_{\mathcal{E}}[\sigma_0(P_1),\sigma_{-1}(P_2)]\right)+L\left(\operatorname{tr}_{\mathcal{E}}[\sigma_{-1}(P_1),\sigma_0(P_2)]\right)=0. \tag{6.10}$$

Thus ρ_L is a trace on the algebra $\Psi^0(M,\mathcal{E})$. We shall call such a trace a *subleading-symbol trace*. The noncommutative residue is such a trace, for we have

$$\operatorname{Res}P=\int_{S^*M}\operatorname{tr}_{\mathcal{E}}\sigma_{-1}(P)(x,\xi)dxd\xi \qquad \forall P\in\Psi^0(M,\mathcal{E}), \tag{6.11}$$

where $dxd\xi$ is the Liouville measure of $S^*M=T^*M/\mathbb{R}_+$.

On the other hand, as in (5.2) we can construct a cross-section $\sigma\rightarrow Q_\sigma$ from $C^\infty(S^*M,\operatorname{End}\mathcal{E})$ to $\Psi^{-1}(M,\mathcal{E})$ such that $\sigma_{-1}(Q_\sigma)=\sigma$ $\forall\sigma\in C^\infty(S^*M,\operatorname{End}\mathcal{E})$. More precisely, for $\sigma\in C^\infty(S^*M,\operatorname{End}\mathcal{E})$ we define Q_σ to be

$$Q_\sigma=\sum\varphi_i[\tau_i^*\kappa_i^*q_i(x,D)]\psi_i, \quad q_i(x,\xi)=(1-\chi(\xi))|\xi|^{-1}(\kappa_{i*}\tau_{i*}\sigma)(x,\frac{\xi}{|\xi|}), \tag{6.12}$$

where the notation is the same as in (5.2).

Lemma 6.2. *Let $\sigma\in C^\infty(S^*M,\operatorname{End}\mathcal{E})$. Then:*

1) The symbol $\sigma_{-1}(P_\sigma)$ vanishes everywhere.

2) There exist $\sigma_1,\ldots,\sigma_{2N}$ in $C^\infty(S^*M,\operatorname{End}\mathcal{E})$ and R_1,R_2 in $\Psi^{-2}(M,\mathcal{E})$ so that

$$P_\sigma=P_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}}}+[P_{\sigma_1},P_{\sigma_2}]+\ldots+[P_{\sigma_{2N-1}},P_{\sigma_{2N}}]+R_1, \tag{6.13}$$

$$Q_\sigma=Q_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}}}+[Q_{\sigma_1},Q_{\sigma_2}]+\ldots+[Q_{\sigma_{2N-1}},Q_{\sigma_{2N}}]+R_2. \tag{6.14}$$

Proof. As in (5.2) the symbol $p_i(x,\xi)$ has no homogeneous component of degree -1, from (6.1) and (6.8) we get

$$\sigma_{-1}\left(\varphi_i[\tau_i^*\kappa_i^*p_i(x,D)]\psi_i\right)=\varphi_i[\tau_i^*\kappa_i^*\sigma_{-1}(p_i(x,D))]\psi_i=0. \tag{6.15}$$

Hence $\sigma_{-1}(P_\sigma)=\sum\sigma_{-1}[\varphi_i\tau_i^*\kappa_i^*p_i(x,D)\psi_i]=0$.

By Lemma 5.1 there exist sections $\sigma_1,\ldots,\sigma_{2N}$ in $C^\infty(S^*M,\operatorname{End}\mathcal{E})$ such that $\sigma=\frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}}+\sum[\sigma_{2j-1},\sigma_{2j}]$. Then Q_σ and $Q_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}}}+\sum[Q_{\sigma_{2j-1}},Q_{\sigma_{2j}}]$ are ΨDOs of order -1 with the same principal symbol, so they agree modulo an operator in $\Psi^{-2}(M,\mathcal{E})$. Hence Q_σ is of the form (6.14).

Thanks to (5.4) there exists R_1 in $\Psi^{-1}(M,\mathcal{E})$ such that

$$P_\sigma=P_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}}}+\sum_{1\leq j\leq N}[P_{\sigma_{2j-1}},P_{\sigma_{2j}}]+R_1, \tag{6.16}$$

Let us show that R_{-1} has order ≤ -2, which will prove (6.13). Indeed, by the first part of the proof $\sigma_{-1}(P_\sigma)=\sigma(P_{\frac{1}{r}(\operatorname{tr}_{\mathcal{E}}\sigma)\operatorname{id}_{\mathcal{E}}})=0$. Likewise, using (6.2) we get $\sigma_{-1}(P_{\sigma_j}P_{\sigma_{j+1}})=\sigma_0(P_{\sigma_j})\sigma_{-1}(P_{\sigma_{j+1}})+\sigma_{-1}(P_{\sigma_j})\sigma_0(P_{\sigma_{j+1}})=0$. Therefore, we see that R_1 is a linear combination of zeroth order ΨDOs whose symbols of degree -1 vanish, so $\sigma_{-1}(R_1)=0$. Since R_1 has order ≤ -1 this means that R_1 belongs to $\Psi^{-2}(M,\mathcal{E})$, giving (6.13). The proof is complete. □

We are ready to prove the main result of this section.

Theorem 6.3. *Assume that* $\dim M = 1$*. Then:*

1) Any trace on $\Psi^0(M,\mathcal{E})$ *can be uniquely written as the sum of a leading-symbol trace and of a subleading-symbol trace.*

2) An operator $P \in \Psi^0(M,\mathcal{E})$ *is a sum of commutators in* $\Psi^0(M,\mathcal{E})$ *if and only if* $\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P)(x,\xi)$ *and* $\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P)(x,\xi)$ *both vanish everywhere.*

Proof. First, if $P \in \Psi^0(M,\mathcal{E})$ belongs to $[\Psi^0(M,\mathcal{E}),\Psi^0(M,\mathcal{E})]$ then $\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P)(x,\xi)$ vanishes everywhere and by (6.3) $\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P)(x,\xi)$ is identically zero too.

Conversely, let $P \in \Psi^0(M,\mathcal{E})$. Since by Lemma 6.2 we have $\sigma_{-1}(P_{\sigma_0(P)}) = 0$ we see that the symbols of degrees 0 and -1 of $P - P_{\sigma_0(P)} - Q_{\sigma_{-1}(P)}$ both vanish everywhere, and hence $P - P_{\sigma_0(P)} - Q_{\sigma_{-1}(P)}$ has order ≤ -2. Combining this with the second part of Lemma 6.2 shows that there exists R in $\Psi^{-2}(M,\mathcal{E})$ such that

$$P = P_{\frac{1}{r}(\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P))\,\mathrm{id}_{\mathcal{E}}} + Q_{\frac{1}{r}(\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P))\,\mathrm{id}_{\mathcal{E}}} + R, \tag{6.17}$$

Since in dimension 1 Proposition 4.5 implies that $\Psi^{-2}(M,\mathcal{E})$ is contained in the commutator space $[\Psi^0(M,\mathcal{E}),\Psi^0(M,\mathcal{E})]$, we see that

$$P = P_{\frac{1}{r}(\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P))\,\mathrm{id}_{\mathcal{E}}} + Q_{\frac{1}{r}(\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P))\,\mathrm{id}_{\mathcal{E}}} \quad \bmod [\Psi^0(M,\mathcal{E}),\Psi^0(M,\mathcal{E})]. \tag{6.18}$$

In particular, if $\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P)(x,\xi)$ and $\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P)(x,\xi)$ vanish everywhere, then P is a sum of commutators in $\Psi^0(M,\mathcal{E})$.

Next, let τ be a trace on $\Psi^0(M,\mathcal{E})$, and let L_1 and L_2 be the linear forms on $C^\infty(S^*M)$ defined by

$$L_1(\sigma) = \tau(P_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}}) \quad \text{and} \quad L_2(\sigma) = \tau(Q_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}}) \quad \text{for all } \sigma \in C^\infty(S^*M). \tag{6.19}$$

Let $P \in \Psi^0(M,\mathcal{E})$. Then it follows from (6.18) that $\tau(P)$ is equal to

$$\tau[P_{\frac{1}{r}(\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P))\,\mathrm{id}_{\mathcal{E}}}] + \tau[Q_{\frac{1}{r}(\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P))\,\mathrm{id}_{\mathcal{E}}}] = L_1(\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P)) + L_2(\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P)), \tag{6.20}$$

showing that $\tau = \tau_{L_1} + \rho_{L_2}$.

Let (L_1', L_2') be a pair of linear forms on $C^\infty(S^*M)$ such that $\tau = \tau_{L_1'} + \rho_{L_2'}$. Let $\sigma \in C^\infty(S^*M)$. As $\sigma_{-1}(P_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}})$ vanishes everywhere $\rho_{L_2'}(P_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}}) = 0$, and hence $L_1(\sigma) = \tau(P_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}}) = \tau_{L_1'}(P_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}}) = L_1'(\sigma)$, showing that $L_1' = L_1$. Similarly $L_2(\sigma) = \tau(Q_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}}) = \rho_{L_2'}(P_{\frac{1}{r}\sigma\,\mathrm{id}_{\mathcal{E}}}) = L_2'(\sigma)$, so we see that $L_2' = L_2$. This proves the uniqueness of the decomposition $\tau = \tau_{L_1} + \rho_{L_2}$. The proof is complete. $\square$

Finally, when $\mathcal{E}$ is the trivial line bundle the condition that $\mathrm{tr}_{\mathcal{E}}\,\sigma_0(P)(x,\xi)$ and $\mathrm{tr}_{\mathcal{E}}\,\sigma_{-1}(P)(x,\xi)$ both vanish everywhere means that P has order ≤ -2. Since in dimension 1 Proposition 4.5 implies that $\Psi^{-2}(M) \subset [\Psi^0(M),\Psi^0(M)]$ we obtain:

Corollary 6.4. *If* $\dim M = 1$*, then* $[\Psi^0(M),\Psi^0(M)] = \Psi^{-2}(M)$.

References

[AM] Albert, A. A.; Muckenhoupt, B.: *On matrices of trace zeros.* Michigan Math. J. **4** (1957), 1–3.

[BG] Beals, R.; Greiner, P.: *Calculus on Heisenberg manifolds.* Ann. Math. Studies 119, Princeton Univ. Press, 1988.

[CM] Connes, A.; Moscovici, H.: *The local index formula in noncommutative geometry.* Geom. Funct. Anal. **5** (1995), no. 2, 174–243.

[FGLS] Fedosov, B.V.; Golse, F.; Leichtnam, E.; Schrohe, E.: *The noncommutative residue for manifolds with boundary.* J. Funct. Anal. **142** (1996), no. 1, 1–31.

[GVF] Gracia-Bondía, J.M.; Várilly, J.C.; Figueroa, H.: *Elements of noncommutative geometry.* Birkhäuser Boston, Boston, MA, 2001.

[Gu1] Guillemin, V.: *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues.* Adv. in Math. **55** (1985), no. 2, 131–160.

[Gu2] Guillemin, V.: *gauged Lagrangian distributions.* Adv. Math. **102** (1993), no. 2, 184–201.

[Gu3] Guillemin, V.: *Residue traces for certain algebras of Fourier integral operators.* J. Funct. Anal. **115** (1993), no. 2, 391–417.

[Hö1] Hörmander, L.: *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Grundlehren der Mathematischen Wissenschaften, 256. Springer, Berlin, 1990.

[Hö2] Hörmander, L.: *The analysis of linear partial differential operators. III. Pseudodifferential operators.* Grundlehren der Mathematischen Wissenschaften, 274. Springer, Berlin, 1994.

[Ka] Kahan, W.: *Only commutators can have trace zero.* Preprint, June 1999. Available online at `http://www.cs.berkeley.edu/~wkahan/MathH110/trace0.pdf`.

[KV] Kontsevich, M.; Vishik, S.: *Geometry of determinants of elliptic operators.* Progr. Math., 131, 173–197, Birkhäuser Boston, 1995.

[Le] Lesch, M.: *On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols.* Ann. Global Anal. Geom. **17** (1999), no. 2, 151–187.

[LP] Lescure, J.M.; Paycha, S.: *Uniqueness of multiplicative determinants on elliptic pseudodifferential operators.* Proc. London Math. Soc. (3) **94** (2007) 772–812.

[MSS] Maniccia, L.; Schrohe, E., Seiler, J.: *Uniqueness of the Kontsevich-Vishik Trace.* Proc. Amer. Math. Soc. **136** (2008), no. 2, 747–752

[MMS] Mathai, V.; Melrose, R.; Singer, I.: *Fractional index theory.* J. Differential Geom. **74** (2006) 265–292.

[Me] Melrose, R.B.: *The Atiyah-Patodi-Singer index theorem.* A.K. Peters, 1993.

[Pa] Paycha, S.: *The noncommutative residue and canonical trace in the light of Stokes' and continuity properties.* E-print, arXiv, June 2007.

[PR1] Paycha, S.; Rosenberg, S.: *Curvature on determinant bundles and first Chern forms.* J. Geom. Phys. **45** (2003), no. 3-4, 393–429.

[PR2] Paycha, S.; Rosenberg, S.: *Traces and characteristic classes on loop groups.* IRMA Lect. Math. Theor. Phys., 5, 185–212, de Gruyter, Berlin, 2004.

[Po] Ponge, R.: *Noncommutative residue for Heisenberg manifolds and applications in CR and contact geometry.* J. Funct. Anal. **252** (2007) 399–463.

[Sc] Schrohe, E.: *Noncommutative residues and manifolds with conical singularities.* J. Funct. Anal. **150** (1997), no. 1, 146–174.

[Sh] Shoda, K.: *Über den Kommutator der Matrizen.* J. Math. Soc. Japan **3** (1951) 78–81.

[Ta] Taylor, M.E: *Partial differential equations. II. Qualitative studies of linear equations.* Applied Mathematical Sciences, 116. Springer-Verlag, New York, 1996.

[Ug] Ugalde, W.J.: *A Construction of critical GJMS operators using Wodzicki's residue.* Comm. Math. Phys. **261** (2006), no. 3, 771–788.

[Va] Vassout, S.: *Feuilletages et résidu non commutatif longitudinal.* PhD thesis, University of Paris 7, 2001.

[Wo1] Wodzicki, M.: *Local invariants of spectral asymmetry.* Invent. Math. **75** (1984), no. 1, 143–177.

[Wo2] Wodzicki, M.: *Spectral asymmetry and noncommutative residue* (in Russian), Habilitation Thesis, Steklov Institute, (former) Soviet Academy of Sciences, Moscow, 1984.

[Wo3] Wodzicki, M.: *Noncommutative residue. I. Fundamentals.* Lecture Notes in Math., 1289, 320–399, Springer, Berlin, 1987.

[Wo4] Wodzicki, M.: *Report on the cyclic homology of symbols.* Preprint, IAS Princeton, Jan. 87. Available online at `http://math.berkeley.edu/~wodzicki`.

[Wo5] Wodzicki, M.: Personal communication.

Department of Mathematics, University of Toronto, 40 St George Street, Toronto, ON M5S 2E4, Canada.

E-mail address: `ponge@math.utoronto.ca`