ORBIT EQUIVALENCE AND THE ASYMPTOTIC GROWTH OF THE NUMBER OF SOFIC APPROXIMATIONS

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Abstract. Given an ergodic probability measure preserving dynamical system $\Gamma \curvearrowright (X, \mu)$, where Γ is a finitely generated countable group, we show that the asymptotic growth of the number of finite models for the dynamics, in the sense of sofic approximations, is an invariant of orbit equivalence. We then prove an additivity formula for free products with amenable (possibly trivial) amalgamation. In particular, we obtain purely combinatorial proofs of several results in orbit equivalence theory.

1. Introduction

In this paper we introduce a new invariant of probability measure preserving dynamical systems using the idea of sofic approximation. This invariant, combinatorial in nature, is defined by counting the number of sofic models on finite sets to within a given precision, measuring the asymptotic growth as the finite sets get larger and larger, and taking a limiting value of these quantities as the precision gets better and better (see Section 2 for a detailed discussion).

Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure preserving (p.m.p.) action of a finitely generated group Γ, and let F be a finite dynamical generating set for $\Gamma \curvearrowright (X, \mu)$, in the sense of Definition 2.4. For instance, if the action $\Gamma \curvearrowright (X, \mu)$ has a finite generating partition $\mathscr{P}$, then F can be taken to be the union of the partition $\mathscr{P}$ and a finite generating set of Γ, viewed as a set of measure-preserving (partial) isomorphisms of (X, μ). We associate to F a value $s(F)$ in $\{-\infty\} \cup [0, \infty[$. Our first result is the following.

Theorem 1.1. *The value $s(F)$ is an invariant of orbit equivalence. Namely, it depends only on the orbit partition of the action $\Gamma \curvearrowright (X, \mu)$.*

We recall that the *orbit partition* of an action $\Gamma \curvearrowright (X, \mu)$ is the measured equivalence relation R on (X, μ) whose classes are the orbits of the action $\Gamma \curvearrowright (X, \mu)$. Two actions are *orbit equivalent* to each other if they have isomorphic orbit partition (see [KM04] for a recent survey on orbit equivalence). The notion of finite dynamical generation considered in Definition 2.4 is more general than the usual dynamical notion requiring the existence of a finite generating partition for $\Gamma \curvearrowright (X, \mu)$. In fact, it is precisely the orbit equivalence generalization of this notion, in that it

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only requires $\Gamma \curvearrowright (X, \mu)$ to be orbit equivalent to an action of a finitely generated group having such a partition. We observe in Section 2 that this holds for instance if $\Gamma \curvearrowright (X, \mu)$ is ergodic. On the other hand, it is more restrictive than the usual notion of generating sets for equivalence relations (also called graphings, see Definition 2.2), in that it requires generation of the measure algebra of (X, μ) in addition to generation of the classes of the relation R. It is natural to reformulate Theorem 1.1 in terms of abstract p.m.p. equivalence relations admitting a finite dynamical generating set, and this is done in Section 4, see Theorem 4.1. The two formulations are equivalent, as any abstract p.m.p. equivalence relation is the orbit partition of an action [FM77, Theorem 1]. We shall denote by $s(R)$ the common value of $s(F)$ over all finite dynamical generating sets F.

Under a mild technical assumption called "s-regularity" we then show (see Theorem 7.1) the following additivity formula for amalgamated free products (see [Gab00] Déf. IV.6 for the definition):

Theorem 1.2. *Assume that the p.m.p. equivalence relation R is an amalgamated free product of the form $R = R_1 *_{R_3} R_2$, where the finitely generated relations R_1 and R_2 are s-regular and R_3 is amenable. Then R is s-regular and*

$$s(R) = s(R_1) + s(R_2) - 1 + \mu(D).$$

where D is a fundamental domain of the finite component of R_3.

Remark 1.3. Given a nonprincipal ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$, we can modify the definition of s to obtain another invariant $s_\omega(R) \leq s(R)$. If R is s-regular, then $s_\omega(R) = s(R)$. With this variation one recovers the formula $s_\omega(R) = s_\omega(R_1) + s_\omega(R_2) - 1 + \mu(D)$ for any amalgamated product $R = R_1 *_{R_3} R_2$, where R_1 and R_2 have a finite dynamical generating set and R_3 is amenable.

These results provide a new approach to orbit equivalence theory (for p.m.p. actions) that relies essentially on counting arguments. For example, they imply that two free groups $\mathbf{F}_p$ and $\mathbf{F}_q$ with $p \neq q$ have no orbit equivalent free ergodic p.m.p. action [Gab00], or that every treeable ergodic p.m.p. equivalence relation (in the sense of [Gab00, Définition I.2]) is sofic [EL10] (see also [Pau10]) and has the expected cost (as computed in [Gab00]). Every treeable ergodic p.m.p. equivalence relation is s-regular (see Corollary 7.6).

Let us now give some more details and historical background on free entropy, orbit equivalence, and sofic approximations.

Given a finite von Neumann algebra M, Voiculescu introduced several quantities, in particular *free entropy* and *free entropy dimension*, defined by taking the asymptotic growth rate of the volume of certain matricial microstates associated to a finite set F of self-adjoint elements in M (see in particular [Voi96] and [Voi98]); for an equivalent packing dimension approach to free entropy dimension, see [Jun03]. Our invariant is a combinatorial analogue of these quantities for dynamical systems. Voiculescu's free entropy theory allowed him to solve several longstanding open problems on finite von Neumann algebras. The computation of his invariants, that we shall simply denote $\delta(F)$ here, has by now been done for many finite sets $F \subset M$. We refer to the introduction of [BDJ08] for a recent overview of results relevant to the present paper. The analogue of Theorem 1.1 is not known for arbitrary finite generating sets of a finite von Neumann algebra M, although it is known that $\delta(F) \leq 1$ for all finite generating sets $F \subset M$ when M is the hyperfinite II_1 factor, or (much) more generally, when M is strongly 1-bounded [Jun07]. Despite much recent progress, the question of distinguishing the II_1 factors of two free groups $\mathbf{F}_p$ and $\mathbf{F}_q$ with $p \neq q$ up to isomorphism remains open.

The *cost* of a p.m.p. equivalence relation, denoted $\mathrm{cost}(R)$, was studied by Gaboriau in his breakthrough paper [Gab00]. One of the main results of [Gab00] is the additivity formula for the cost,

$$\mathrm{cost}(R) = \mathrm{cost}(R_1) + \mathrm{cost}(R_2) - 1 + \mu(D),$$

for a free product $R = R_1 *_{R_3} R_2$ of finitely generated equivalence relations over an amenable subrelation R_3. In the context of Voiculescu's free entropy, a similar formula for certain finite sets of a free product $M = M_1 *_{M_3} M_2$ amalgamated over an amenable von Neumann algebra M_3 is established in [BDJ08]. Furthermore, a relative version of Voiculescu's free entropy theory was shown by Shlyakhtenko in [Shl03] to provide another orbit equivalence invariant $\delta(R)$. Shlyakhtenko proves that

$$\delta(R) = \delta(R_1) + \delta(R_2)$$

whenever $R = R_1 * R_2$ is a free product of finitely generated p.m.p. equivalence relations.

The relation between these invariants is unclear in general. In all known cases, we have $\underline{s}(R) = s_\omega(R) = s(R) = \delta(R) = \mathrm{cost}(R)$, but it seems unlikely that these equalities hold in general. They do hold if R is treeable, as a consequence of the fact that these invariants take the same value for amenable equivalence relations, and behave similarly under free products. For example, if R is the orbit equivalence relation of a free p.m.p. action of a free group $\mathbf{F}_p$ on p generators, then $s(R) = \delta(R) = \mathrm{cost}(R) = p$. Note that, in particular, these invariants cannot distinguish between any two different actions of $\mathbf{F}_p$ up to orbit equivalence (there are uncountably many such actions [GP05]), but it is not known whether this remains the case for more general acting groups. We note that the equality $s(R) = \mathrm{cost}(R)$ (resp. $\delta(R) = \mathrm{cost}(R)$) implies in particular that R is sofic (resp. that R is hyperlinear, namely, that the von Neumann algebra of R embeds into an ultrapower of the hyperfinite II_1 factor).

Sofic groups were introduced by Gromov in [Gro99, 4.G] (see also [Wei00]) and have generated a wealth of activity in recent years. While it seems difficult to construct groups that are not sofic, many of the conjectures that are formulated for all countable groups are known to be true for sofic groups (see e.g. [Gro99, ES05]).

The invariant s measures "how sofic" the system under consideration is. Understandably, it is maximal for treeable equivalence relations, that is, in the freest case (a similar invariant for groups would give $s(\mathbf{F}_p) = p$, which is the maximal value for a p-generated group). The notion of sofic equivalence relations was introduced and studied in [EL10] (the definition we are using in the present paper is taken from [Oz09] and was studied in [Pau10]). Sofic approximations have already found several applications to dynamical systems, a spectacular one being Bowen's construction of measure-conjugacy entropy invariants for actions of sofic groups [Bow10]. Besides the analogy with Voiculescu's free entropy dimension mentioned already, it is interesting to note that the proof of Theorem 1.1 also has some similarity with the proof of invariance of entropy under change of finite generating partition [Bow10, Theorem 2.2], see especially the proof of Theorem 2.6 in [KL10]. We will study these aspects in a subsequent paper. In the other direction, we mention that, by considering actions by Bernoulli shifts, our results can also be applied to finitely generated groups (although this is not the shortest way to purely group-theoretic results). For example, they imply that an amalgamated free product of sofic groups over an amenable group is sofic, a recent result obtained independently by [ES10] and [Pau10] (see also [ES06] for the case of free products with no amalgamation and [CD10] for the case of amalgamation over monotileably amenable subgroups).

Finally, we note that an invariant similar to s was defined independently (and prior to us) for countable groups by M. Abert, L. Bowen, and N. Nikolov (see Item 14 in Section 4 of [AS09]). The cases of groups and group actions *per se* offer different problems than the ones considered in the present paper and deserve to be studied in their own right.

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2. Definition of $s(F)$, dynamical generating sets, and s-regularity

Throughout (X, μ) denotes a standard probability space. We refer to [KM04] for a recent reference on measured equivalence relations.

Let R be a measured equivalence relation on (X, μ). We write $[R]$ for the full group of R and $[\![R]\!]$ for the set of all partial measure-preserving transformations of X whose graph is included in R, where we identify two transformations if they coincide almost surely in the usual way. The product st of two transformations

$s, t \in [\![R]\!]$ is defined to be $(st)x := s(t(x))$ for all x in $\operatorname{dom} st = t^{-1}(\operatorname{ran} t \cap \operatorname{dom} s)$, where $\operatorname{dom} s$ and $\operatorname{ran} s$ denote respectively the domain and the range of $s \in [\![R]\!]$. We write $1 \in [R]$ for the identity transformation of X, and $0 \in [\![R]\!]$ for the negligible transformation of X (with empty domain). The set $[\![R]\!]$ is an inverse semigroup under composition and inverse. The equivalence relation R is said to *preserve the measure* μ if $\mu(\operatorname{dom} s) = \mu(\operatorname{ran} s)$ for all $s \in [\![R]\!]$. By "p.m.p. equivalence relation on (X, μ)" we mean a measured equivalence relation on the probability space (X, μ) which preserves the measure μ.

Let R be a p.m.p. equivalence relation on (X, μ). The *uniform distance* $|s - t|$ between two elements $s, t \in [\![R]\!]$ is defined by

$$|s - t| := \mu\{x \in \operatorname{dom} s \cup \operatorname{dom} t \mid s(x) \neq t(x)\},$$

with the convention that $s(x) \neq t(x)$ if $x \in \operatorname{dom} s \,\Delta\, \operatorname{dom} t$. See Lemma 3.1 below for some elementary properties of the distance $|\cdot|$ (note that the restriction of $|\cdot|$ to $[R]$ is the usual uniform distance with respect to which $[R]$ is a Polish group). We also set

$$\tau(s) = \tau_R(s) := \mu\{x \in X \mid s(x) = x\},$$

that is, the restriction to $[\![R]\!]$ of the normalized trace on the von Neumann algebra LR of R (viewing elements of $[\![R]\!]$ as partial isometries in LR in the usual way).

Given p.m.p. equivalence relations R and R', a finite set $F \subset [\![R]\!]$, integers $n, d \geq 1$ and a $\delta > 0$, we say that a map $\varphi : [\![R]\!] \to [\![R']\!]$ is (F, n, δ)-*multiplicative* if

$$|\varphi(s_1 \cdots s_n) - \varphi(s_1) \cdots \varphi(s_n)| < \delta$$

for all $(s_1, \ldots, s_n) \in F_\pm^{\times n}$, and (F, n, δ)-*trace-preserving* if

$$|\tau_{R'}(\varphi(s)) - \tau_R(s)| < \delta.$$

for all $s \in F_\pm^n$, where we write $F_\pm$ for the finite subset of $[\![R]\!]$ defined by $F_\pm := F \cup \{s^{-1},\, s \in F\} \cup \{1\}$, and we use the notations $F^{\times n}$ and F^n to denote respectively the n-fold Cartesian product of F and the finite subset of $[\![R]\!]$ consisting of all products of n elements of F, with the convention that $F_\pm^n$ refers to the subset $(F_\pm)^n$. We note that these two notions are local in the sense that they only involve knowledge of the values of φ on the finite set F^n.

Let d be an integer. We denote by $[d]$ the symmetric group on d elements and let $[\![d]\!]$ be the associated inverse semigroup of partial permutations, i.e., the inverse semigroup associated to the full equivalence relation on the set with d elements, endowed with the uniform probability measure.

Given F a finite subset of $[\![R]\!]$, $n \in \mathbb{N}$, and $\delta > 0$, we define $\mathrm{SA}(F, n, \delta, d)$ to be the set of all unital maps $\varphi : [\![R]\!] \to [\![d]\!]$ which are (F, n, δ)-multiplicative and (F, n, δ)-trace-preserving. Elements of $\mathrm{SA}(F, n, \delta, d)$ are called (sofic) *microstates* for R. We write $\mathrm{NSA}(F, n, \delta, d)$ for the number of distinct restrictions of elements of $\mathrm{SA}(F, n, \delta, d)$ to the set F.

Definition 2.1. We set

$$\begin{aligned} s(F,n,\delta) &= \limsup_{d\to\infty} \frac{1}{d\log d} \log \mathrm{NSA}(F,n,\delta,d), \\ s(F,n) &= \inf_{\delta>0} s(F,n,\delta), \\ s(F) &= \inf_{n\in\mathbb{N}} s(F,n). \end{aligned}$$

We similarly define $\underline{s}(F,n,\delta)$, $\underline{s}(F,n)$, and $\underline{s}(F)$ by replacing the limit supremum in the first line with a limit infimum. Given a nonprincipal ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$, we set

$$s_\omega(F,n,\delta) = \lim_{d\to\omega} \frac{1}{d\log d} \log \mathrm{NSA}(F,n,\delta,d),$$

and define $s_\omega(F,n)$ and $s_\omega(F)$ as above by taking infima over $\delta > 0$ and $n \in \mathbb{N}$. In particular, $\underline{s}(F) \le s_\omega(F) \le s(F)$.

We trust that our notation $s(F)$ for the invariant and $s \in F$ for the element will not cause confusion.

Definition 2.2. A set $F \subset [\![R]\!]$ is called a *generating set* (or *graphing*) of R if for almost every $(x,y) \in R$ there exists an $n \in \mathbb{N}$ and a n-tuple $(s_1,\dots,s_n) \in F_{\pm}^{\times n}$ such that $y = s_n \cdots s_1(x)$. An equivalence relation is said to be *finitely generated* if it admits a finite generating set.

For example, the Connes–Feldman–Weiss theorem [CFW81] states that every amenable equivalence relation R is *singly generated*, namely generated by a single transformation s in $[R]$. In other words, for almost every $(x,y) \in R$, we can find an $n \in \mathbb{Z}$ such that $y = s^n(x)$.

Levitt introduced the notion of *cost* for generating a relation R, which was much studied recently (see [Lev95, Gab00]) and which we recall now. Given a countable subset $F \subset [\![R]\!]$, set $\mathrm{cost}(F) := \sum_{s\in F} \mu(\mathrm{dom}\, s)$. This is usually not an orbit equivalence invariant, but we can define one as follows.

Definition 2.3. Given a p.m.p. equivalence relation R, set

$$\mathrm{cost}(R) = \inf_F \mathrm{cost}(F)$$

where the infimum is taken over all countable generating sets of R.

Let us now introduce a different notion of generating set for equivalence relations, and define the notion of s-regularity. Note that we have a partial additive structure on $[\![R]\!]$, with neutral element $0 \in [\![R]\!]$, coming from the additive structure on the von Neumann algebra LR. Namely, if $s_1,\dots,s_k$ are elements of $[\![R]\!]$ with pairwise disjoint domains and pairwise disjoint ranges (in which case we say that the s_i are pairwise orthogonal), then $\sum_{i=1}^k s_k \in [\![R]\!]$ defined to be the partial isomorphism which coincides with s_i on $\mathrm{dom}\, s_i$ and is undefined elsewhere. Given $F \subseteq [\![R]\!]$ we write $\mathbf{\Sigma}F$ for the set of all elements in $[\![R]\!]$ which can be written as a finite sum of elements in F. That is, $\mathbf{\Sigma}F$ is the set of all sums $\sum_{i=1}^{k} s_k$ where $s_1, \ldots, s_k$ are pairwise orthogonal elements of F.

Definition 2.4. A set $F \subseteq [\![R]\!]$ is called a *dynamical generating set* for R if for every $t \in [\![R]\!]$ and $\varepsilon > 0$ there are an $n \in \mathbb{N}$ and $s \in \mathbf{\Sigma}F_{\pm}^{n}$ such that $|t - s| < \varepsilon$. We say that R is *dynamically finitely generated* if it admits a finite dynamical generating set.

Example 2.5. Let $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ be a Bernoulli action of $\mathbb{Z}$, where $\{0,1\}^{\mathbb{Z}}$ is endowed with an invariant product probability measure. Let s be the automorphism of $\{0,1\}^{\mathbb{Z}}$ corresponding to the generator of $\mathbb{Z}$ and, for $i = 0, 1$, let p_i be projection onto the Borel subset of $\{0,1\}^{\mathbb{Z}}$ consisting of all sequence whose 0 coordinate is i. Then $F = \{s, p_0, p_1\}$ is a dynamical generating set for $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$. More generally, if $\Gamma \curvearrowright (X, \mu)$ is a p.m.p. action of a finitely generated group which admits a finite generating partition $\mathscr{P}$, then the union of the finite set of projections associated to $\mathscr{P}$ and a finite generating set of Γ (viewed as a subset of $\mathrm{Aut}(X, \mu)$) forms a dynamical generating set for the orbit equivalence relation associated to $\Gamma \curvearrowright (X, \mu)$.

Proposition 2.6. *An ergodic p.m.p. equivalence relation is finitely generated if and only if it is dynamically finitely generated.*

Proof. It is clear that any dynamical generating set is a generating set in the sense of Definition 2.2. For the reverse implication, suppose that the p.m.p. equivalence relation R is finitely generated. Then there are partial transformations $s_1, \ldots, s_n$ which generate R in the sense of Definition 2.2 and such that s_1 is an ergodic automorphism of (X, μ). Indeed, we may just take any finite generating set of R and add to it an ergodic transformation in $[R]$, whose existence is guaranteed by the ergodicity assumption on R (see e.g. [KM04]). By Dye's theorem [Dye59], the automorphism s_1 is orbit equivalent to a Bernoulli shift $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$. Thus, as in the previous example, we can find a dynamical generating set F of the subrelation of R generated by s_1. Then the set $F \cup \{s_2, \ldots, s_n\}$ is a dynamical generating set for R. □

We will also need the following simple lemma.

Lemma 2.7. *Assume that the equivalence relation R is generated by two subrelations R_1 and R_2, and that F_1 and F_2 are dynamical generating sets of R_1 and R_2. Then $F_1 \cup F_2$ is a dynamical generating set of R.*

We now introduce the notion of regularity for p.m.p. equivalence relations mentioned in the introduction. This notion is comparable to regularity in the context of free entropy, as defined in [Voi96, Definition 3.6] and [BDJ08, Definition 2.1].

Definition 2.8. Let $F \subset [\![R]\!]$ be a finite set. The set F is said to be *regular* if $\underline{s}(F) = s(F)$. A finitely generated equivalence relation R is said to be *s-regular* if it admits a finite dynamical generating set which is regular.

Remark 2.9. It is a corollary of Theorem 4.1 that if the equivalence relation R admits a regular finite dynamical generating set, then all finite dynamical generating sets for R are regular. This is the case for example if R is amenable, or more generally if R is treeable (see Corollary 7.6). We do not know of an example of a finite generating set of an equivalence relation which is not regular.

3. Preliminary technical lemmas

In this section we first establish several lemmas that will be used in the course of proving Theorem 1.1 and Theorem 1.2.

Lemma 3.1. *Let R be a p.m.p. equivalence relation, and let $r, s, t \in [\![R]\!]$. Then*

(1) $|s-t| = \mu(\operatorname{dom} s \,\Delta\, \operatorname{dom} t) + \tau(s^{-1}st^{-1}t) - \tau(st^{-1})$
$\qquad = \tau(s^{-1}s) + \tau(t^{-1}t) - 2\tau(s^{-1}st^{-1}t) - \tau(st^{-1})$,
(2) *if* $s^{-1}st^{-1}t = 0$, *then* $|s-t| = \mu(\operatorname{dom} s) + \mu(\operatorname{dom} t)$,
(3) $|s-t| = |s^{-1} - t^{-1}|$,
(4) $|\tau(s) - \tau(t)| \le |s-t|$,
(5) $|rs - rt| = |ps - pt| \le |s-t|$ *where* $p = r^{-1}r$,
(6) $|sr - tr| = |sp - tp| \le |s-t|$ *where* $p = rr^{-1}$.

Furthermore, if for some $\delta > 0$ we have

(7) $|s-p| < \delta$, *where p is an projection, then there exists an projection $p' \le s^{-1}s$ such that $|p-p'| < \delta$.*
(8) $|sts - s| < \delta$ *and* $|tst - t| < \delta$, *then* $|t - s^{-1}| < 3\delta$.

This lemma is elementary and we leave the proof to the reader. Let us establish (8) for example. Let $p = ts$, so that $\operatorname{dom} p \subset \operatorname{dom} s$. Since by assumption $|sp - s| < \delta$ we have, for all $x \in \operatorname{dom} s$ outside a subset of measure at most δ, that $sp(x) = s(x)$. In particular $x \in \operatorname{dom} p$ and $p(x) = x$, and thus $|s^{-1}s - p| < \delta$. Similarly, $|ss^{-1} - st| < \delta$. Now the second inequality shows that $|pt - t| < \delta$ and so we conclude that

$$|t - s^{-1}| < |pt - s^{-1}| + \delta \le |s^{-1}st - s^{-1}| + 2\delta = |st - ss^{-1}| + 2\delta < 3\delta.$$

Lemma 3.2. *Let R be a p.m.p. equivalence relation. Fix a finite set $F \subset [\![R]\!]$, integers $n, d \ge 1$, a $\delta > 0$, and a microstate $\varphi \in \mathrm{SA}(F, n, \delta, d)$. Then we have*

(1) $|\varphi(\prod_{i=1}^k s_i) - \prod_1^k \varphi(s_i)| \le (k+1)\delta$ *for all* $(s_1, \dots, s_k) \in (F_\pm^{\lfloor n/k \rfloor})^{\times k}$ *where* $2 \le k \le n/2$,
(2) $|\varphi(s^{-1}) - \varphi(s)^{-1}| \le 50\delta$ *for all* $s \in F_\pm^{n-2}$,
(3) $|\varphi(\prod_{i=1}^k s_i^{\varepsilon_i}) - \prod_1^k \varphi(s_i)^{\varepsilon_i}| \le (50k_0 + k + 1)\delta$ *for all* $(s_1, \dots, s_k) \in (F_\pm^{\lfloor n/k \rfloor})^{\times k}$ *and* $(\varepsilon_1, \dots, \varepsilon_k) \in \{\pm 1\}^{\times k}$, *where* $2 \le k \le n/2$ *and k_0 is the number of indices i such that* $\varepsilon_i = -1$,
(4) $|\varphi(s) - \varphi(t)| \le |s-t| + 400\delta$ *for all* $s, t \in F_\pm^{\lfloor n/4 \rfloor}$.

Proof. (1) Write $s_i = \prod_{j=1}^{k_i} s_{i,j}$ where $s_{i,j} \in F_\pm$ and $k_i \leq \lfloor n/k \rfloor$. We have:

$$\begin{aligned}\left|\varphi\Big(\prod_{i=1}^{k} s_i\Big) - \prod_{i=1}^{k}\varphi(s_i)\right| &\leq \left|\varphi\Big(\prod_{i=1}^{k}\prod_{j=1}^{k_i} s_{i,j}\Big) - \prod_{i=1}^{k}\prod_{j=1}^{k_i}\varphi(s_{i,j})\right| \\ &\quad + \left|\prod_{i=1}^{k}\prod_{j=1}^{k_i}\varphi(s_{i,j}) - \prod_{i=1}^{k}\varphi\Big(\prod_{j=1}^{k_i} s_{i,j}\Big)\right| \\ &< \delta + \sum_{i=1}^{k}\left|\prod_{j=1}^{k_i}\varphi(s_{i,j}) - \varphi\Big(\prod_{j=1}^{k_i} s_{i,j}\Big)\right| \\ &< (k+1)\delta.\end{aligned}$$

(2) Let $s \in F_\pm^{\lfloor n/3 \rfloor}$. Since $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$, we get

$$|\varphi(s) - \varphi(s)\varphi(s^{-1})\varphi(s)| < 4\delta \text{ and } |\varphi(s^{-1}) - \varphi(s^{-1})\varphi(s)\varphi(s^{-1})| < 4\delta.$$

Applying Lemma 3.1, we see that $|\varphi(s^{-1}) - \varphi(s)^{-1}| < 12\delta$. Take now $s \in F_\pm^{n-2}$ and write $s = s_1 s_2 s_3$ where $s_i \in F_\pm^{\lfloor n/3 \rfloor}$. We have

$$|\varphi(s) - \varphi(s_1)\varphi(s_2)\varphi(s_3)| < 4\delta \text{ and } |\varphi(s^{-1}) - \varphi(s_3^{-1})\varphi(s_2^{-1})\varphi(s_1^{-1})| < 4\delta.$$

Using again Lemma 3.1, it follows that $|\varphi(s^{-1}) - \varphi(s)^{-1}| \leq (8 + 3 \cdot 12)\delta = 44\delta$, yielding the result. Assertion (3) is immediate. Let us show (4). For all $s, t \in F_\pm^{n/4}$ we have

$$|s - t| = \tau(s^{-1}s) + \tau(t^{-1}t) - 2\tau(s^{-1}st^{-1}t) - \tau(st^{-1})$$

and

$$\begin{aligned}|\varphi(s) - \varphi(t)| = {}& \operatorname{tr}(\varphi(s)^{-1}\varphi(s)) + \operatorname{tr}(\varphi(t)^{-1}\varphi(t)) \\ & - 2\operatorname{tr}(\varphi(s)^{-1}\varphi(s)\varphi(t)^{-1}\varphi(t)) - \operatorname{tr}(\varphi(s)\varphi(t)^{-1})\end{aligned}$$

Hence

$$|\varphi(s) - \varphi(t)| \leq |s - t| + 54\delta + 54\delta + 212\delta + 54\delta = |s - t| + 374\delta.$$

This proves the lemma. ☐

Let R be a p.m.p. equivalence relation on (X, μ). It will be convenient to extend the additive structure on $[\![R]\!]$ to any k-tuple of elements. This can be done as follows (note that this extension does not coincide anymore with the additive structure on the von Neumann algebra LR).

Definition 3.3. Let $s_1, \ldots, s_k \in [\![R]\!]$ be partial isomorphisms. We define $\sum_{i=1}^{k} s_i$ to be the partial isomorphism

$$\sum_{i=1}^{k} s_i := \sum_{i=1}^{k} s_i \pi_i(s_1, \ldots, s_n)$$

where $\pi_i(s_1,\dots,s_n)$ is the projection defined by

$$\pi_i(s_1,\dots,s_n) = \left(s_i^{-1}s_i\prod_{j\neq i}(1-s_j^{-1}s_j)\right)\times s_i^{-1}\left(s_is_i^{-1}\prod_{j\neq i}(1-s_js_j^{-1})\right)s_i.$$

It is clear that this extends the definition of addition in the case that the s_i are pairwise orthogonal.

Fix a finite set $F\subset [\![R]\!]$, integers $n,d\geq 1$, a $\delta>0$.

Lemma 3.4. *Let $s_1,\dots,s_k\in F^{\lfloor n/4\rfloor}$ be pairwise orthogonal elements, and let $\varphi\in \mathrm{SA}(F,n,\delta,d)$ be a microstate. Then we have*

$$|\varphi(s_i)\pi_i(\varphi(s_1),\dots,\varphi(s_k))-\varphi(s_i)|<220(k-1)\delta$$

Proof. Since $s_i^{-1}s_is_j^{-1}s_j=0$ we have using Lemma 3.2 that

$$\mathrm{tr}(\varphi(s_i)^{-1}\varphi(s_i)\varphi(s_j)^{-1}\varphi(s_j))<106\delta.$$

Using a similar inequality for $s_is_i^{-1}s_js_j^{-1}$, we deduce that

$$\mathrm{tr}(\pi_i(\varphi(s_1),\dots,\varphi(s_k)))>\mathrm{tr}(\varphi(s_i)^{-1}\varphi(s_i))-212(k-1)\delta,$$

and the lemma follows. □

Definition 3.5. Given a microstate $\varphi\in\mathrm{SA}(F,n,\delta,d)$, a *linear extension of* φ is a map $\bar\varphi\in\mathrm{SA}(F,n,\delta,d)$ which coincides with φ on $F_\pm^n$ and whose values on each $s\in\mathbf{\Sigma}F_\pm^n\backslash F_\pm^n$ is given by $\bar\varphi(s):=\sum_{i=1}^k\varphi(s_i)$, where $s=\sum_{i=1}^k s_i$ is some decomposition of s with the $s_i\in F$ pairwise orthogonal.

Thus, a given φ could have several different linear extensions, but for our purposes any of them will do.

Lemma 3.6. *Let $m\geq 1$ be an integer and suppose that $\delta,\delta'>0$ satisfy $500(2|F|+1)^{3mn}\delta<\delta'$. Then for every $\varphi\in\mathrm{SA}(F,4mn,\delta,d)$, and for every linear extension $\bar\varphi$ φ, we have $\bar\varphi\in\mathrm{SA}(\mathbf{\Sigma}F_\pm^m,n,\delta',d)$.*

Proof. Let $t_1,\dots t_n\in\mathbf{\Sigma}F_\pm^m$, which we write as sums $t_i=\sum_{\boldsymbol{s}\in F_\pm^{\times m}}\gamma_{i,\boldsymbol{s}}\check{\boldsymbol{s}}$ where $\check{\boldsymbol{s}}$ means $s_1\cdots s_m$ for $\boldsymbol{s}=(s_1,\dots,s_m)\in F_\pm^{\times m}$, where the choices of $\gamma_{i,\boldsymbol{s}}\in\{0,1\}$ make this a sum of pairwise orthogonal elements as given in Definition 3.5. Let $\mathscr{F}_i$ be the family of all $\boldsymbol{s}\in F_\pm^{\times m}$ such that $\gamma_{i,\boldsymbol{s}}\neq 0$, and $\mathscr{F}$ the family of all tuples $(\boldsymbol{s}_1,\dots,\boldsymbol{s}_n)\in(F_\pm^{\times m})^{\times n}$ such that $\prod_{i=1}^n\gamma_{i,\boldsymbol{s}_i}\neq 0$. Note that the family of products $\prod_{i=1}^n\check{\boldsymbol{s}}_i\in F_\pm^{mn}$, taken over all $(\boldsymbol{s}_1,\dots,\boldsymbol{s}_n)\in\mathscr{F}$, is a subset of $F_\pm^{mn}$ (resp. the family of elements $\check{\boldsymbol{s}}\in F_\pm^m$, taken over all $\boldsymbol{s}\in\mathscr{F}_i$, is a subset of $F_\pm^m$) which consists of pairwise orthogonal elements whose sum coincides with $t_1\cdots t_n$ (resp. with t_i). Given a $(\boldsymbol{s}_1,\dots,\boldsymbol{s}_n)\in\mathscr{F}$, we let $\pi_{\varphi,(\boldsymbol{s}_1,\dots,\boldsymbol{s}_n)}(\mathscr{F})$ be the projection associated to the family $\{\varphi\bigg(\prod_{i=1}^n\check{\boldsymbol{t}}_i\bigg),(\boldsymbol{t}_1,\dots,\boldsymbol{t}_n)\in\mathfrak{F}\}$ and the element $\varphi(\prod_{i=1}^n\check{\boldsymbol{s}}_i)$ as in Definition

3.3. Similarly, given an $\boldsymbol{s} \in \mathscr{F}_i$, we let $\pi_{\varphi,\boldsymbol{s}}(\mathscr{F}_i)$ be the projection associated to the family $\{\varphi(\check{\boldsymbol{t}}), \boldsymbol{t} \in \mathfrak{F}_i\}$ and $\varphi(\check{\boldsymbol{s}})$. Using Lemma 3.4, we have

$$|\varphi(\check{\boldsymbol{s}})\pi_{\varphi,\boldsymbol{s}}(\mathscr{F}_i) - \varphi(\check{\boldsymbol{s}})| < 220((2|F|+1)^m - 1)\delta$$

for all $\boldsymbol{s} \in \mathscr{F}_i$, and

$$\left|\varphi\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right)\pi_{\varphi,(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)}(\mathscr{F}) - \varphi\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right)\right| < 220((2|F|+1)^{mn} - 1)\delta.$$

for all $(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n) \in \mathscr{F}$. Then

$$\begin{aligned}
\left|\bar{\varphi}\left(\prod_i t_i\right) - \prod_{i=1}^n \bar{\varphi}(t_i)\right| &= \left|\bar{\varphi}\left(\prod_{i=1}^n \sum_{\boldsymbol{s}\in F_\pm^{\times m}} \gamma_{i,\boldsymbol{s}}\check{\boldsymbol{s}}\right) - \prod_{i=1}^n \bar{\varphi}\left(\sum_{\boldsymbol{s}\in F_\pm^{\times m}} \gamma_{i,\boldsymbol{s}}\check{\boldsymbol{s}}\right)\right| \\
&= \left|\bar{\varphi}\left(\sum_{(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)\in\mathscr{F}} \prod_{i=1}^n \check{\boldsymbol{s}}_i\right) - \prod_{i=1}^n \bar{\varphi}\left(\sum_{\boldsymbol{s}\in F_\pm^m} \gamma_{i,\boldsymbol{s}}\check{\boldsymbol{s}}\right)\right| \\
&= \left|\sum_{(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)\in\mathscr{F}} \varphi\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right)\pi_{\varphi,(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)}(\mathscr{F}) - \prod_{i=1}^n \sum_{\boldsymbol{s}\in\mathscr{F}_i} \varphi(\check{\boldsymbol{s}})\pi_{\varphi,\boldsymbol{s}}(\mathscr{F}_i)\right| \\
&\leq \sum_{(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)\in\mathscr{F}} \left|\varphi\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right)\pi_{\varphi,(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)}(\mathscr{F}) - \prod_{i=1}^n \varphi(\check{\boldsymbol{s}}_i)\pi_{\varphi,\boldsymbol{s}_i}(\mathscr{F}_i)\right| \\
&< \sum_{(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)\in\mathscr{F}} \left(\left|\varphi\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right) - \prod_{i=1}^n \varphi(\check{\boldsymbol{s}}_i)\right| + 440((2|F|+1)^{2mn} - 1)\delta\right) \\
&< (2|F|+1)^{mn}(1+n)\delta + 440(2|F|+1)^{3mn}\delta \\
&\leq 500(2|F|+1)^{3mn}\delta.
\end{aligned}$$

Also

$$\begin{aligned}
&\left|\operatorname{tr}\circ\bar{\varphi}\left(\prod_i t_i\right) - \tau\left(\prod_i t_i\right)\right| \\
&\quad = \left|\sum_{(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)\in\mathscr{F}} \operatorname{tr}\left(\varphi\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right)\pi_{\varphi,(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)}(\mathscr{F})\right) - \tau\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right)\right| \\
&\quad < \left|\sum_{(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_n)\in\mathscr{F}} \operatorname{tr}\circ\varphi\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right) - \tau\left(\prod_{i=1}^n \check{\boldsymbol{s}}_i\right)\right| + 220(2|F|+1)^{2mn}\delta \\
&\quad < (2|F|+1)^{mn}(1+n)\delta + 220(2|F|+1)^{2mn}\delta \\
&\quad \leq 500(2|F|+1)^{3mn}\delta.
\end{aligned}$$

□

Let R be a p.m.p. equivalence relation on (X, μ). Given a finite set $F \subset [\![R]\!]$ we define a pseudometric on the set of all unital linear maps $[\![R]\!] \to [\![d]\!]$ by

$$|\varphi - \psi|_F := \max_{s \in F} |\varphi(s) - \psi(s)|.$$

Let $\varepsilon \geq 0$. We write $N_\varepsilon(\mathrm{SA}(F, n, \delta, d))$ for the ε-covering number of $\mathrm{SA}(F, n, \delta, d)$ with respect to $|\cdot|_F$, namely, the minimal number of ε-balls required to cover $\mathrm{SA}(F, n, \delta, d)$. Note that $\mathrm{NSA}(F, n, \delta, d) = N_0(\mathrm{SA}(F, n, \delta, d))$. We then set

$$\begin{aligned} s_\varepsilon(F, n, \delta) &= \limsup_{d \to \infty} \frac{1}{d \log d} \log N_\varepsilon(\mathrm{SA}(F, n, \delta, d)), \\ s_\varepsilon(F, n) &= \inf_{\delta > 0} s_\varepsilon(F, n, \delta), \\ s_\varepsilon(F) &= \inf_{n \in \mathbb{N}} s_\varepsilon(F, n). \end{aligned}$$

We similarly define ε-covering constants for $\underline{s}$ and s_ω.

Lemma 3.7. *Let $\kappa > 0$. Then there is an $\varepsilon > 0$ such that*

$$|\{t \in [\![d]\!] : |t - s| < \varepsilon\}| \leq d^{\kappa d}.$$

for all $d \in \mathbb{N}$ and $s \in [\![d]\!]$.

Proof. Let $\varepsilon > 0$. Let $d \in \mathbb{N}$ and $s \in [\![d]\!]$. For every $t \in [\![d]\!]$ satisfying $|s - t| < \varepsilon$ the cardinality of the set of all $j \in \{1, \ldots, d\}$ such that $t(j) \neq s(j)$ is at most εd. Thus the set of all $t \in [\![d]\!]$ such that $|t - s| < \varepsilon$ has cardinality at most $\sum_{j=0}^{\lfloor \varepsilon d \rfloor} \binom{d}{j} j!$. This sum is bounded above by $\lfloor \varepsilon d \rfloor \binom{d}{\lfloor \varepsilon d \rfloor} (\lfloor \varepsilon d \rfloor)! \leq \varepsilon d^{1 + \varepsilon d}$, which for small enough ε is less that $d^{\kappa d}$, independently of d. □

Lemma 3.8. *Let F be a finite subset of $[\![R]\!]$. Let $\kappa > 0$. Then there is an $\varepsilon > 0$ such that*

$$s(F, n) \leq s_\varepsilon(F, n) + \kappa$$

for all $n \in \mathbb{N}$. The same inequality holds for $\underline{s}$ and s_ω.

Proof. This is a straightforward consequence of Lemma 3.7. □

Lemma 3.9. *Let F be a finite subset of $[\![R]\!]$. Then*

$$s(F) = \lim_{\varepsilon \to 0} s_\varepsilon(F).$$

The same equality holds for $\underline{s}$ and s_ω.

Proof. Since $N_\varepsilon(\cdot)$ is increasing as ε decreases, $s_\varepsilon(F)$ is increasing as $\varepsilon \to 0$. By taking an infimum over n, the previous lemma shows that for all $\kappa > 0$ there is an $\varepsilon > 0$ such that

$$s(F) \leq s_\varepsilon(F) + \kappa.$$

Thus $s(F) \leq \lim_{\varepsilon \to 0} s_\varepsilon(F)$. The other inequality is clear. □

4. INVARIANCE UNDER ORBIT EQUIVALENCE

Theorem 4.1. *Let R be a p.m.p. equivalence relation and let E and F be finite dynamical generating sets. Then $s(E) = s(F)$, $\underline{s}(E) = \underline{s}(F)$, and $s_\omega(E) = s_\omega(F)$.*

Proof. Let us show the first equality. By symmetry it suffices to prove that $s(E) \leq s(F)$. Let $\varepsilon > 0$ and let $\delta > 0$ be smaller than $\varepsilon/4$. Since F is generating, by increasing n if necessary we may assume that for every $s \in E$ there is an $s' \in \mathbf{\Sigma}F_\pm^n$ for which $|s - s'| < \delta$. Take a $\delta' > 0$ such that $500\delta' < \delta$ and $\delta' < \varepsilon/502000(2|F|+1)^n$. Since E is generating there exists an $m \in \mathbb{N}$ such that for all $s \in \mathbf{\Sigma}F_\pm^n$, there exists an element $\theta(s) \in \mathbf{\Sigma}E_\pm^m$ for which $|s - \theta(s)| < \delta'/(n+1)$. Observe that the map $\theta : s \mapsto \theta(s)$ is (F, n, δ')-multiplicative. Indeed, let $s_1, \dots s_n \in F_\pm$ and set $s = s_1 \cdots s_n$. Then we have

$$\begin{aligned} |\theta(s) - \theta(s_1)\cdots\theta(s_n)| &\leq |\theta(s) - s| + |s_1 \cdots s_n - \theta(s_1)\cdots\theta(s_n)| \\ &\leq |\theta(s) - s| + \sum_{i=1}^n |\theta(s_i) - s_i| < \delta'. \end{aligned}$$

Take a $\delta'' > 0$ such that $500(2|E|+1)^{3mn}\delta'' < \delta'$. Let $\varphi \in \mathrm{SA}(E, 4mn, \delta'', d)$. By Lemma 3.6, any linear extension $\bar\varphi$ of φ belongs to $\mathrm{SA}(\mathbf{\Sigma}E_\pm^m, n, \delta', d)$. Write $\varphi^\natural$ for $\bar\varphi \circ \theta$. We note that $\varphi^\natural \in \mathrm{SA}(F, n, 500\delta', d)$. Indeed, take $t_1, \dots, t_n \in F_\pm$ and set $t = t_1 \cdots t_n$. Then, using Lemma 3.2,

$$\begin{aligned} |\varphi^\natural(t) - \varphi^\natural(t_1)\cdots\varphi^\natural(t_n)| &= |\bar\varphi(\theta(t)) - \bar\varphi(\theta(t_1)\cdots\theta(t_n))| \\ &\qquad + |\bar\varphi(\theta(t_1)\cdots\theta(t_n)) - \bar\varphi(\theta(t_1))\cdots\bar\varphi(\theta(t_n))| \\ &\leq |\theta(t) - \theta(t_1)\cdots\theta(t_n)| + 400\delta' + \delta' \\ &< 402\delta'. \end{aligned}$$

and

$$\begin{aligned} |\operatorname{tr}\circ\varphi^\natural(t) - \tau(t)| &\leq |\operatorname{tr}\circ\bar\varphi(\theta(t)) - \tau(\theta(t))| + |\tau(\theta(t)) - \tau(t)| \\ &< \delta' + |\theta(t) - t| < 2\delta'. \end{aligned}$$

Now consider microstates φ and ψ in $\mathrm{SA}(E, mn, \delta'', d)$ and suppose that $\varphi^\natural$ and $\psi^\natural$ coincide on F. Then by Lemma 3.2 $|\varphi^\natural(s^{-1}) - \psi^\natural(s^{-1})| < 100\delta'$ for all s in $F_\pm$ and thus, for $t_1, \dots, t_n \in F_\pm$ such that $t = t_1 \cdots t_n$, we have

$$\begin{aligned} |\varphi^\natural(t) - \psi^\natural(t)| \leq &\left|\varphi^\natural(t) - \prod_{i=1}^n \varphi^\natural(t_i)\right| \\ &+ \left|\prod_{i=1}^n \varphi^\natural(t_i) - \prod_{i=1}^n \psi^\natural(t_i)\right| + \left|\prod_{i=1}^n \psi^\natural(t_i) - \psi^\natural(t)\right| \\ < \delta' + \sum_{i=1}^n &|\varphi^\natural(t_i) - \psi^\natural(t_i)| + \delta' \end{aligned}$$

$$< (100n+2)\delta'.$$

If $t = \sum_{i=1}^{k} t_i \in \mathbf{\Sigma} F_{\pm}^n$, we have, by Lemma 3.4

$$|\varphi^\natural(t_i)\pi_i(\varphi^\natural(t_1),\dots,\varphi^\natural(t_k)) - \varphi^\natural(t_i)| < 220 \times 500(k-1)\delta' \leq 110000(2|F|+1)^n\delta',$$

and similarly

$$|\psi^\natural(t_i)\pi_i(\psi^\natural(t_1),\dots,\psi^\natural(t_k)) - \psi^\natural(t_i)| < 110000(2|F|+1)^n\delta'.$$

whence

$$\begin{aligned}
|\varphi^\natural(t) - \psi^\natural(t)| &= \left| \sum_{i=1}^{k} \varphi^\natural(t_i)\pi_i(\varphi^\natural(t_1),\dots,\varphi^\natural(t_k)) - \sum_{i=1}^{k} \psi^\natural(t_i)\pi_i(\psi^\natural(t_1),\dots,\psi^\natural(t_k)) \right| \\
&\leq \sum_{i=1}^{k} |\varphi^\natural(t_i)\pi_i(\varphi^\natural(t_1),\dots,\varphi^\natural(t_k)) - \psi^\natural(t_i)\pi_i(\psi^\natural(t_1),\dots,\psi^\natural(t_k))| \\
&\leq (110000(2|F|+1)^n\delta' + (100n+2)\delta' + 110000(2|F|+1)^n\delta')k \\
&\leq 250000(2|F|+1)^{2n}\delta'.
\end{aligned}$$

Let $s \in E$. We can find $s' \in \mathbf{\Sigma} F_{\pm}^n$ such that $|s - s'| < \delta$. Then $|\theta(s') - s'| < \delta'$, and thus $|s - \theta(s')| < \delta + \delta'$. Hence,

$$|\varphi(s) - \varphi^\natural(s')| = |\bar{\varphi}(s) - \bar{\varphi}(\theta(s'))| < |s - \theta(s')| + 400\delta' < \delta + 401\delta'$$

Similarly, $|\psi(s) - \psi^\natural(s')| < \delta + 401\delta'$, so that

$$\begin{aligned}
|\varphi(s) - \psi(s)| &\leq |\varphi(s) - \varphi^\natural(s')| + |\varphi^\natural(s') - \psi^\natural(s')| + |\psi(s) - \psi^\natural(s')| \\
&< 2\delta + 802\delta' + 250000(2|F|+1)^{2n}\delta' \\
&< \varepsilon/2 + 251000(2|F|+1)^{2n}\delta' < \varepsilon.
\end{aligned}$$

Consequently, $|\varphi - \psi|_E < \varepsilon$ and thus

$$N_\varepsilon(\mathrm{SA}(E, 4mn, \delta'', d)) \leq \mathrm{NSA}(F, n, 500\delta', d) \leq \mathrm{NSA}(F, n, \delta, d),$$

By taking a limit supremum over d, an infimum over $\delta > 0$ and over $n \in \mathbb{N}$, we conclude that $s_\varepsilon(E) \leq s(F)$. Hence, Lemma 3.9 shows that $s(E) \leq s(F)$.

The same proof applies to $\underline{s}$ and s_ω. □

In view of Theorem 4.1, we introduce the following isomorphism invariant for dynamically finitely generated (e.g. finitely generated ergodic) equivalence relations.

Definition 4.2. Let R be a p.m.p. equivalence relation on (X,μ). Assume that R is dynamically finitely generated and let F be a finite dynamical generating set. Then we set

$$s(R) := s(F).$$

We end this section with the proof of the formula $s(R) \leq \mathrm{cost}(R)$ mentioned in the introduction.

Lemma 4.3. *Let R be a p.m.p. equivalence relation on (X, μ) and let F be a finite subset of $[\![R]\!]$. Then $s(F) \leq \mathrm{cost}(F)$.*

Proof. We show that $s(F, 4) \leq \mathrm{cost}(F)$, from which the lemma follows readily. Let $\delta > 0$ and $d \in \mathbb{N}$. Take a $\varphi \in \mathrm{SA}(F, 4, \delta, d)$. Then for every $s \in F$ we have by Lemma 3.2

$$\begin{aligned} |\operatorname{tr}(\varphi(s)\varphi(s)^{-1}) - \tau(ss^{-1})| &< |\operatorname{tr}(\varphi(s)\varphi(s)^{-1}) - \operatorname{tr}(\varphi(ss^{-1})| \\ &\qquad + |\operatorname{tr}(\varphi(ss^{-1}) - \tau(ss^{-1})| \\ &< 52\delta + \delta = 53\delta. \end{aligned}$$

Thus, we have $|F|$ partial permutations $\varphi(s)$, $s \in F$, of the set with d elements whose respective domains A_s and ranges B_s have cardinality $(\mathrm{cost}(s) \pm 53\delta)d$. Since we have $\binom{d}{|A_s|}$ ways to choose A_s and B_s, and $|A_s|!$ ways to choose a bijection from A_s to B_s, we obtain

$$\mathrm{NSA}(F, 4, \delta, d) \leq \prod_{s \in F} \binom{d}{|A_s|}^2 |A_s|! = \prod_{s \in F} \frac{d!^2}{|A_s|!(d - |A_s|)!^2}$$

An easy computation using the Stirling formula shows that, taking the limit supremum over d,

$$\begin{aligned} s(F, 4, \delta) &\leq \sum_{s \in F} 2 - (\mathrm{cost}(s) - 53\delta) - 2(1 - \mathrm{cost}(s) - 53\delta) \\ &< \mathrm{cost}(F) + 160|F|\delta. \end{aligned}$$

The claim follows by taking the infimum over δ. □

Proposition 4.4. *Let R be a finitely generated ergodic p.m.p. equivalence relation on (X, μ). Then $s(R) \leq \mathrm{cost}(R)$.*

Proof. Let $\varepsilon > 0$. Since R is ergodic, we may choose a generating set F of R (in the sense of Definition 2.2) with

$$\mathrm{cost}(F) < \mathrm{cost}(R) + \varepsilon$$

which contains an ergodic automorphism of (X, μ) (see e.g. [Gab00, Lemma III.5]). Since this automorphism is orbit equivalent to a Bernoulli shift (by Dye's theorem), we may replace it in F by two partial isomorphisms of (X, μ) of cost $\frac{1}{2}$ each, which generate the same subrelation of R (compare Example 2.5). The resulting generating set F' has the same cost as that of F, and is dynamically generating. Thus

$$s(R) = s(F') \leq \mathrm{cost}(F') \leq \mathrm{cost}(R) + \varepsilon.$$

Hence the result by taking an infimum over $\varepsilon > 0$. □

5. Microstates and finite inverse semigroups

In this section G denotes a finite inverse semigroup, which we assume to be principal for convenience. Namely, G is of the form $G = [\![R_G]\!]$ where R_G is an equivalence relation on a finite set X_G. The set X_G can be taken to be the set of minimal projections in G, in which case R_G is the equivalence relation on X_G generated by von Neumann equivalence of projections: $p \sim q$ if and only if $p = s^{-1}s$ and $q = ss^{-1}$ for some $s \in G$. We endow X_G with any invariant probability measure with rational values, and G with the corresponding tracial state τ.

Given a generating set E of G, a p.m.p. equivalence relation R on a probability space (X, μ), and a unital trace-preserving embedding $G \subset [\![R]\!]$, and a finite subset $F \subset [\![R]\!]$ containing E, we denote by $\mathrm{SA}_G(F, n, \delta, d)$ the set of all maps $\varphi \in \mathrm{SA}(F, n, \delta, d)$ for which the restriction $\varphi_{|G}$ is a trace-preserving embedding. We write $\mathrm{NSA}_G(F, n, \delta, d)$ for the number of restrictions to F of elements of $\mathrm{SA}_G(F, n, \delta, d)$.

Lemma 5.1. *There exist integers n_G, m_G, d_G depending only on G such that, for every p.m.p. measured equivalence relation R, generating set E of G, unital trace-preserving embedding $G \subset [\![R]\!]$, finite subset $F \subset [\![R]\!]$ containing E, $\varepsilon \geq 0$, $\delta > 0$, integers n and $d > \delta^{-1}$, we have*

$$N_{m_G\delta+\varepsilon}(\mathrm{SA}(F, n_G + n, \delta, d_G d)) \leq N_\varepsilon(\mathrm{SA}_G(F, n_G + n, m_G n\delta, d_G d)).$$

In particular,

$$s(F) = \inf_{n\in\mathbb{N}} \inf_{\delta>0} \limsup_{d\to\infty} \frac{1}{d_G d \log d_G d} \log \mathrm{NSA}_G(F, n, \delta, d_G d).$$

The number d_G can be chosen to be any integer such that, for any $d \in \mathbb{N}$, there exists a unital trace-preserving embedding $G \subset [\![d_G d]\!]$. Furthermore, the same results hold for $\underline{s}(F)$ and $s_\omega(F)$ provided one replaces the limit supremum by $\liminf_{d\to\infty}$ and $\lim_{d\to\omega}$ respectively.

Proof. Let d_G be such that $d_G\tau(p) \in \mathbb{N}$ for all $p \in X_G$, and let n_G be at least 4 times the length of the longest reduced word over any generating set of G. We set $t_G = \min_{p\in X_G} \tau(p)$ and let m_G be a integer larger than $1000 n_G |G|^3 |X_G| t_G^{-1}$. Take E, F, n, δ, d as in the statement of the lemma, and consider for all $s, t \in G$ and $\varphi \in \mathrm{SA}(F, n_G + n, \delta/m_G, d_G d)$ the set

$$V^\varphi_{s,t} := \{i \in \{1, \dots, d_G d\} \mid \varphi(st^{-1})i = \varphi(s)\varphi(t)^{-1}i\}.$$

By the definition of n_G we have $|\varphi(st^{-1}) - \varphi(s)\varphi(t^{-1})| < 51\delta$, and thus

$$\left|V^\varphi_{s,t}\right| > (1 - 51\delta) d_G d$$

for all $s, t \in G$. Denoting $V^\varphi := \bigcap_{s,t\in G} V^\varphi_{s,t}$, it follows that $|V^\varphi| > (1 - 51|G|^2\delta) d_G d$. Let p_φ be the projection onto the subset $\{i \in V^\varphi \mid \varphi(G)i \subset V^\varphi\}$ of V^φ. We have

$\operatorname{tr}(p_\varphi) > 1 - 51|G|^3\delta$ and the restriction to G of the compression $p_\varphi \varphi p_\varphi : [\![R]\!] \to p_\varphi [\![d_G d]\!] p_\varphi$ of φ is a morphism of inverse semigroups. For any $p \in X_G$ we have

$$\operatorname{tr}(p_\varphi \varphi(p)) > \operatorname{tr}(\varphi(p)) - 51|G|^3\delta > \tau(p) - 52|G|^3\delta.$$

Thus, there exists a projection $p' \leq p_\varphi \varphi(p)$ such that $\operatorname{tr}(p') > \tau(p) - 52|G|^3\delta$. Furthermore, by replacing each p' with $p' \times \prod_{q \neq p} q'$ if necessary, we may assume that the projections p' satisfy $p'q' = 0$ for all $p \neq q \in X_G$. In doing so we obtain

$$\operatorname{tr}(p') > \tau(p) - 52|G|^3|X_G|\delta \geq \tau(p)(1 - 52|G|^3|X_G|t_G^{-1}\delta).$$

Let k be the largest integer such that $k < (1 - 52|G|^3|X_G|t_G^{-1}\delta)d$. Since $d_G\tau(p) \in \mathbb{N}$, we have $d_G d\left(\frac{k}{d}\tau(p)\right) \in \mathbb{N}$ for all $p \in X_G$, and therefore we may assume, by further restricting p' if necessary, that $\operatorname{tr}(p') = \frac{k}{d}\tau(p)$ for all $p \in X_G$. In particular the restriction to G of the compression

$$p'_\varphi \varphi p'_\varphi : [\![R]\!] \to p'_\varphi [\![d_G d]\!] p'_\varphi,$$

where $p'_\varphi := \sum_{p \in X_G} p'$, is a trace scaling isomorphism and, since we have $k \geq (1 - 52|G|^3|X_G|t_G^{-1}\delta)d - 1$,

$$\operatorname{tr}(p'_\varphi) \geq \sum_{p \in X_G} (1 - 52|G|^3|X_G|t_G^{-1}\delta - 1/d)\tau(p) \geq 1 - 52|G|^3|X_G|t_G^{-1}\delta - 1/d.$$

Furthermore, by construction $d_G d \operatorname{tr}(1 - p'_\varphi) \in \mathbb{N}$ is a multiple of d_G, so that we can choose a trace scaling isomorphism $\tilde{\varphi}$ between G and an inverse semigroup of $(1 - p'_\varphi)[\![d_G d]\!](1 - p'_\varphi)$ such that $\operatorname{tr}(\tilde{\varphi}(p)) = (1 - \frac{k}{d})\tau(p)$ for all $p \in X_G$.

Consider then the map $\varphi^\natural$ defined by

$$\varphi^\natural(s) := \varphi(s) \text{ if } s \in [\![R]\!] \setminus G,$$
$$\varphi^\natural(s) := p'_\varphi \varphi p'_\varphi + \tilde{\varphi} \text{ if } s \in G.$$

It is clear that the restriction of $\varphi^\natural$ to G is a trace-preserving isomorphism. We now show that $\varphi^\natural \in \mathrm{SA}_G(F, n_G + n, m_G n\delta, d_G d)$. Note first that for all $s \in G$

$$\begin{aligned}|\varphi(s) - \varphi^\natural(s)| &\leq |(1 - p'_\varphi)\varphi(s)(1 - p'_\varphi) - (1 - p'_\varphi)\varphi^\natural(s)(1 - p'_\varphi)| \\ &\quad + |(1 - p'_\varphi)\varphi(s)p'_\varphi| + |p'_\varphi\varphi(s)(1 - p'_\varphi)| \\ &\leq 156|G|^3|X_G|t_G^{-1}\delta + 3/d.\end{aligned}$$

Therefore,

$$|\varphi - \varphi^\natural|_{F_\pm} \leq 156|G|^3|X_G|t_G^{-1}\delta + 3/d.$$

Now choose $m \leq n_G + n$ and let $(s_1, \ldots, s_m) \in F_\pm^m$. Let us assume first that $s := \prod_{i=1}^m s_i \notin G$. Then we have

$$\begin{aligned}\left|\varphi^\natural(s) - \prod_{i=1}^m \varphi^\natural(s_i)\right| &\leq \left|\varphi(s) - \prod_{i=1}^m \varphi(s_i)\right| + \sum_{i=1}^m |\varphi(s_i) - \varphi^\natural(s_i)| \\ &< \delta + (156|G|^3|X_G|t_G^{-1}\delta + 3/d)(n_G + n)\end{aligned}$$

while, clearly, $|\operatorname{tr}(\varphi^\natural(s)) - \tau(s)| < \delta$. Assume next that $s \in G$. Then since $|\varphi(s) - \varphi^\natural(s)| \le 156|G|^3|X_G|t_G^{-1}\delta + 3/d$ we obtain similarly

$$\begin{aligned}\left|\varphi^\natural(s) - \prod_{i=1}^m \varphi^\natural(s_i)\right| &\le |\varphi(s) - \varphi^\natural(s)| + \left|\varphi(s) - \prod_{i=1}^m \varphi^\natural(s_i)\right| \\ &< \delta + (156|G|^3|X_G|t_G^{-1}\delta + 3/d)(n_G + n + 1)\end{aligned}$$

and $|\operatorname{tr}(\varphi^\natural(s)) - \tau(s)| < \delta + 156|G|^3|X_G|t_G^{-1}\delta + 3/d$. Since $d > \delta^{-1}$, these estimates show that $\varphi^\natural \in \mathrm{SA}_G(F, n_G + n, m_G n\delta, d_G d)$. Furthermore, if $\varphi, \psi \in \mathrm{SA}(F, n, \delta, d)$ are such that $|\varphi^\natural - \psi^\natural|_F \le \varepsilon$, then

$$\begin{aligned}|\varphi - \psi|_F &\le |\varphi - \varphi^\natural|_F + |\varphi^\natural - \psi^\natural|_F + |\psi^\natural - \psi|_F \\ &< \varepsilon + 312|G|^3|X_G|t_G^{-1}\delta + 6/d < \varepsilon + m_G\delta.\end{aligned}$$

This shows the first assertion of the lemma. For the second, let ε be any real number such that $\varepsilon > m_G\delta$. Then by the first assertion

$$\begin{aligned}N_\varepsilon(\mathrm{SA}(F, n_G + n, \delta, d_G d)) &\le N_{m_G\delta}(\mathrm{SA}(F, n_G + n, \delta, d_G d)) \\ &\le \mathrm{NSA}_G(F, n_G + n, m_G n\delta, d_G d).\end{aligned}$$

Taking a log, a limit infimum over d, and infima over δ and n, we obtain

$$s_\varepsilon(F) \le \inf_{n\in\mathbb{N}} \inf_{\delta>0} \limsup_{d\to\infty} \frac{1}{d_G d \log d_G d} \log \mathrm{NSA}_G(F, n, \delta, d_G d).$$

Thus, by Lemma 3.9,

$$s(F) \le \inf_{n\in\mathbb{N}} \inf_{\delta>0} \limsup_{d\to\infty} \frac{1}{d_G d \log d_G d} \log \mathrm{NSA}_G(F, n, \delta, d_G d).$$

The other inequality is clear, and the case of $\underline{s}$, s_ω is treated similarly. □

We note the following easy corollary, which can also be proved directly.

Corollary 5.2. *Let R be a p.m.p. equivalence relation on (X, μ) and $F \subset [\![R]\!]$ be a finite subset containing the identity. Assume that F generates a finite inverse subsemigroup G of $[\![R]\!]$ (namely, there exists $n_0 \in \mathbb{N}$ such that $F_\pm^{n_0} = F_\pm^n$ for all $n \ge n_0$). Let $D \subset X$ be a fundamental domain for G. Then*

$$\underline{s}(F) = s(F) = \underline{s}(G) = s(G) = 1 - \mu(D).$$

In particular, if R has finite classes and D denotes a fundamental domain, then all finite generating subsets $F \subset [\![R]\!]$ are regular and satisfy $s(F) = 1 - \mu(D)$.

Proof. Lemma 5.1 shows that

$$s(F) = \inf_{n\in\mathbb{N}} \inf_{\delta>0} \limsup_{d\to\infty} \frac{1}{d_G d \log d_G d} \log \mathrm{NSA}_G(F, n, \delta, d_G d).$$

Since $F^n = G$ for all $n \ge n_0$, we need to estimate the number of unital trace-preserving embeddings of G into $[\![d_G d]\!]$. By the definition of d_G there exists at least one such embedding $\varphi : G \to [\![d_G d]\!]$. Then, for any permutation $\theta \in [d_G d]$, the conjugate $\theta\varphi\theta^{-1}$ is another such embedding, and it is easy to see that any such embedding is obtained in this way. Let $[d_G d]_G$ be the subgroup of all permutations in $[d_G d]$ which commute with $\varphi(G)$. Clearly, if $\theta_1\varphi\theta_1^{-1}$ and $\theta_2\varphi\theta_2^{-1}$ coincide on F, then they coincide on G and thus $\theta_1^{-1}\theta_2 \in [d_G d]_G$. Hence, the total number of embeddings is

$$\frac{(d_G d)!}{|[d_G d]_G|}.$$

Denote by $\mu_k \in [0,1]$ the measure of a fundamental domain of the subrelation of R'_3 consisting of all R'_3-classes of cardinality $k \in \mathbb{N}$. Then we have $\mu(D') = \sum_{k\geq 1} \mu_k$ and $|[d_G d]_G| = \prod_{k\geq 1}(\mu_k d_G d)!$ (both the sum and the product are uniformly finite in d), so that $\log |[d_G d]_G| \sim \mu(D') d_G d \log d_G d$ as d tends to infinity, using Stirling's formula. Thus we obtain

$$s(F,n) = 1 - \mu(D)$$

for all $n \geq n_0$. Taking a limit infimum instead of a limit supremum, we see that $\underline{s}(F,n) = 1 - \mu(D)$, yielding the lemma. □

6. Asymptotic freeness and measure concentration

We now prove several useful lemmas related to asymptotic freeness. They are based on standard techniques, including the measure concentration property of symmetric groups. We refer to [Voi91, Theorem 3.9], [Voi98, Theorem 2.7], and more recently [BDJ08, Section 3], for analogous results in the context of free entropy.

Let G be a finite principal inverse semigroup with fixed tracial state, as in the previous section. Given $d \in \mathbb{N}$ and a unital trace-preserving embedding $j_d : G \to [\![d]\!]$, we write $[d]_G$ for the set of permutations in $[d] \subset [\![d]\!]$ that commute with $j_d(G)$. We endow $[d]_G$ with the Hamming distance $|s-t|_d := |\{i = 1,\dots,d \mid s(i) \neq t(i)\}|/d$ and the uniform probability measure $\mathbb{P}_{G,d}$ assigning $1/|[d]_G|$ to every permutation.

The following is a straightforward generalization of the fact that the symmetric groups $[d]$ endowed with the Hamming distance and the uniform probability measure (which corresponds to $G = \{e\}$) form a Lévy family (see [Mau79] and [GM83, Section 3.6]).

Lemma 6.1. *Fix integers $d_1 < d_2 < d_3 < \dots$ and unital trace-preserving embeddings $j_{d_k} : G \to [\![d_k]\!]$. Then $([d_k]_G, |\cdot|_{d_k}, \mathbb{P}_{G,d_k})_k$ is a Lévy family.*

Let G be a finite principal inverse semigroup with fixed tracial state. Given a p.m.p. equivalence relation R on (X,μ) and a unital trace-preserving embedding $G \subset [\![R]\!]$, we denote by Res_G the trace-preserving restriction map

$$\mathrm{Res}_G : [\![R]\!] \to [\![G]\!]$$
$$s \mapsto sp$$

where $p \in [\![R]\!]$ is the maximal projection (possibly 0) such that $sp \in [\![G]\!]$.

Lemma 6.2. *Let G be a finite principal inverse semigroup with fixed tracial state and let n be a positive integer. There exists a constant $C_{G,n}$ such that for any $\varepsilon > 0$, integers $d_1 < d_2 < d_3 < \cdots$, unital trace-preserving embeddings $j_{d_k} : G \to [\![d_k]\!]$, and partial transformations $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in [\![d_k]\!]$, we have*

$$\int_{[d_k]_G} |\mathrm{Res}_{j_{d_k}(G)}(\varphi_1\theta\psi_1\theta^{-1}\cdots\varphi_n\theta\psi_n\theta^{-1})| \,\mathrm{d}\mathbb{P}_{G,d_k}(\theta) < C_{G,n} \max_i(|\mathrm{Res}_{j_{d_k}(G)}(\varphi_i)|, |\mathrm{Res}_{j_{d_k}(G)}(\psi_i)|) + \varepsilon$$

This lemma is a generalization of [Nic93, Theorem 4.1] (see also [CD10, Theorem 2.1]) and can be proved by using similar techniques.

Lemma 6.3. *Let G be a finite principal inverse semigroup with fixed tracial state and let n be a positive integer. There exists a constant $C_{G,n}$ such that for any $\varepsilon > 0$, any integers $d_1 < d_2 < d_3 < \cdots$, any unital trace-preserving embeddings $j_{d_k} : G \to [\![d_k]\!]$, and any $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in [\![d_k]\!]$, the sets*

$$\Omega_{d_k,\varepsilon}(\varphi_i, \psi_i) := \{\theta \in [d_k]_G \mid |\mathrm{Res}_{j_{d_k}(G)}(\varphi_1\theta\psi_1\theta^{-1}\cdots\varphi_n\theta\psi_n\theta^{-1})| < C_{G,n} \max_i(|\mathrm{Res}_{j_{d_k}(G)}(\varphi_i)|,\ |\mathrm{Res}_{j_{d_k}(G)}(\psi_i)|) + \varepsilon\}$$

satisfy $\mathbb{P}_{G,d_k}(\Omega_{d_k,\varepsilon}(\varphi_i, \psi_i)) \to 1$ as $k \to \infty$.

Proof. Let $\eta > 0$ and denote by $V_\eta(\Omega_{d_k,\varepsilon})$ the η-neighborhood of $\Omega_{d_k,\varepsilon}$ for the Hamming distance. If $\theta' \in V_\eta(\Omega_{d_k,\varepsilon})$, then

$$|\varphi_1\theta'\psi_1\theta'^{-1}\ldots\varphi_n\theta'\psi_n\theta^{-1} - \varphi_1\theta\psi_1\theta^{-1}\cdots\varphi_n\theta\psi_n\theta^{-1}| < \sum_{i=1}^{n} |\theta'\psi_i\theta'^{-1} - \theta\psi_i\theta^{-1}| < 2n\eta.$$

Thus if $\varepsilon > 0$ and $\eta < \varepsilon/4n$, then $V_\eta(\Omega_{d_k,\varepsilon/2}) \subset \Omega_{d_k,\varepsilon}$. Since

$$|\mathrm{Res}_{j_{d_k}(G)}(\varphi_1\theta\psi_1\theta^{-1}\cdots\varphi_n\theta\psi_n\theta^{-1})| \le 1,$$

Lemma 6.2 shows that

$$\liminf_{k\to\infty} \mathbb{P}_{G,d_k}(\Omega_{d_k,\varepsilon/2}) > 0.$$

By the Lévy property, $\liminf_{k\to\infty} \mathbb{P}_{G,d_k}(V_\eta(\Omega_{d_k,\varepsilon})) = 1$ and thus

$$\lim_{k\to\infty} \mathbb{P}_{G,d_k}(V_\eta(\Omega_{d_k,\varepsilon})) = 1,$$

proving the lemma. □

7. The free product formula

Let R be a p.m.p. equivalence relation on (X,μ).

Theorem 7.1. *Assume that $R = R_1 *_{R_3} R_2$ is a free product of equivalence relations amalgamated over an amenable subrelation R_3. Assume that R_1 and R_2 are dynamically finitely generated. Then we have*

$$s_\omega(R) = s_\omega(R_1) + s_\omega(R_2) - 1 + \mu(D).$$

where D is a fundamental domain of the finite component of R_3. If furthermore R_1 and R_2 are s-regular, then R is s-regular and

$$s(R) = s(R_1) + s(R_2) - 1 + \mu(D).$$

The proof follows directly from the following two lemmas.

Lemma 7.2. *Assume that $R = R_1 *_{R_3} R_2$ is a free product of equivalence relations amalgamated over an amenable subrelation R_3. Assume that R_1 and R_2 are dynamically finitely generated. Then,*

$$\underline{s}(R) \geq \underline{s}(R_1) + \underline{s}(R_2) - 1 + \mu(D).$$

Furthermore, the same inequality holds for s_ω instead of $\underline{s}$.

Proof. Let $F_1 \subset [\![R_1]\!]$ and $F_2 \subset [\![R_2]\!]$ be dynamical generating sets. For convenience, we assume that F_1 and F_2 are symmetric and contain the identity. Furthermore, by conjugating F_2 with a suitable element of $[R_2]$, we may assume that $F_1 \cap F_2$ contains only the identity.

Set $F := F_1 \cup F_2$ and let $n \in \mathbb{N}$. Any $s \in F^n$ can be written as a product $s = \check{\boldsymbol{s}}_1 \cdots \check{\boldsymbol{s}}_\ell$ where $\ell \leq n$ and the $\boldsymbol{s}_i$ alternate membership in ${F_1}^{\times n}$ and ${F_2}^{\times n}$. Let W_s be the corresponding set of all tuples $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_\ell)$ for which $s = \check{\boldsymbol{s}}$ and ℓ is minimal. By the amalgamated free product assumption, the index $\ell = \ell_s$ is independent of the word $\boldsymbol{s} \in W_s$ representing s. Furthermore, for any two such words $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_\ell), \boldsymbol{s}' = (\boldsymbol{s}'_1, \ldots, \boldsymbol{s}'_\ell) \in W_s$, there exist $k_1^{\boldsymbol{s},\boldsymbol{s}'}, \ldots, k_{\ell_s+1}^{\boldsymbol{s},\boldsymbol{s}'} \in [\![R_3]\!]$ such that $\check{\boldsymbol{s}}_i = k_i^{\boldsymbol{s},\boldsymbol{s}'} \check{\boldsymbol{s}}'_i (k_{i+1}^{\boldsymbol{s},\boldsymbol{s}'})^{-1}$ for all $i = 1 \ldots \ell_s$, where we have $k_1^{\boldsymbol{s},\boldsymbol{s}'} = k_{\ell_s+1}^{\boldsymbol{s},\boldsymbol{s}'} = 1$ if $\ell_s > 1$. Let p_s, q_s be the projections defined by $q_s s = \mathrm{Res}_{[\![R_3]\!]}(s)$ and $p_s s = s - q_s s$. For any $\boldsymbol{s} \in W_s$, there exist projections $p_{s,\boldsymbol{s}_i}$, $q_{s,\boldsymbol{s}_i}$ such that $q_{s,\boldsymbol{s}_i} \check{s}_i \leq \mathrm{Res}_{[\![R_3]\!]}(\check{\boldsymbol{s}}_i)$ and $p_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i \leq \boldsymbol{s}_i - q_{s,\boldsymbol{s}_i} \boldsymbol{s}_i$, and

$$p_s s = \prod_{i=1}^{n} p_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i, \text{ and } q_s s = \prod_{i=1}^{n} q_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i.$$

These projections are defined by recurrence as follows: choose first $p_{s,\boldsymbol{s}_1} = p_s$, $q_{s,\boldsymbol{s}_1} = q_s$, and for each i let $p_{s,\boldsymbol{s}_{i+1}}, q_{s,\boldsymbol{s}_{i+1}}$ be the smallest projections such that $p_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i = p_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i p_{s,\boldsymbol{s}_{i+1}}$ and $q_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i = q_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i q_{s,\boldsymbol{s}_{i+1}}$, respectively. To simplify notation, we assume below that we actually have the equality $q_{s,\boldsymbol{s}_i} \check{s}_i = \mathrm{Res}_{[\![R_3]\!]}(\check{\boldsymbol{s}}_i)$ and $p_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i = \boldsymbol{s}_i - q_{s,\boldsymbol{s}_i} \boldsymbol{s}_i$. The general case can be easily handled by subdividing $p_s = \sum_{\ell=1}^{\ell_s} p_s^\ell$ and $p_{s,\boldsymbol{s}_i} =$

$\sum_{\ell=1}^{\ell_s} p_{s,\boldsymbol{s}_i}^\ell$ according to suitable restrictions $p_s^\ell s$ of length ℓ, where ℓ varies among all possible lengths $\ell = 1, \ldots, \ell_s$. Let

$$K_n = \bigcup_{s \in F_\pm^n} \Bigg(\{p_s, q_s, q_s s\} \cup \bigcup_{\boldsymbol{s} \in W_s} \{p_{s,\boldsymbol{s}_i}, q_{s,\boldsymbol{s}_i}, q_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i\} \\ \cup \bigcup_{\boldsymbol{s},\boldsymbol{s}' \in W_s} \{(k_i^{\boldsymbol{s},\boldsymbol{s}'})^{\pm 1} \mid i = 1 \ldots \ell_s + 1\} \Bigg).$$

Clearly, K_n is a finite subset of $[\![R_3]\!]$ for all n. We set $K := K_{3n}$.

Let $\delta > 0$ and let D be a fundamental domain of the finite component of R_3. Let $\varepsilon > 0$ be smaller than $\delta/400n$. A direct application of the Connes–Feldman-Weiss theorem [CFW81] shows that there exists a finite principal inverse semigroup $G \subset [\![R_3]\!]$, as in Section 5, with a fundamental domain D' such that $\mu(D \setminus D') < \varepsilon$ and, for any $k \in K$, there exists a $g \in \boldsymbol{\Sigma} G$ for which $g \leq k$ and $|k - g| < \varepsilon$. Set $F_i' = F_i \cup G$.

Let C be the constant (depending only on G, F_i, and n) given by Lemma 6.3, and m_G, n_G and d_G be the integers (depending only on G, F_i, and n) as defined in Lemma 5.1. Let $\kappa > 0$. Since both F_1 and F_2 are dynamical generating sets, we can find an integer n_0 such that for all $g \in G$ and $i = 1, 2$, there exists an $s \in \boldsymbol{\Sigma} F_i^{n_0}$ such that $|g - s| < \kappa/8$.

Let $\delta' > 0$ be smaller than $\delta/80C$, $\delta/150000n$ and $\kappa/6000 m_G |F|^{2n}$, let δ'' be smaller than $\delta'/500(2|F|+1)^{3n}$, and let n' be greater than both n_0 and $36(n_G + n)$. Choose microstates $\varphi_1 \in \mathrm{SA}_G(F_1', n', \delta'', d_G d)$ and $\varphi_2 \in \mathrm{SA}_G(F_2', n', \delta'', d_G d)$. By Lemma 5.1 we have, for all $d > m_G n \delta''^{-1}$,

$$\begin{aligned} N_{\kappa/2}(\mathrm{SA}_G(L, n', \delta'', d_G d)) &\geq N_{m_G \delta' + \kappa/2}(\mathrm{SA}(L, n', \delta''/m_G n, d_G d)) \\ &\geq N_\kappa(\mathrm{SA}(L, n', \delta''/m_G n, d_G d)) \end{aligned}$$

in either one of the two cases $L = F_1' \subset [\![R_1]\!]$ or $L = F_2' \subset [\![R_2]\!]$. We denote by $\bar{\varphi}_i$ any linear extension of φ_i. Since the φ_i are trace-preserving isomorphisms in restriction to G, the maps $\bar{\varphi}_i$ are trace-preserving isomorphism in restriction to $\boldsymbol{\Sigma} G$. Thus, by Lemma 3.6, $\bar{\varphi}_i \in \mathrm{SA}_{\boldsymbol{\Sigma} G}(\boldsymbol{\Sigma} F_i', n'/4, \delta', d_G d)$. Furthermore, since the restrictions of $\bar{\varphi}_1$ and $\bar{\varphi}_2$ to G are trace-preserving embeddings, we have $|\varphi_1(D')| = |\varphi_2(D')|$, and any bijection $\rho : \varphi_1(D') \to \varphi_2(D')$ extends by G-equivariance to a permutation (still denoted) $\rho \in [d_G d]$ implementing an isomorphism between the inverse semigroups $\varphi_1(G)$ and $\varphi_2(G)$. We let $\bar{\varphi}_2'(s) = \rho^{-1} \bar{\varphi}_2(s) \rho$ for $s \in [\![R_2]\!]$, so that $\bar{\varphi}_1$ and $\bar{\varphi}_2'$ coincide on $\boldsymbol{\Sigma} G$.

Let $[d_G d]_G$ be the subgroup of bijections in $[d_G d]$ which commute with $\varphi_1(G) = \varphi_2'(G)$. Given $\theta \in [d_G d]_G$, we construct a map φ_θ on F^n as follows. For each $s \in F^n$, we fix a word $\boldsymbol{s}^s = (\boldsymbol{s}^s_1, \ldots, \boldsymbol{s}^s_{\ell_s}) \in W_s$, and set

$$\varphi_\theta(s) := \prod_{i=1}^{\ell_s} \tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}^s_i)$$

where $\tilde{\varphi}_1 := \bar{\varphi}_1$, $\tilde{\varphi}_2 := \theta\bar{\varphi}'_2\theta^{-1}$, and $j^s_i = 1$ if $\boldsymbol{s}^s_i \in F_1^{\times n}$ and $j^s_i = 2$ if $\boldsymbol{s}^s_i \in F_2^{\times n}$. This defines φ_θ for all $s \in F^n$, except when $s \in [\![R_3]\!]$. In that case we set $\varphi_\theta(s) := \varphi_1(s)$.

Let us show that $\varphi_\theta \in \mathrm{SA}(F, n, \delta, d_G d)$ for sufficiently many θ. We divide the proof into two claims.

Claim 1. *φ_θ is $(F, 3n, \delta/16)$-multiplicative.*

Proof of Claim 1. Let $\boldsymbol{s} = (s_1, \ldots s_{3n}) \in F^{\times 3n}$ and rewrite the word $s = \check{\boldsymbol{s}}$ as $s = \check{\boldsymbol{s}}_1 \cdots \check{\boldsymbol{s}}_{k_s}$ where $(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{k_s}) \in W_s$. Thus $\boldsymbol{s}_i = (s_{l_i}, \ldots, s_{l_i+1})$ for some indices $l_1 = 1 < l_2 < \ldots < l_{\ell_s+1} = 3n$. Assume first that $\ell_s = 1$. In that case, s and all s_i belong to K. Choose $g, g_i \in \boldsymbol{\Sigma} G$ such that $g \leq s$, $g_i \leq s_i$ and $|s - g| < \varepsilon$, $|s_i - g_i| < \varepsilon$. Then $|g - \prod_{i=1}^{3n} g_i| < (3n+1)\varepsilon$ and thus

$$\begin{aligned} |\varphi_\theta(s) - \varphi_\theta(s_1)\cdots\varphi_\theta(s_{3n})| &< \left|\varphi_1\left(\prod_{i=1}^{3n} g_i\right) - \prod_{i=1}^{3n}\varphi_\theta(s_i)\right| + (3n+1)\varepsilon + 400\delta' \\ &\leq \sum_{i=1}^{3n} |\varphi_1(g_i) - \varphi_\theta(s_i)| + (3n+1)\varepsilon + 400\delta' \\ &= \sum_{i=1}^{3n} |\tilde{\varphi}_{j^s_i}(g_i) - \varphi_{j^s_i}(s_i)| + (3n+1)\varepsilon + 400\delta' \\ &< 6(\varepsilon + 800\delta')n < \delta/16. \end{aligned}$$

We now assume that $\ell_s > 1$. Let $k_i \in K$ be such that $\check{\boldsymbol{s}}_i = k_i \check{\boldsymbol{s}}^s_i k_{i+1}^{-1}$, and choose $g_i \in \boldsymbol{\Sigma} G$ such that $g_i \leq k_i$ and $|g_i - k_i| < \varepsilon$, where $g_1 = g_{\ell_s+1} = 1$. In particular, $|\check{\boldsymbol{s}}_i - g_i\check{\boldsymbol{s}}^s_i g_{i+1}^{-1}| < 2\varepsilon$ and thus, since $n'/4 > 9(n_G + n)$ we get using Lemma 3.2 that

$$\begin{aligned} |\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}_i) - \tilde{\varphi}_{j^s_i}(g_i)\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}^s_i)\tilde{\varphi}_{j^s_i}(g_{i+1}^{-1})| &< |\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}_i) - \tilde{\varphi}_{j^s_i}(g_i\check{\boldsymbol{s}}^s_i g_{i+1}^{-1})| + 4\delta' \\ &< |\check{\boldsymbol{s}}_i - g_i\check{\boldsymbol{s}}^s_i g_{i+1}^{-1}| + 404\delta' < 2\varepsilon + 404\delta'. \end{aligned}$$

Since the restriction of $\tilde{\varphi}_{j^s_i}$ to $\boldsymbol{\Sigma} G$ is an isomorphism, we obtain

$$\begin{aligned} \left|\prod_{i=1}^{\ell_s}\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}_i) - \varphi_\theta(s)\right| &= \left|\prod_{i=1}^{\ell_s}\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}_i) - \prod_{i=1}^{\ell_s}\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}^s_i)\right| \\ &= \left|\prod_{i=1}^{\ell_s}\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}_i) - \prod_{i=1}^{\ell_s}\tilde{\varphi}_{j^s_i}(g_i)\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}^s_i)\tilde{\varphi}_{j^s_i}(g_{i+1}^{-1})\right| \\ &\leq \sum_{i=1}^{\ell_s} |\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}_i) - \tilde{\varphi}_{j^s_i}(g_i)\tilde{\varphi}_{j^s_i}(\check{\boldsymbol{s}}^s_i)\tilde{\varphi}_{j^s_i}(g_{i+1}^{-1})| \end{aligned}$$

$$< 3(2\varepsilon + 404\delta')n.$$

For each index i, we have $|\prod_{l=l_i}^{l_{i+1}} \tilde{\varphi}_{j_i^s}(s_l) - \tilde{\varphi}_{j_i^s}(\check{\boldsymbol{s}}_i)| < \delta'$. Furthermore, $\varphi_\theta(t) = \tilde{\varphi}_{j_i^s}(t)$ for every $t \in F_{j_i}$, so that

$$|\varphi_\theta(s) - \varphi_\theta(s_1)\cdots\varphi_\theta(s_n)| < 3(4\varepsilon + 1208\delta')n < \delta/32 + \delta/32 = \delta/16,$$

as claimed. □

Claim 2. *For any large enough d, the set of $\theta \in [d_G d]_G$ such that φ_θ is (F, n, δ)-trace-preserving has cardinality at least $\frac{1}{2}|[d_G d]_G|$.*

Proof of Claim 2. Fix $s \in F^n$. Let $\tilde{p}_s, \tilde{q}_s, \tilde{p}_{s,\boldsymbol{s}_i}, \tilde{q}_{s,\boldsymbol{s}_i} \in \boldsymbol{\Sigma} G$ be such that $\tilde{p}_s \leq p_s$, $\tilde{q}_s \leq q_s$, $\tilde{p}_{s,\boldsymbol{s}_i} \leq p_{s,\boldsymbol{s}_i}$, $\tilde{q}_{s,\boldsymbol{s}_i} \leq q_{s,\boldsymbol{s}_i}$, and $|\tilde{p}_s - p_s| < \varepsilon$, $|\tilde{q}_s - q_s| < \varepsilon$, $|\tilde{p}_{s,\boldsymbol{s}_i} - p_{s,\boldsymbol{s}_i}| < \varepsilon$, $|\tilde{q}_{s,\boldsymbol{s}_i} - q_{s,\boldsymbol{s}_i}| < \varepsilon$,

$$\tilde{p}_s s = \prod_{i=1}^{n} \tilde{p}_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i, \text{ and } \tilde{q}_s s = \prod_{i=1}^{n} \tilde{q}_{s,\boldsymbol{s}_i} \check{\boldsymbol{s}}_i.$$

We may assume that $\tilde{p}_s = \tilde{p}_{s,\boldsymbol{s}_1}$ and, as $q_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i \in K$, that $\tilde{q}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i \in \boldsymbol{\Sigma} G$. Since

$$|\tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i - \tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i \tilde{p}_{s,\boldsymbol{s}_{i+1}}| \leq 3\varepsilon,$$

we have

$$\begin{aligned}
&\left|\tilde{\varphi}_1(\tilde{p}_s)\varphi_\theta(s) - \prod_{i=1}^{\ell_s} \tilde{\varphi}_{j_i^s}(\tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right| \\
&\quad = \left|\tilde{\varphi}_{j_1^s}(\tilde{p}_s)\tilde{\varphi}_{j_1^s}(\check{\boldsymbol{s}}_1^s)\cdots\tilde{\varphi}_{j_{\ell_s}^s}(\check{\boldsymbol{s}}_{\ell_s}^s) - \prod_{i=1}^{\ell_s} \tilde{\varphi}_{j_i^s}(\tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right| \\
&\quad \leq \left|\tilde{\varphi}_{j_1^s}(\tilde{p}_s\check{\boldsymbol{s}}_1^s)\cdots\tilde{\varphi}_{j_{\ell_s}^s}(\check{\boldsymbol{s}}_{\ell_s}^s) - \prod_{i=1}^{\ell_s} \tilde{\varphi}_{j_i^s}(\tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right| + 3\delta' \\
&\quad \leq \left|\tilde{\varphi}_{j_1^s}(\tilde{p}_{s,\boldsymbol{s}_1}\check{\boldsymbol{s}}_1^s\tilde{p}_{s,\boldsymbol{s}_2})\cdots\tilde{\varphi}_{j_{\ell_s}^s}(\check{\boldsymbol{s}}_{\ell_s}^s) - \prod_{i=1}^{\ell_s} \tilde{\varphi}_{j_i^s}(\tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right| + 3\varepsilon + 403\delta' \\
&\quad \leq \left|\tilde{\varphi}_{j_1^s}(\tilde{p}_{s,\boldsymbol{s}_1}\check{\boldsymbol{s}}_1^s)\tilde{\varphi}_{j_2^s}(\tilde{p}_{s,\boldsymbol{s}_2})\tilde{\varphi}_{j_2^s}(\check{\boldsymbol{s}}_2^s)\cdots\tilde{\varphi}_{j_{\ell_s}^s}(\check{\boldsymbol{s}}_{\ell_s}^s) - \prod_{i=1}^{\ell_s} \tilde{\varphi}_{j_i^s}(\tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right| + 3\varepsilon + 406\delta' \\
&\quad \leq \left|\tilde{\varphi}_{j_2^s}(\tilde{p}_{s,\boldsymbol{s}_2})\tilde{\varphi}_{j_2^s}(\check{\boldsymbol{s}}_2^s)\cdots\tilde{\varphi}_{j_{\ell_s}^s}(\check{\boldsymbol{s}}_{\ell_s}^s) - \prod_{i=2}^{\ell_s} \tilde{\varphi}_{j_i^s}(\tilde{p}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right| + 3\varepsilon + 406\delta' \\
&\quad \leq (3\varepsilon + 406\delta')n.
\end{aligned}$$

A similar computation shows that

$$\left|\tilde{\varphi}_1(\tilde{q}_s)\varphi_\theta(s) - \prod_{i=1}^{\ell_s} \tilde{\varphi}_1(\tilde{q}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right| < (3\varepsilon + 406\delta')n.$$

Applying Lemma 6.3 to $[d_G d]_G$, we see that for any large enough d, the set of permutations $\theta \in [d_G d]_G$ satisfying

$$|\operatorname{tr}(\tilde{\varphi}_1(\tilde{p}_s)\varphi_\theta(s))| < \delta/2$$

has cardinality at least $\frac{1}{2}|[d_G d]_G|$. Indeed, for any d large enough and for at least half of the $\theta \in [d_G d]_G$ we have

$$\begin{aligned}|\mathrm{Res}_{\varphi_1(G)}(\tilde{\varphi}_1(\tilde{p}_s)\varphi_\theta(s))| &< \left|\mathrm{Res}_{\varphi_1(G)}\left(\prod_{i=1}^{\ell_s}\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right)\right| + (3\varepsilon + 406\delta')n \\ &< C\max_i(|\mathrm{Res}_{\varphi_1(G)}\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)|) + \delta/16 + (3\varepsilon + 406\delta')n\end{aligned}$$

Let $g \in G$ and let $f \leq \varphi_1(g)\varphi_1(g)^{-1}$, $f \leq \tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)^{-1}$ be such that

$$f\varphi_1(g) = \mathrm{Res}_{\varphi_1(G)}\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s) = f\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s).$$

Since $|\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\tilde{\varphi}_1(g)^{-1} - \tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s g^{-1})| < 4\delta'$, we see that $|f - f\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s g^{-1})| < 4\delta'$. On the other hand, since $\tau(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s g^{-1}) = 0$ by our definition of $\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}$, we have $|\operatorname{tr}(\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s g^{-1}))| < \delta'$. Therefore $|\operatorname{tr}(f)| < 5\delta'$, and we deduce that

$$|\mathrm{Res}_{\varphi_1(G)}\tilde{\varphi}_{j_i^s}(\tilde{p}_{\boldsymbol{s},\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)| < 5\delta'.$$

Thus

$$|\mathrm{Res}_{\varphi_1(G)}(\varphi_{j_1^s}(\tilde{p}_s)\varphi_\theta(s))| < 5C\delta' + \delta/16 + (3\varepsilon + 406\delta')n < \delta/4$$

as claimed. In particular, $|\operatorname{tr}(\tilde{\varphi}_1(\tilde{p}_s)\varphi_\theta(s))| < \delta/2$.

Since $|\tilde{p}_s + \tilde{q}_s - ss^{-1}| < 2\varepsilon$, where $ss^{-1} \in F^n \cap [\![R_3]\!] \subset K$, we have

$$|\tilde{\varphi}_1(ss^{-1}) - (\tilde{\varphi}_1(\tilde{p}_s) + \tilde{\varphi}_1(\tilde{q}_s))| < 2\varepsilon + 400\delta'.$$

On the other hand, since $ss^{-1}s \in F^{3n}$, we have by Claim 1

$$|\varphi_\theta(s) - \tilde{\varphi}_1(ss^{-1})\varphi_\theta(s)| < \delta/4$$

and

$$|\varphi_\theta(s) - (\tilde{\varphi}_1(\tilde{p}_s) + \tilde{\varphi}_1(\tilde{q}_s))\varphi_\theta(s)| < \delta/4 + 2\varepsilon + 400\delta'.$$

Thus

$$\begin{aligned}|\operatorname{tr}\circ\varphi_\theta(s) - \tau(s)| &\leq |\operatorname{tr}(\tilde{\varphi}_1(\tilde{q}_s)\varphi_\theta(s)) - \tau(q_s s)| + |\operatorname{tr}(\tilde{\varphi}_1(\tilde{p}_s)\varphi_\theta(s)) - \tau(p_s s)| \\ &\qquad + \delta/4 + 2\varepsilon + 400\delta' \\ &\leq |\operatorname{tr}(\tilde{\varphi}_1(\tilde{q}_s)\varphi_\theta(s)) - \tau(q_s s)| + 3\delta/4 + 2\varepsilon + 400\delta' \\ &\leq \left|\operatorname{tr}\left(\prod_{i=1}^{\ell_s}\tilde{\varphi}_1(\tilde{q}_{s,\boldsymbol{s}_i}\check{\boldsymbol{s}}_i^s)\right) - \tau(q_s s)\right| + 5\varepsilon n + 806\delta' n + 3\delta/4 \\ &\leq |\operatorname{tr}(\tilde{\varphi}_1(\tilde{q}_s s)) - \tau(q_s s)| + 5\varepsilon n + 807\delta' n + 3\delta/4 \\ &\leq |\tau(\tilde{q}_s s) - \tau(q_s s)| + 5\varepsilon n + 808\delta' n + 3\delta/4 \\ &\leq 6\varepsilon + \delta/8 + 3\delta/4 < \delta.\end{aligned}$$

This proves Claim 2. □

Summarizing, for every $\varphi_1 \in \mathrm{SA}_G(F_1', n', \delta'', d_G d)$, $\varphi_2 \in \mathrm{SA}_G(F_2', n', \delta'', d_G d)$ and at least half of the permutations $\theta \in [d_G d]_G$, we get a map $\varphi_\theta \in \mathrm{SA}_G(F, n, \delta, d_G d)$. Clearly, if the maps φ_θ and $\psi_{\theta'}$ associated to two triples $(\varphi_1, \varphi_2, \theta)$ and $(\psi_1, \psi_2, \theta')$ coincide on F, then φ_1, ψ_1 coincide on F_1 and $\theta\varphi_2'\theta^{-1}$ and $\theta'\psi_2'\theta'^{-1}$ coincide on F_2. In particular, we can find a permutation $\gamma \in [d_G d]$ such that φ_2 and $\gamma\psi_2\gamma^{-1}$ coincide on F_2. Furthermore, for every $g \in G$, we can find an element $v \in \mathbf{\Sigma} {F_1}^{n_0}$ such that $|g - v| < \kappa/8$. Suppose that $v \in {F_1}^{n_0}$. Since $\varphi_1 = \psi_1$ on F_1, we have, writing $v = \prod_{i=1}^{n_0} v_i$ as a product of elements of F_1, that

$$|\varphi_1(v) - \psi_1(v)| \le \left|\varphi_1(v) - \prod_{i=1}^{n_0} \varphi_1(v_i)\right| + \left|\prod_{i=1}^{n_0} \psi_1(v_i) - \psi_1(v)\right| < 2\delta'.$$

If $v = \sum_{i=1}^k v_i \in \mathbf{\Sigma} F_1^{n_0}$, we have, by Lemma 3.4,

$$|\varphi_1(v_i)\pi_i(\varphi_1(v_1), \dots, \varphi_1(v_k)) - \varphi_1(v_i)| < 220(k-1)\delta'$$

and similarly

$$|\psi_1(v_i)\pi_i(\psi_1(v_1), \dots, \psi_1(v_k)) - \psi_1(v_i)| < 220(k-1)\delta'.$$

whence

$$\begin{aligned}
|\bar\varphi_1(v) - \bar\psi_1(v)| &= \left|\sum_{i=1}^k \varphi_1(v_i)\pi_i(\varphi_1(v_1), \dots, \varphi_1(v_k)) - \sum_{i=1}^k \psi_1(v_i)\pi_i(\psi_1(v_1), \dots, \psi_1(v_k))\right| \\
&\le \sum_{i=1}^k |\varphi_1(v_i)\pi_i(\varphi_1(v_1), \dots, \varphi_1(v_k)) - \psi_1(v_i)\pi_i(\psi_1(v_1), \dots, \psi_1(v_k))| \\
&\le (220(k-1)\delta' + 2\delta' + 220(k-1)\delta')k \\
&\le 500|F_1|^{2n}\delta',
\end{aligned}$$

and

$$\begin{aligned}
|\varphi_1(g) - \psi_1(g)| &\le |\bar\varphi_1(g) - \bar\varphi_1(v)| + |\bar\varphi_1(v) - \bar\psi_1(v)| + |\bar\psi_1(v) - \bar\psi_1(g)| \\
&< 2|g - v| + 800\delta' + 500|F_1|^{2n}\delta' \\
&< \kappa/4 + 802\delta' < \kappa/2.
\end{aligned}$$

Therefore, $|\varphi_1 - \psi_1|_{F_1'} < \kappa/2$, and a similar computation for φ_2 and $\gamma\psi_2\gamma^{-1}$ shows that $|\varphi_2 - \gamma\psi_2\gamma^{-1}|_{F_2'} < \kappa/2$. We deduce that

$$\begin{aligned}
&\mathrm{NSA}(F, n, \delta, d_G d) \\
&\qquad \ge N_{\kappa/2}(\mathrm{SA}_G(F_1', n', \delta'', d_G d)) N_{\kappa/2}(\mathrm{SA}_G(F_2', n', \delta'', d_G d)) \cdot \frac{|[d_G d]_G|}{2(d_G d)!}
\end{aligned}$$

$$\geq N_\kappa(\mathrm{SA}(F'_1, n', \delta''/m_G n, d_G d))N_\kappa(\mathrm{SA}(F'_2, n', \delta''/m_G n, d_G d)) \cdot \frac{|[d_G d]_G|}{2(d_G d)!}$$

As in Corollary 5.2, we have

$$\log |[d_G d]_G| \sim \mu(D')d_G d \log d_G d$$

as $d \to \infty$, using the Stirling formula. Since $\mu(D') \geq \mu(D) - \varepsilon$ we obtain, by taking the limit infimum over d,

$$\begin{aligned}\underline{s}(F, n, \delta) &\geq \underline{s}_\kappa(F'_1, n', \delta''/m_G n) + \underline{s}_\kappa(F'_2, n', \delta''/m_G n) + \mu(D') - 1 \\ &\geq \underline{s}_\kappa(F'_1) + \underline{s}_\kappa(F'_2) - 1 + \mu(D) - \varepsilon.\end{aligned}$$

Note that the same inequality holds for s_ω as well, by taking a limit along the ultrafilter ω instead of a limit infimum. Thus, taking the limit over κ (using Lemma 3.9) and ε we get:

$$\underline{s}(F, n, \delta) \geq \underline{s}(F'_1) + \underline{s}(F'_2) - 1 + \mu(D),$$

Since F, F'_1 and F'_2 are dynamical generating sets of R, R_1 and R_2 respectively (using Lemma 2.7), the lemma is now a direct consequence of Theorem 4.1. □

Lemma 7.3. *Assume that $R = R_1 *_{R_3} R_2$ is a free product of equivalence relations amalgamated over an amenable subrelation R_3. Assume that R_1 and R_2 are dynamically finitely generated, and let D be a fundamental domain of the finite component of R_3. Then we have:*

$$s(R) \leq s(R_1) + s(R_2) - 1 + \mu(D).$$

Furthermore, the same inequality holds for s_ω instead of s.

Proof. Let $F_1 \subset [\![R_1]\!]$ and $F_2 \subset [\![R_2]\!]$ be finite dynamical generating sets and let $\varepsilon > 0$. As in Lemma 7.2, the Connes–Feldman-Weiss theorem gives us a finite inverse semigroup $G \subset [\![R_3]\!]$ of support X with a fundamental domain D' such that $\mu(D') \leq \mu(D) + \varepsilon$. Set $F'_i = F_i \cup G$. By Lemma 5.1, there exists an integer d_G depending only on G such that

$$s(L) = \inf_{n \in \mathbb{N}} \inf_{\delta > 0} \limsup_{d \to \infty} \frac{1}{d_G d \log d_G d} \log \mathrm{NSA}_G(L, n, \delta, d_G d),$$

in either one of the cases $L = F'_1 \subset [\![R_1]\!]$, $L = F'_2 \subset [\![R_2]\!]$ or $L = F'_1 \cup F'_2 \subset [\![R]\!]$.

Clearly, every $\varphi \in \mathrm{SA}_G(F'_1 \cup F'_2, n, \delta, d_G d)$ gives by restriction to $[\![R_1]\!] \subset [\![R]\!]$ and $[\![R_2]\!] \subset [\![R]\!]$ two maps $\varphi_1 \in \mathrm{SA}_G(F'_1, n, \delta, d_G d)$ and $\varphi_2 \in \mathrm{SA}_G(F'_2, n, \delta, d_G d)$ respectively. Consider the map Θ defined by

$$\begin{aligned}\Theta : \mathrm{SA}_G(F'_1 \cup F'_2, n, \delta, d) \times [d_G d] &\to \mathrm{SA}_G(F'_2, n, \delta, d) \times \mathrm{SA}_G(F'_2, n, \delta, d) \\ (\varphi, \theta) &\mapsto (\varphi_1, \theta\varphi_2\theta^{-1})\end{aligned}$$

If (φ, θ) and (φ', θ') have the same image under Θ, then $\varphi_1 = \varphi'_1$ on F'_1 and $\theta\varphi_2\theta^{-1} = \theta'\varphi'_2\theta'^{-1}$ on F'_2. Let $\rho = \theta'^{-1}\theta$. The first condition implies that $\varphi_{|G} = \varphi'_{|G}$ and the second that $\rho\varphi_{|G}\rho^{-1} = \varphi'_{|G} = \varphi_{|G}$. In particular, ρ belongs to the commutant $[d_G d]_G$ of $\varphi(G)$ in $[d_G d]$. It follows that

$$\mathrm{NSA}_G(F_1' \cup F_2', n, \delta, d_G d) \leq \mathrm{NSA}_G(F_1', n, \delta, d_G d)\mathrm{NSA}_G(F_2', n, \delta, d_G d) \cdot \frac{|[d_G d]_G|}{(d_G d)!}$$

so that, as in Lemma 7.2, we obtain

$$s(F_1' \cup F_2') + 1 - \mu(D') \leq s(F_1') + s(F_2').$$

Thus $s(F_1' \cup F_2') \leq s(F_1') + s(F_2') - 1 + \mu(D) + \varepsilon$, and the lemma follows from Theorem 4.1. ☐

Remark 7.4. As the above proof clearly shows, the following stronger result holds: assume that $R = R_1 *_{R_3} R_2$ is a free product of finitely generated equivalence relation amalgamated over a common subrelation R_3, and let $R_3' \subset R_3$ be an amenable subrelation. Then we have:

$$s(R) \leq s(R_1) + s(R_2) - 1 + \mu(D)$$

where D is a fundamental domain of the finite component of R_3'. In particular if R_3 is diffuse, then $s(R) \leq s(R_1) + s(R_2) - 1$.

Finally, we note that if the relation R_3 is trivial, in which case $R = R_1 * R_2$ is a free product, then the finite inverse semigroup G in both the proof of Lemma 7.2 and Lemma 7.3 is trivial, and thus we can remove the assumption that F_1 and F_2 are dynamical generating sets. Therefore, in this case, we have the following result.

Theorem 7.5. *Assume that $R = R_1 * R_2$ is a free product of R_1 and R_2. Let F_1 and F_2 be finite sets of $[\![R_1]\!]$ and $[\![R_2]\!]$ respectively. Then $\underline{s}(F_1 \cup F_2) \geq \underline{s}(F_1) + \underline{s}(F_2)$, $s(F_1 \cup F_2) \leq s(F_1) + s(F_2)$ and $s_\omega(F_1 \cup F_2) = s_\omega(F_1) + s_\omega(F_2)$ for any nonprincipal ultrafilter ω. In particular, if F_1 and F_2 are regular, then $F_1 \cup F_2$ is regular and*

$$s(F_1 \cup F_2) = s(F_1) + s(F_2).$$

Corollary 7.6. *Let R be an ergodic p.m.p. equivalence relation. If R is treeable, it is s-regular and $s(R) = \mathrm{cost}(R)$ (in particular, R is sofic).*

Proof. By Hjorth's theorem [Hjo06], we can find a generating set of R of the form $F = \{s_1, s_2, \ldots s_n\}$ where $s_1, \ldots s_{n-1} \in [R]$ and $s_n \in [\![R_1]\!]$, such that s_1 is ergodic and, if R_i denotes the relation generated by s_i, then $R = R_1 * \cdots * R_n$ (see the proof of [KM04, Theorem 28.3]). Since s_1 is ergodic, we may replace it in F by a dynamical generating set consisting of two partial isomorphisms of cost $\frac{1}{2}$. The corresponding finite set F' is then a finite generating set for R and thus by recurrence,

$$s(R) = s(F') = n - 1 + s(\{s_n\}) = n - 1 + \mu(\mathrm{dom}\, s_n) = \mathrm{cost}(R),$$

where the last equality is the subject of Theorem 1 in [Gab00]. ☐

References

[AS09] M. Abert and B. Szegedy, *Residually finite groups, graph limits and dynamics*, Banff reports, available at http://www.birs.ca/workshops/2009/09frg147/report09frg147.pdf (2009).

[Bow10] L. Bowen, *Measure conjugacy invariants for actions of countable sofic groups*, J. Amer. Math. Soc. **23** (2010), no. 1, 217–245.

[BDJ08] N.P. Brown, K. Dykema, and K. Jung, *Free entropy dimension in amalgamated free products*, Proc. London Math. Soc. **97** (2008), no. 2, 339–367.

[CD10] B. Collins and K. Dykema, *Free products of sofic groups with amalgamation over monotileably amenable groups*, Muenster J. Math., to appear (2010).

[CFW81] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergod. Th. Dyn. Sys. **1** (1981), no. 04, 431–450.

[Dye59] H.A. Dye, *On groups of measure preserving transformations. I*, Amer. J. Math. **81** (1959), no. 1, 119–159.

[EL10] G. Elek and G. Lippner, *Sofic equivalence relations*, J. Funct. Anal. **258** (2010), no. 5, 1692–1708.

[ES05] G. Elek and E. Szabo, *Hyperlinearity, essentially free actions and L^2-invariants. The sofic property*, Math. Ann. **332** (2005), no. 2, 421–441.

[ES06] ——, *On sofic groups*, J. Group Th. **9** (2006), no. 2, 161–171.

[ES10] ——, *Sofic representations of amenable groups*, preprint (2010), 1–8.

[FM77] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc (1977), 289–324.

[Gab00] D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), no. 1, 41–98.

[GP05] D. Gaboriau and S. Popa, *An uncountable family of nonorbit equivalent actions of F_n*, J. Amer. Math. Soc. **18** (2005), no. 3, 547–559.

[Gro99] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. **1** (1999), no. 2, 109–197.

[GM83] M. Gromov and V.D. Milman, *A topological application of the Isoperimetric Inequality*, Amer. J. Math. **105** (1983), no. 4, 843–854.

[Hjo06] G. Hjorth, *A lemma for cost attained*, Ann. Pure Appl. Logic **143** (2006), no. 1-3, 87–102.

[Jun03] K. Jung, *A free entropy dimension lemma*, Pacific J. Math., **177** (2003), 265–271.

[Jun07] K. Jung, *Strongly 1-bounded von Neumann algebras*, Geom. Funct. Anal. **17** (2007), no. 4, 1180–1200.

[KL10] D. Kerr and H. Li, *Entropy and the variational principle for actions of sofic groups*, preprint (2010).

[KM04] A.S. Kechris and B.D. Miller, *Topics in orbit equivalence*, Lect. Note Math. **1852** (2004).

[Lev95] G. Levitt: *On the cost of generating an equivalence relation*, Ergod. Th. Dyn. Sys. **15** (1995), 1173–1181.

[Mau79] B. Maurey, *Construction de suites symétriques*, Compt. Rend. Acad. Sci. Paris **288** (1979), 679–681.

[Nic93] A. Nica, *Asymptotically free families of random unitaries in symmetric groups*, Pacific J. Math **157** (1993), no. 2, 295–310.

[Oz09] Ozawa N., *Hyperlinearity, sofic groups and applications to group theory*, Notes from a 2009 talk available at http://people.math.jussieu.fr/~pisier/taka.talk.pdf

[Pau10] L. Păunescu, *On sofic actions and equivalence relations*, arXiv1002.0605v4 (2010).

[Shl03] D. Shlyakhtenko, *Microstates free entropy and cost of equivalence relations*, Duke Math. J. **118** (2003), no. 3, 375–426.

[Voi91] D. Voiculescu, *Limit laws for random matrices and free products*, Invent. Math. **104** (1991), no. 1, 201–220.

[Voi96] ———, *The analogues of entropy and of Fisher's information measure in free probability theory III: The absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), no. 1, 172–199.

[Voi98] ———, *A strengthened asymptotic freeness result for random matrices with applications to free entropy*, Int. Math. Res. Not. **1998** (1998), no. 1, 41–63.

[Wei00] B. Weiss, *Sofic groups and dynamical systems*, Sankhyà: The Indian Journal of Statistics, Series A **62** (2000), 350–359.