

A NEW SHORT PROOF OF THE LOCAL INDEX FORMULA OF ATIYAH-SINGER

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ABSTRACT. In this talk we present a new short proof of the local index formula of Atiyah-Singer for Dirac operators ([AS1], [AS2]) which, as a byproduct and unlike Getzler's short proof, allows us to compute the CM cyclic cocycle for Dirac spectral triples.

INTRODUCTION

The aim of the talk is to present the results of [Po] where a new short proof of the local index formula of Atiyah-Singer for Dirac operators ([AS1], [AS2]) was given. The point is that this proof is as simple as Getzler's short proof [Ge1] but, as a byproduct and unlike Getzler's proof, it allows us to compute the CM cocycle for Dirac spectral triples. This is interesting because:

- The probabilistic arguments in (part of) Getzler's short proof don't go through to compute the CM cocycle;
- The previous computations of the CM cocycle (Connes-Moscovici [CM], Chern-Hu [CH], Lescuré [Le]) made use of the asymptotic ΨDO calculus used by Getzler in its first proof [Ge2], which we can bypass here.

The main ingredients of the proof are:

- Getzler's rescaling as in [Ge2];
- Greiner's approach of the heat kernel asymptotics [Gr].

The talk is divided into four parts. In the first one we recall Greiner's approach of the heat kernel asymptotics and in the second one we prove the Atiyah-Singer index formula. The third part reviews the framework for the local index formula in noncommutative geometry following [CM]. Then in the last part we compute the CM cocycle for Dirac spectral triples.

1. GREINER'S APPROACH OF THE HEAT KERNEL ASYMPTOTICS

Here we let M^n be a compact Riemannian manifold, let \mathcal{E} be a Hermitian bundle over M , and consider a second order selfadjoint elliptic differential operator $\Delta : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$ such that the principal symbol of Δ is positive definite. This has the effect that Δ is bounded from below and generates a bounded heat semi-group $e^{-t\Delta}$, $t \geq 0$, on $L^2(M, \mathcal{E})$. In fact, for $t > 0$ the operator $e^{-t\Delta}$ is smoothing, i.e. has a smooth distribution kernel.

On the other hand, the heat semigroup $e^{-t\Delta}$, $t \geq 0$, allows us to invert the heat equation, since if we consider the operator $Q_0 : C_c^\infty(M \times \mathbb{R}) \rightarrow \mathcal{D}'(M \times \mathbb{R})$ given by

$$(1.1) \quad Q_0 u(x, t) = \int_0^\infty e^{-s\Delta} u(x, t-s) ds, \quad u \in C_c^\infty(M \times \mathbb{R}, \mathcal{E}),$$

then, for $u \in C_c^\infty(M \times \mathbb{R}, \mathcal{E})$ we have

$$(1.2) \quad (\Delta + \partial_t) Q_0 u = Q_0 (\Delta + \partial_t) u = u \quad \text{in } \mathcal{D}'(M \times \mathbb{R}, \mathcal{E}).$$

Observe also that at the level of distribution kernels (1.1) implies that:

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1. Q_0 has the *Volterra property*, i.e. it has a distribution kernel of the form $K_{Q_0}(x, y, t - s)$ with $K_{Q_0}(x, y, t) = 0$ for $t < 0$.

2. For $t > 0$ we have

$$(1.3) \quad K_{Q_0}(x, y, t) = k_t(x, y).$$

The equality (1.3) leads us to use ΨDO techniques for looking at the asymptotics of $k_t(x, x)$ as $t \rightarrow 0^+$. For achieving this aim the relevant ΨDO calculus is the Volterra calculus developed independently by Greiner [Gr] and Piriou [Pi].

In the sequel we let U be an open subset of \mathbb{R}^n and let \mathbb{C}_- be the complex halfplane $\{\Im \tau > 0\}$ with closure $\bar{\mathbb{C}}_-$. Then the Volterra symbols can be defined as follows.

Definition 1.1. $S_{v,m}(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of smooth functions $q_m(x, \xi, \tau)$ on $U_x \times (\mathbb{R}_{(\xi, \tau)}^{n+1} \setminus 0)$ so that $q_m(x, \xi, \tau)$ can be extended to a smooth function on $U_x \times [(\mathbb{R}_\xi^n \times \bar{\mathbb{C}}_-) \setminus 0]$ in such way to be analytic with respect to $\tau \in \mathbb{C}_-$ and to be homogeneous of degree m , i.e. $q_m(x, \lambda \xi, \lambda^2 \tau) = \lambda^m q_m(x, \xi, \tau)$ for any $\lambda \in \mathbb{R} \setminus 0$.

In fact, Definition 1.1 is intimately related to the Volterra property, for we have:

Lemma 1.2 ([BGS, Prop. 1.9]). Any homogeneous Volterra symbol $q(x, \xi, \tau) \in S_{v,m}(U \times \mathbb{R}^{n+1})$ can be extended into a unique distribution $g(x, \xi, \tau) \in C^\infty(U) \hat{\otimes} \mathcal{S}'(\mathbb{R}^{n+1})$ in such way to be homogeneous with respect to the covariables (ξ, τ) and so that $\check{q}(x, y, t) := \mathcal{F}_{(\xi, \tau) \rightarrow (y, t)}^{-1}[g](x, y, t) = 0$ for $t < 0$.

Definition 1.3. $S_v^m(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of the smooth functions $q(x, \xi, \tau)$ on $U_x \times \mathbb{R}_{(\xi, \tau)}^{n+1}$ that have an asymptotic expansion $q \sim \sum_{j \geq 0} q_{m-j}$, $q_{m-j} \in S_{v,m-j}(U \times \mathbb{R}^{n+1})$, where \sim means that, for any integer N and for any compact $K \subset U$, there exists a constant $C_{NK\alpha\beta k} > 0$ such that

$$(1.4) \quad |\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j < N} q_{m-j})(x, \xi, \tau)| \leq C_{NK\alpha\beta k} (|\xi| + |\tau|^{1/2})^{m-|\beta|-wk-N},$$

for $x \in K$ and $|\xi| + |\tau|^{1/2} \geq 1$.

Definition 1.4. A Volterra ΨDO of order m is a continuous operator Q from $C_c^\infty(M \times \mathbb{R}, \mathcal{E})$ to $C^\infty(M \times \mathbb{R}, \mathcal{E})$ such that:

- Q has the Volterra property, i.e. has a distribution kernel of the form $K_Q(x, y, t - s)$ with $K_Q(x, y, t) = 0$ for $t < 0$;
- The distribution kernel of Q is smooth off the diagonal;
- Locally we can write $Q = q(x, D_x, D_t) + R$ for some Volterra symbol q of order m and some smoothing operator R .

Example 1.5. The heat operator $\Delta + \partial_t$ is a Volterra ΨDO of order 2 and its principal symbol is $p_2(x, \xi) + i\tau$ (where p_2 denotes the principal symbol of Δ).

Example 1.6. Let q_l be a homogeneous Volterra symbol on $U \times \mathbb{R}^n \times \mathbb{R}$ and define $q_l(x, D_x, D_t)$ to be the operator with kernel $\check{q}_l(x, x - y, t - s)$. Then $q_l(x, D_x, D_t)$ is a Volterra ΨDO of order l .

Using the Volterra ΨDO calculus we can prove:

Proposition 1.7 ([Gr], [Pi]). The heat operator $\Delta + \partial_t$ is invertible on smooth sections and its inverse is a Volterra ΨDO of order -2 .

On the other hand, we have the following.

Lemma 1.8 ([Gr]). Let $Q \in$ be a Volterra ΨDO of order m on $U \times \mathbb{R}$. Then, with asymptotics in $C^\infty(U)$, we have

$$(1.5) \quad K_Q(x, x, t) \sim_{t \rightarrow 0^+} t^{-(\frac{n}{2} + [\frac{m}{2}] + 1)} \sum t^l \check{q}_{2[\frac{m}{2}] - 2l}(x, 0, 1),$$

where $q \sim \sum_{j \geq 0} q_{m-j}$ denotes the symbol of Q .

Applying this to $Q = (\Delta + \partial_t)^{-1}$ and using (1.3) we get:

Proposition 1.9 ([Gr]). *In $C^\infty(M, |\Lambda|(M) \otimes \text{End } \mathcal{E})$ we have:*

$$(1.6) \quad k_t(x, x) \sim_{t \rightarrow 0^+} t^{-\frac{n}{2}} \sum t^j a_l(\Delta)(x), \quad a_l(\Delta)(x) = \check{q}_{-2-2l}(x, 0, 1),$$

where $q \sim \sum_{j \geq 0} q_{-2-j}$ denotes the symbol of any Volterra Ψ DO parametrix for $\Delta + \partial_t$.

The interest of this approach is twofold. First, to determine the asymptotics of $k_t(x, x)$ at a point x_0 we only need a Volterra parametrix Q for $\Delta + \partial_t$ near x_0 , for we have:

$$(1.7) \quad k_t(x_0, x_0) = K_Q(x_0, x_0, t) + O(t^\infty).$$

Second, we can differentiate the heat kernel asymptotics as follows. Let P be a differential operator of order m and let $h_t(x, y)$ denote the kernel of $Pe^{-t\Delta}$, $t > 0$. Then:

$$(1.8) \quad h_t = P_x k_t = P_x K_{(\Delta + \partial_t)^{-1}} = K_{P(\Delta + \partial_t)^{-1}}.$$

Therefore by applying Lemma 1.8 to the operator $P(\Delta + \partial_t)^{-1}$ we get a differentiable version of Proposition 1.9 as follows.

Proposition 1.10. *With asymptotics in $C^\infty(M, |\Lambda|(M))$ we have*

$$(1.9) \quad h_t(x, x) \sim_{t \rightarrow 0^+} t^{-\left(\frac{n}{2} + \left[\frac{m}{2}\right]\right)} \sum_{l \geq 0} t^l b_l(P, \Delta)(x), \quad b_l(P, \Delta)(x) = \check{q}_{2\left[\frac{m}{2}\right] - 2 - 2l}(x, 0, 1),$$

where $q \sim \sum_{j \geq 0} q_{m-2-j}$ denotes the symbol of $P(\Delta + \partial_t)^{-1}$.

2. PROOF OF THE ATIYAH-SINGER INDEX FORMULA

Let M^n be a compact Riemannian spin manifold with spin bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, let \mathcal{E} be a Hermitian bundle equipped with a unitary connection and consider the Dirac operator with coefficients in \mathcal{E} ,

$$(2.1) \quad \mathcal{D}_{\mathcal{E}} : C^\infty(M, \mathcal{S} \otimes \mathcal{E}) \rightarrow C^\infty(M, \mathcal{S} \otimes \mathcal{E}).$$

Then the Atiyah-Singer index formula is the following.

Theorem 2.1 ([AS1], [AS2]). *We have:*

$$(2.2) \quad \text{ind } \mathcal{D}_{\mathcal{E}} = (2i\pi)^{-\frac{n}{2}} \int_M [\hat{A}(R_M) \wedge \text{Ch}(F^{\mathcal{E}})]^{(n)},$$

where $\hat{A}(R^M) = \det^{\frac{1}{2}}\left(\frac{R^M/2}{\sinh(R^M/2)}\right)$ is the \hat{A} -form of the Riemann curvature R^M and $\text{Ch}(F^{\mathcal{E}}) = \text{Tr} \exp(-F^{\mathcal{E}})$ is the Chern form of the curvature $F^{\mathcal{E}}$ of \mathcal{E} .

Let $k_t(x, y)$ denote the heat kernel of $\mathcal{D}_{\mathcal{E}}^2$. Then by the McKean-Singer formula we have

$$(2.3) \quad \text{ind } \mathcal{D}_{\mathcal{E}} = \int \text{Str } k_t(x, x) \quad \text{for any } t > 0,$$

where $\text{Str} = \text{Tr}_{\mathcal{S}^+ \otimes \mathcal{E}} - \text{Tr}_{\mathcal{S}^- \otimes \mathcal{E}}$ denotes the supertrace. Therefore, the index formula of Atiyah-Singer follows from:

Theorem 2.2. *As $t \rightarrow 0^+$ we have*

$$(2.4) \quad \text{Str } k_t(x, x) = [\hat{A}(R_M) \wedge \text{Ch}(F^{\mathcal{E}})]^{(n)} + O(t),$$

with $O(t)$ in $C^\infty(M, |\Lambda|(M))$.

This theorem, also called Local Index Theorem, was first proved by Patodi, Gilkey and Atiyah-Bott-Patodi ([ABP], [Gi]), and then in a purely analytic fashion by Getzler ([Ge1], [Ge2]) and Bismut [Bi] (see also [BGV], [Ro]). Moreover, as it is a purely local statement it holds *verbatim* for (geometric) Dirac operators acting on a Clifford bundle. Thus it allows us to recover, on the one hand, the Gauss-Bonnet, signature and Riemann-Roch theorems ([ABP], [BGV], [LM], [Ro]) and, on the other hand, the full index theorem of Atiyah-Singer ([ABP], [LM]).

Proof of Theorem 2.2. First, it is enough to prove Theorem 2.2 at a point x_0 . Furthermore, to this end we only need a Volterra parametrix Q for $\mathcal{D}_\mathcal{E}^2 + \partial_t$ in local trivializing coordinates near x_0 , for we have

$$(2.5) \quad k_t(x_0, x_0) = K_Q(x_0, x_0, t) + O(t^\infty).$$

In fact, we can use normal coordinates centered at x_0 and trivializations of \mathcal{S} and \mathcal{E} by means of parallel translation along the geodesics out of the origin. This allows us to replace $\mathcal{D}_\mathcal{E}$ by a Dirac operator \mathcal{D} on \mathbb{R}^n acting on the trivial bundle with fiber $\mathcal{S}_n \otimes \mathbb{C}^p$, where \mathcal{S}_n is the space of spinors on \mathbb{R}^n .

Now, recall that as algebra $\text{End } \mathcal{S}_n$ can be identified with the Clifford algebra $\text{Cl}(n)$. The latter is isomorphic to the exterior algebra $\Lambda(n) = \bigoplus_{i=0}^n \Lambda^i(n)$,

$$(2.6) \quad \Lambda(n) \xrightarrow{c} \text{Cl}(n) \xrightarrow{\sigma=c^{-1}} \Lambda(n),$$

so that for $\xi, \eta \in \Lambda(n)$ we have

$$(2.7) \quad \sigma[c(\xi^{(i)})c(\eta^{(j)})] = \xi^{(i)} \wedge \eta^{(j)} \text{ mod } \Lambda^{i+j-2},$$

$$(2.8) \quad \zeta^{(l)} = \text{component of } \zeta \in \Lambda(n) \text{ in } \Lambda^l(n).$$

Here c is called the Clifford quantification map and σ the Clifford symbol map.

Lemma 2.3 (Getzler [Ge1]). *For any a in $\text{End } \mathcal{S}_n$ we have:*

$$(2.9) \quad \text{Str } a = (-2i)^{\frac{n}{2}} \sigma[a]^{(n)}.$$

From all this we see that as $t \rightarrow 0^+$ we have:

$$(2.10) \quad \text{Str } k_t(0, 0) = (-2i)^{\frac{n}{2}} \sigma[K_Q(0, 0, t)]^{(n)} + O(t^\infty),$$

where Q is any Volterra ΨDO parametrix for $\mathcal{D}^2 + \partial_t$.

Now, recall that the Getzler rescaling [Ge2] assigns the following degrees:

$$(2.11) \quad \deg \partial_j = \frac{1}{2} \deg \partial_t = \deg c(dx^j) = -\deg x^j = 1,$$

while $\deg B = 0$ for any $B \in M_p(\mathbb{C})$. It defines a filtration of Volterra ΨDO 's with coefficients in $\text{End}(\mathcal{S}_n \otimes \mathbb{C}^p) \simeq \text{Cl}(\mathbb{R}^n) \otimes M_p(\mathbb{C})$ as follows.

Let $q \sim \sum_{l \leq m'} q_l$ be a Volterra symbol with values in $\text{End } \mathcal{S}_n \otimes M_p(\mathbb{C})$. Then taking components in each subspace $\Lambda^{(j)}(n)$ and using Taylor expansions near $x = 0$ allows us to get formal expansions,

$$(2.12) \quad \sigma[q] \sim \sum_{l \leq m'} \sigma[q_l] \sim \sum_{l,j} \sigma[q_l]^{(j)} \sim \sum_{l,j,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_l(0, \xi, \tau)]^{(j)}.$$

Observe that each symbol $\frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_l(0, \xi, \tau)]^{(j)}$ is Getzler homogeneous of degree $-|\alpha| + l + j$. Therefore, we have

$$(2.13) \quad \sigma[q(x, \xi, \tau)] \sim \sum_{j \geq 0} q_{(m-j)}(x, \xi, \tau), \quad q_{(m)} \neq 0,$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m - j$.

Definition 2.4. Using (2.13) we set-up the following definitions:

- The integer m is the Getzler order of Q ,
- The symbol $q_{(m)}$ is the principal Getzler homogeneous symbol of Q ,
- The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ is the model operator of Q .

Example 2.5. The operator \mathcal{D}^2 has Getzler order 2 and its model operator is

$$(2.14) \quad \mathcal{D}_{(2)}^2 = H_R + F^{\mathcal{E}}(0), \quad H_R = - \sum_{i=1}^n (\partial_i - \frac{1}{4} R_{ij}^M(0) x^j)^2,$$

$$(2.15) \quad R_{ij}^M(0) = \frac{1}{2} \langle R^M(\partial_i, \partial_j) \partial_k, \partial_l \rangle(0) dx^k \wedge dx^l, \quad F^{\mathcal{E}}(0) = \frac{1}{2} F^{\mathcal{E}}(\partial_k, \partial_l)(0) dx^k \wedge dx^l.$$

This uses the behavior near $x = 0$ in normal coordinates of the metric and of the coefficients of the Levi-Civita connection in the synchronous frame (cf. [Ge2], [Po]).

The interest of the above definitions stems from the two results below.

Lemma 2.6 ([Po]). For $j = 1, 2$ let Q_j be a Volterra Ψ DO of Getzler order m_j . Then

$$(2.16) \quad Q_1 Q_2 = c(Q_{(m_1)} Q_{(m_2)}) + O_G(m_2 + m_2 - 1),$$

where $O_G(m_2 + m_2 - 1)$ has Getzler order $\leq m_2 + m_2 - 1$.

Lemma 2.7 ([Po]). Let Q have Getzler order m and model operator $Q_{(m)}$.

- If $m - j$ is odd, then $\sigma[K_Q(0, 0, t)]^{(j)} = O(t^{\frac{j-m-n-1}{2}})$.
- If $m - j$ is even, then $\sigma[K_Q(0, 0, t)]^{(j)} = t^{\frac{j-m-n}{2}-1} K_{Q_{(m)}}(0, 0, 1)^{(j)} + O(t^{\frac{j-m-n}{2}})$.

In particular, for $m = -2$ and $j = n$ we get:

$$(2.17) \quad \sigma[K_Q(0, 0, t)]^{(n)} = K_{Q_{(-2)}}(0, 0, 1)^{(n)} + O(t).$$

In particular, using Lemma 2.6 we can easily get:

Lemma 2.8 ([Po]). Let Q be a Volterra Ψ DO parametrix for $\mathcal{D}^2 + \partial_t$. Then:

- 1) Q has Getzler order 2 and model operator $Q_{(-2)} = (H_R + F^{\mathcal{E}}(0) + \partial_t)^{-1}$.
- 2) $K_{Q_{(-2)}}(x, 0, t) = G_R(x, t) \wedge \exp(-tF^{\mathcal{E}}(0))$, where $G_R(x, t)$ is the fundamental solution of $H_R + \partial_t$, i.e. $(H_R + \partial_t)G_R(x, t) = \delta(x, t)$, where $\delta(x, t)$ is the Dirac distribution at the origin.

Noticing that H_R is a harmonic oscillator associated to the curvature at $x = 0$ we get:

Lemma 2.9 (Melher's Formula). We have:

$$(2.18) \quad G_R(x, t) = \chi(t)(4\pi t)^{-\frac{n}{2}} \hat{A}(tR^M(0)) \exp(-\frac{1}{4t} \langle \frac{tR^M(0)/2}{\tanh(tR^M(0)/2)} x, x \rangle),$$

where $\chi(t)$ is the characteristic function of the interval $(0, +\infty)$.

Now, from (2.10) and Lemma 2.7 we deduce

$$(2.19) \quad \text{Str } k_t(0, 0) = (-2i)^{\frac{n}{2}} \sigma[K_Q(0, 0, t)]^{(n)} + O(t^\infty) = (-2i)^{\frac{n}{2}} K_{Q_{(-2)}}(0, 0, 1)^{(n)} + O(t).$$

Thus using Lemma 2.8 and Lemma 2.9 we get:

$$(2.20) \quad \text{Str } k_t(0, 0) = (-2i)^{\frac{n}{2}} [G_R(0, 1) \wedge \exp(-F^{\mathcal{E}}(0))]^{(n)} + O(t), \\ = (2i\pi)^{-\frac{n}{2}} [\hat{A}(R_M(0)) \wedge \text{Ch}(F^{\mathcal{E}}(0))]^{(n)} + O(t).$$

Comparing this with the heat kernel asymptotics (1.6) for \mathcal{D}^2 we get Theorem 2.2, and so complete the proof of the Atiyah-Singer index formula. \square

The main new feature in the previous proof of the Atiyah-Singer index formula is the use of Lemma 2.7 which, by very elementary considerations on Getzler orders shows that the convergence of the supertrace of the heat kernel is a consequence of a general fact about Volterra ΨDO 's. It incidentally gives a differentiable version of Theorem 2.2 as follows.

Proposition 2.10 ([Po]). *Let \mathcal{P} be a differential operator of Getzler order m and let $h_t(x, y)$ denote the distribution kernel of $\mathcal{P}e^{-t\mathcal{D}_M}$.*

- If m is odd, then $\text{Str } h_t(x, x) = O(t^{-\frac{m+1}{2}})$.

- If m is even, then $\text{Str } h_t(x, x) = t^{-\frac{m}{2}} B_0(\mathcal{D}_{\mathcal{E}}^2, \mathcal{P})(x) + O(t^{-\frac{m}{2}+1})$, where in synchronous normal coordinates centered at x_0 we have $B_0(\mathcal{D}_{\mathcal{E}}^2, \mathcal{P})(0) = (-2i\pi)^{\frac{n}{2}} [(\mathcal{P}_{(m)} G_R)(0, 1) \wedge \text{Ch}(F^{\mathcal{E}}(0))]^{(n)}$, with $\mathcal{P}_{(m)}$ denoting the model operator of \mathcal{P} .

Proof. By Proposition 1.10 we have

$$(2.21) \quad h_t(x, x) = K_{\mathcal{P}(\mathcal{D}_{\mathcal{E}}^2 + \partial_t)^{-1}}(x, x, t) \sim_{t \rightarrow 0^+} \sum_{j \geq 0} t^{-(\frac{n}{2} + [\frac{m'}{2}])} b_j(P, \Delta)(x),$$

with asymptotics in $C^\infty(M, |\Lambda| \otimes \text{End } \mathcal{S} \otimes \text{End } \mathcal{E})$. On the other hand, using Lemma 2.6 we see that $\mathcal{P}(\mathcal{D}_{\mathcal{E}}^2 + \partial_t)^{-1}$ has Getzler order $m - 2$. Combining all this with Lemma 2.7 gives the result. \square

3. THE LOCAL INDEX FORMULA IN NONCOMMUTATIVE GEOMETRY

Rather than at topological or geometrical levels the local index formula holds in full generality in a purely operator theoretic setting ([CM]; see also [Hi]). This uses two main tools, namely, spectral triples [CM] and cyclic cohomology [Co].

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where:

- \mathcal{H} is a Hilbert space together with a \mathbb{Z}_2 -grading $\gamma : \mathcal{H}_+ \oplus \mathcal{H}_- \rightarrow \mathcal{H}_- \oplus \mathcal{H}_+$;
- \mathcal{A} is an involutive unital algebra represented in \mathcal{H} and commuting with the \mathbb{Z}_2 -grading γ ;
- D is a selfadjoint unbounded operator on \mathcal{H} s.t. $[D, a]$ is bounded $\forall a \in \mathcal{A}$ and of the form,

$$(3.1) \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D_{\pm} : \mathcal{H}^{\mp} \rightarrow \mathcal{H}^{\pm}.$$

In addition we assume that \mathcal{A} is *smooth* in the sense that \mathcal{A} is contained in $\cap_{k \geq 0} \text{dom } \delta^k$.

Recall that the datum of D above defines an index map $\text{ind}_D : K_*(\mathcal{A}) \rightarrow \mathbb{Z}$ so that

$$(3.2) \quad \text{ind}_D[e] = \text{ind } eD^+e,$$

for any selfadjoint idempotent $e \in M_q(\mathcal{A})$.

Example 3.1 (Dirac spectral triple). A typical spectral triple is a triple $(C^\infty(M), L^2(M, \mathcal{S}), \mathcal{D}_M)$, where M is a compact Riemannian spin manifold of even dimension, \mathcal{S} is the spin bundle of M and $\mathcal{D}_M : C^\infty(M, \mathcal{S}) \rightarrow C^\infty(M, \mathcal{S})$ is the Dirac operator on M . Moreover, under the Serre-Swan isomorphism $K_0(C^\infty(M)) \simeq K^0(M)$ for any Hermitian bundle over M we have:

$$(3.3) \quad \text{ind}_{\mathcal{D}_M}[\mathcal{E}] = \text{ind } \mathcal{D}_{\mathcal{E}}.$$

Next, the *cyclic cohomology groups* $HC^*(\mathcal{A})$ are obtained from the spaces,

$$(3.4) \quad C^k(\mathcal{A}) = \{(k+1)\text{-linear forms on } \mathcal{A}\}, \quad k \in \mathbb{N},$$

by restricting the Hochschild coboundary,

$$(3.5) \quad b\psi(a^0, \dots, a^{k+1}) = (-1)^{k+1} \psi(a^{k+1}a^0, \dots, a^k) + \sum (-1)^j \psi(a^0, \dots, a^j a^{j+1}, \dots, a^{k+1}),$$

to cyclic cochains, i.e. those such that

$$(3.6) \quad \psi(a^1, \dots, a^k, a^0) = (-1)^k \psi(a^0, a^1, \dots, a^k).$$

Equivalently, $HC^*(\mathcal{A})$ can be obtained as the second filtration of the (b, B) -bicomplex of (arbitrary) cochains, where the horizontal differential B is given by

$$(3.7) \quad B = AB_0 : C^{m+1}(\mathcal{A}) \rightarrow C^m(\mathcal{A}), \quad (A\phi)(a^0, \dots, a^m) = \sum (-1)^{mj} \psi(a^j, \dots, a^{j-1}),$$

$$(3.8) \quad B_0\psi(a^0, \dots, a^m) = \psi(1, a^0, \dots, a^m).$$

The *periodic cyclic cohomology* is the inductive limit of the groups $HC^k(\mathcal{A})$ with respect to the the cup product with the generator of $HC^2(\mathbb{C})$. In terms of the (b, B) -bicomplex this is the cohomology of the short complex,

$$(3.9) \quad C^{\text{ev}}(\mathcal{A}) \xrightarrow{b+B} C^{\text{odd}}(\mathcal{A}), \quad C^{\text{ev/odd}}(\mathcal{A}) = \bigoplus_{k \text{ even/odd}} C^k(\mathcal{A}),$$

the cohomology groups of which are denoted $HC^{\text{ev}}(\mathcal{A})$ and $HC^{\text{odd}}(\mathcal{A})$.

We have a pairing between $HC^{\text{ev}}(\mathcal{A})$ and $K_0(\mathcal{A})$ such that, for any cocycle $\varphi = (\varphi_{2k})$ in $C^{\text{ev}}(\mathcal{A})$ and any selfadjoint idempotent e in $M_q(\mathcal{A})$, we have

$$(3.10) \quad \langle [\varphi], [e] \rangle = \sum_{k \geq 0} (-1)^k \frac{(2k)!}{k!} \varphi_{2k} \# \text{Tr}(e, \dots, e),$$

where $\varphi_{2k} \# \text{Tr}$ is the $(2k+1)$ -linear map on $M_q(\mathcal{A}) = M_q(\mathbb{C}) \otimes \mathcal{A}$ given by

$$(3.11) \quad \varphi_{2k} \# \text{Tr}(\mu^0 \otimes a^0, \dots, \mu^{2k} \otimes a^{2k}) = \text{Tr}(\mu^0 \dots \mu^{2k}) \varphi_{2k}(a^0, \dots, a^{2k}),$$

for $\mu^j \in M_q(\mathbb{C})$ and $a^j \in \mathcal{A}$.

Example 3.2. Given a compact manifold M let \mathcal{A} be the algebra $C^\infty(M)$ and let $\mathcal{D}_k(M)$ denote the space of k -dimensional de Rham currents. Then we have a morphism from $\mathcal{D}_{\text{ev/odd}}(M)$ to $C^{\text{ev/odd}}(\mathcal{A})$,

$$(3.12) \quad C = (C_k) \rightarrow \varphi_C = \left(\frac{1}{k!} \psi_{C_k}\right), \quad \psi_{C_k}(f^0, f^1, \dots, f^n) = \langle C_k, f^0 df^1 \wedge \dots \wedge df^k \rangle.$$

such that $b\varphi_C = 0$ and $B\varphi_C = \varphi_{d^t C}$, where d^t denotes the de Rham boundary for currents. Therefore, we get a morphism from the even and odd de Rham's homology groups $H^{\text{ev/odd}}(M)$ to $HC^{\text{ev/odd}}(C^\infty(M))$. This even gives rise to an isomorphism if we restrict ourselves to continuous cyclic cochains.

Furthermore, we have:

$$(3.13) \quad \langle [\varphi_C], \mathcal{E} \rangle = \langle [C], \text{Ch}^* \mathcal{E} \rangle \quad \forall \mathcal{E} \in K^0(M),$$

where $\text{Ch}^* : K^0(M) \rightarrow H^{\text{ev}}(M)$ is the Chern character in cohomology.

Now, we can compute the index map by pairing the K -theory a cyclic cocycle as follows. First, we say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *p-summable* when we have

$$(3.14) \quad \mu_k(D^{-1}) = O(k^{-1/p}) \quad \text{as } k \rightarrow +\infty,$$

where $\mu_k(D^{-1})$ denotes the $(k+1)$ 'th eigenvalue of D^{-1} .

Next, let $\Psi_D^0(\mathcal{A})$ be the algebra generated by the \mathbb{Z}_2 -grading γ and the $\delta^k(a)$'s, $a \in \mathcal{A}$, where δ is the derivation $\delta(T) = [|D|, T]$.

Definition 3.3. *The dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$ is the union set of the singularities of all the zeta functions $\zeta_b(z) = \text{Tr } b|D|^{-z}$, $b \in \Psi_D^0(\mathcal{A})$.*

Assuming p -summability and simple and discrete dimension spectrum we define a residual trace on $\Psi_D^0(\mathcal{A})$ by letting:

$$(3.15) \quad \int b = \text{Res}_{z=0} \text{Tr } b|D|^{-z} \quad \text{for any } b \in \Psi_D^0(\mathcal{A}).$$

This trace is an algebraic analogue of the Wodzicki-Guillemin noncommutative residue trace ([Wo], [Gu]) for the algebra $\Psi_D^0(\mathcal{A})$. Moreover, it is local in the sense of noncommutative geometry since it vanishes on those elements of $\Psi_D^0(\mathcal{A})$ that are traceable.

Theorem 3.4 ([CM]). *Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is p -summable and has a discrete and simple dimension spectrum. Then:*

1) *The following formulas define an even cocycle $\varphi_{\text{CM}} = (\varphi_{2k})$ in the (b, B) -complex of \mathcal{A} .*

- *For $k = 0$, we let*

$$(3.16) \quad \varphi_0(a^0) = \text{finite part of } \text{Tr } \gamma a^0 e^{-tD^2} \text{ as } t \rightarrow 0^+,$$

- *For $k \neq 0$, we let*

$$(3.17) \quad \varphi_{2k}(a^0, \dots, a^{2k}) = \sum_{\alpha} c_{k,\alpha} \int \gamma P_{k,\alpha} |D|^{-2(|\alpha|+k)},$$

$$(3.18) \quad P_{k,\alpha} = a^0 [D, a^1]^{[\alpha_1]} \dots [D, a^{2k}]^{[\alpha_{2k}]},$$

where $\Gamma(|\alpha| + k) c_{k,\alpha}^{-1} = 2(-1)^{|\alpha|} \alpha! (\alpha_1 + 1) \dots (\alpha_1 + \dots + \alpha_{2k} + 2k)$ and the symbol $T^{[j]}$ denotes the j 'th iterated commutator with D^2 .

2) *We have $\text{ind}_D[\mathcal{E}] = \langle [\varphi_{\text{CM}}], \mathcal{E} \rangle$ for any $\mathcal{E} \in K_0(\mathcal{A})$.*

Remark 3.5. Here we have assumed that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ was \mathbb{Z}_2 -graded. In the terminology of [CM] such a spectral triple is *even* and an ungraded spectral triple is said to be *odd*. In the latter case we can similarly compute the Fredholm index with coefficients in $K_1(\mathcal{A})$ by pairing $K_1(\mathcal{A})$ with an *odd* cyclic cocycle, also called CM cocycle, and which is given by formulas of the same type as (3.17)–(3.18) (see [CM] for a more detailed account).

4. COMPUTATION OF THE CM COCYCLE FOR DIRAC SPECTRAL TRIPLES.

Here M^n is a compact Riemannian spin manifold of even dimension and $\mathcal{D}_M : C^\infty(M, \mathcal{S}) \rightarrow C^\infty(M, \mathcal{S})$ is the Dirac operator on M . Then $(C^\infty(M), L^2(M, \mathcal{S}), \mathcal{D}_M)$ is an even n -summable spectral triple with dimension spectrum $\{k \in \mathbb{Z}; k \leq n\}$. Therefore, the CM cocycle is well defined and allows us to compute the index map, so that for any hermitian bundle \mathcal{E} equipped with a unitary connection we have

$$(4.1) \quad \langle [\varphi_{\text{CM}}], [\mathcal{E}] \rangle = \text{ind}_{\mathcal{D}_M}[\mathcal{E}] = \text{ind } \mathcal{D}_{\mathcal{E}}.$$

Proposition 4.1. *The components of the CM cyclic cocycle $\varphi_{\text{CM}} = (\varphi_{2k})$ associated to the spectral triple $(C^\infty(M), L^2(M, \mathcal{S}), \mathcal{D}_M)$ are given by*

$$(4.2) \quad \varphi_{2k}(f^0, \dots, f^{2k}) = \frac{1}{(2k)!} \int_M f^0 df^1 \wedge \dots \wedge df^{2k} \wedge \hat{A}(R_M)^{(n-2k)},$$

for f^0, f^1, \dots, f^n in $C^\infty(M)$.

Proof. First, using the Mellin transform,

$$(4.3) \quad |\mathcal{D}_M|^{-2s} = \frac{1}{\Gamma(2s)} \int_0^\infty t^{s-1} e^{-t\mathcal{D}_M^2} dt, \quad \Re s > 0,$$

we can rewrite $\varphi_{\text{CM}} = (\varphi_{2k})$ as

$$(4.4) \quad \varphi_{2k}(f^0, \dots, f^{2k}) = \sum_{\alpha} \frac{2c_{k,\alpha}}{\Gamma(|\alpha| + k)} \cdot \{\text{coeff. of } t^{-(|\alpha|+k)} \text{ in } \text{Str } \mathcal{P}_{k,\alpha} e^{-t\mathcal{D}_M^2} \text{ as } t \rightarrow 0^+\}.$$

Notice also that $\mathcal{P}_{k,\alpha} = f^0 [\mathcal{D}_M, f^1]^{[\alpha_1]} \dots [\mathcal{D}_M, f^{2k}]^{[\alpha_{2k}]} = f^0 c(df^1)^{[\alpha_1]} \dots c(df^{2k})^{[\alpha_{2k}]}$. Therefore, we can make use of Proposition 2.10 to compute the CM cocycle. Indeed, let $h_{\alpha,t}(x, y)$ denote the kernel of $\mathcal{P}_{k,\alpha} e^{-t\mathcal{D}_M^2}$. Then using Lemma 2.6 and Proposition 2.10 we obtain:

- If $\alpha = 0$, then $\mathcal{P}_{k,0} = f^0 c(df^1) \dots c(df^{2k})$ has Getzler order $2k$ and model operator $f^0 df^1 \wedge \dots \wedge df^{2k}$, so that we have

$$(4.5) \quad \text{Str } h_{0,t}(x, x) = \frac{t^{-k}}{(2i\pi)^{\frac{n}{2}}} f^0 df^1 \wedge \dots \wedge df^{2k} \wedge \hat{A}(R_M)^{(n-2k)} + O(t^{-k+1}).$$

- If $\alpha \neq 0$, then $\mathcal{P}_{k,\alpha}$ has Getzler order $\leq 2(|\alpha| + k) - 1$, and so

$$(4.6) \quad \text{Str } h_{\alpha,t}(x, x) = O(t^{-(\alpha+k)+1}).$$

Combining this with (4.4) we get the formula (4.2) for $\varphi_{2k}(f^0, \dots, f^{2k})$. \square

Finally, let us explain how this enables us to recover the local index formula of Atiyah-Singer. Notice that Formula (4.2) means that φ_{CM} is the image under the map (3.12) of the current that is the Poincaré dual of $\hat{A}(R^M)$. Therefore, using (3.13) we see that for any Hermitian bundle \mathcal{E} with curvature $F^\mathcal{E}$ we have:

$$(4.7) \quad \langle [\varphi_{CM}], [\mathcal{E}] \rangle = \langle [\varphi_{\hat{A}(R^M)}], [\mathcal{E}] \rangle = (2i\pi)^{-\frac{n}{2}} \int_M [\hat{A}(R_M) \wedge \text{Ch}(F^\mathcal{E})]^{(n)}.$$

On the other hand, we have

$$(4.8) \quad \langle [\varphi_{CM}], [\mathcal{E}] \rangle = \text{ind}_{\mathcal{D}_M} [\mathcal{E}] = \text{ind } \mathcal{D}_\mathcal{E}.$$

Thus, we obtain

$$(4.9) \quad \text{ind } \mathcal{D}_\mathcal{E} = (2i\pi)^{-\frac{n}{2}} \int_M [\hat{A}(R_M) \wedge \text{Ch}(F^\mathcal{E})]^{(n)},$$

which is precisely the Atiyah-Singer index formula.

Remark 4.2. When $\dim M$ is odd we can also compute the odd CM cocycle associated to the corresponding Dirac spectral triple (cf. [Po]). In this setting we recover the spectral flow formula of Atiyah-Patodi-Singer [APS].

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