Classification of Fuchsian Systems and their Connection Problem

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- A classification of Fuchsian systems of ODE
- Connection formula — Fuchsian ODE without moduli

Transformation of Fuchsian systems

1. Isomonodromy deformation $\mathbb{C}$

   accessory parameters+geometric moduli, Painlevé VI

2. Adjacent relation $\mathbb{Z}$

   exponents $\leftrightarrow$ local monodromies

   shift operator, Schlesinger transf., special solutions, etc.

3. Symmetry – Group G

   Katz's middle convolution, Yokoyama's extension

   automorphisms of a Kac-Moody root system, Bäcklund transf.

4. ODE $\leftrightarrow$ PDE $M \ni z$

   Fuchsian systems of PDE: extension, restriction, integration

   $\longrightarrow$ confluence of singularities, Laplace transformation, ...

- A classification of Fuchsian systems of ODE
- Connection formula $\longrightarrow$ Harmonic analysis on Sym. sp.

Transformation of Fuchsian systems

1. Isomonodromy deformation $\mathbb{C}$

   compatibility condition for integrability, Painlevé VI

2. Adjacent relation $\mathbb{Z}$

   exponents $\leftrightarrow$ local monodromies

   homomorphisms of $\mathcal{D}$-modules

3. Symmetry – Group G

   tensor+integral, extension+restriction of $\mathcal{D}$-modules

   automorphisms of a Kac-Moody root system

4. ODE $\leftrightarrow$ PDE $M \ni z$

   tensor, integral, extension, restriction of $\mathcal{D}$-modules

   $\longrightarrow$ limits and other transformation of $\mathcal{D}$-modules

§1. Fuchsian systems

$$\mathcal{M}_{\mathbf{A}} : \boxed{\frac{du}{dz} = \sum_{j=1}^{k} \frac{A_j}{z - z_j} u}$$

(Schlesinger's normal form (SNF))

$\mathcal{M}_{\mathbf{A}}$: regular singularities at $z_0 := \infty, z_1, \dots, z_k$

$\mathbf{A} = (A_0, A_1, \dots, A_k) \in M(n, \mathbb{C})_0^{k+1}$ $\quad (A_0 + A_1 + \cdots + A_k = 0)$

A_j : residue matrix at z_j $\quad \Rightarrow$ local monodromy

$$u_j(z) = (I_n + C_{j,1}(z - z_j) + C_{j,2}(z - z_j)^2 + \cdots)(z - z_j)^{A_j}$$

$$u_0(\tfrac{1}{z}) = (I_n + \frac{C_{0,1}}{z} + \frac{C_{0,2}}{z^2} + \cdots)(\tfrac{1}{z})^{A_0} \qquad (C_{j,\nu} \in M(n, \mathbb{C}))$$

$\Leftarrow$ difference of eigenvalues of $A_j \notin \mathbb{Z} \setminus \{0\}$

$\mathbf{A} \sim \mathbf{B} \overset{\text{def}}{\Longleftrightarrow} \exists g \in GL(n, \mathbb{C})$ s.t. $gA_jg^{-1} = B_j \ (\forall j)$

$\mathbf{A}$: irreducible $\overset{\text{def}}{\Longleftrightarrow}$ $[V \subset \mathbb{C}^n,\ A_jV \subset V\ (\forall j) \Rightarrow V = \{0\}$ or $\mathbb{C}^n]$

$p : \boxed{M(n, \mathbb{C})_0^{k+1}/\!\sim} \ \rightarrow \ \boxed{M(n, \mathbb{C})/\!\sim \times \cdots \times M(n, \mathbb{C})/\!\sim}$?

§2. Deligne-Simpson problem and rigidity

- additive Deligne-Simpson problem $(\mathrm{Im}\, p\, ?)$:

Given $B_j \in M(n,\mathbb{C})$ $(j = 0, 1, \ldots, k)$ satisfying $\sum \mathrm{trace}\, B_j = 0$.

$\exists ?$ $A_j \sim B_j$ with irreducible $\mathbf{A} = (A_0, \ldots, A_k) \in M(n,\mathbb{C})_0^{k+1}$

- What is $p^{-1}(\mathbf{B})$?

For irreducible $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$

$$\dim p^{-1}(p(\mathbf{A})) = (k-1)n^2 - \sum_{j=0}^{k} \dim Z_{M(n,\mathbb{C})}(A_j) + 2$$

$$\mathrm{idx}\, \mathbf{A} := \sum_{j=0}^{k} \dim Z_{M(n,\mathbb{C})}(A_j) - (k-1)n^2$$: index of rigidity

$p^{-1}(\mathbf{B})$: accessory parameter $(\dim p^{-1}(\mathbf{B}) > 0)$

Theorem [Katz '95] $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$: irreducible $\Rightarrow$

$\mathbf{A}$ is rigid (i.e. $\#p^{-1}(p(\mathbf{A})) = 1$) $\Longleftrightarrow$ $\mathrm{idx}\, \mathbf{A} = 2$

§3. Katz's middle convolution

1. addition :

$$u(z) \mapsto \prod_j (z - z_j)^{\mu_j} \cdot u(z)$$

2. middle convolution :

$$u(z) \mapsto \int (t - z)^{\lambda - 1} u(t) dt \qquad \text{(Euler transform)}$$

$\rightsquigarrow$ an irreducible $\mathcal{D}$-submodule

§3. Katz's middle convolution (Dettwiler-Reiter)

1. $M_\mu\ : (A_1,\dots,A_k)\mapsto(A_1+\mu_1,\dots,A_k+\mu_k)\in M(n,\mathbb{C})^k$

2. $mc_\lambda : (A_1,\dots,A_k)\mapsto(G_1,\dots,G_k)\mapsto(\bar G_1,\dots,\bar G_k)$

$$G_j := j)\begin{pmatrix} & & \overset{j}{\smile} & & \\ A_1 & \cdots & A_j+\lambda & \cdots & A_k \\ & & & & \end{pmatrix}\in M(kn,\mathbb{C}),\quad \mathcal{K}:=\begin{pmatrix}\ker A_1\\ \vdots\\ \ker A_k\end{pmatrix}\subset\mathbb{C}^{kn}$$

$$\bar G_j := G_j|_{\mathbb{C}^{kn}/(\mathcal{K}+\mathcal{L}_\lambda)},\quad G_0:=-(G_1+\cdots+G_k),\quad \mathcal{L}_\lambda := \ker G_0$$

Theorem [Katz '95, Dettwiler-Reiter '00] $\mathbf{A}$: irreducible

i) $mc_\lambda(\mathbf{A})$: irreducible, $\operatorname{idx} mc_\lambda(\mathbf{A})=\operatorname{idx}\mathbf{A}$

ii) $\mathbf{A}\sim\mathbf{B}\ \Rightarrow\ mc_\lambda(\mathbf{A})\sim mc_\lambda(\mathbf{B})$

iii) $mc_\lambda\circ mc_{\lambda'}(\mathbf{A})\sim mc_{\lambda+\lambda'}(\mathbf{A})$ and $mc_0(\mathbf{A})\sim\mathbf{A}$

Similar for multiplicative version : $\hat A_0\hat A_1\cdots\hat A_k=I_n$

$$k=2\ \Rightarrow\quad \hat G_1=\begin{pmatrix}\hat\lambda\hat A_1 & \hat A_2-I_n\\ 0 & I_n\end{pmatrix},\ \hat G_2=\begin{pmatrix}I_n & 0\\ \hat\lambda(\hat A_1-I_n) & \hat\lambda\hat A_2\end{pmatrix}$$

$$\mathbf{A}' = mc_\lambda(\mathbf{A}+\mu)$$

$\mathcal{M}_{\mathbf{A}} \to \mathcal{M}_{\mathbf{A}'}$ $\rightsquigarrow$ Transformation of the solutions

- $\exists$ Integral expression (Euler type) of sol. for $\forall$rigid irreducible system

  [Haraoka '02, Haraoka-Yokoyama '06], [Dettweiler-Reiter '07]

- $\mathcal{M}_{\mathbf{A}'}$: Monodromy $\leftarrow$ multiplicative middle conv. of that of $\mathcal{M}_{\mathbf{A}}$

  [Katz '95], [Dettweiler-Reiter '07]

- Isomonodromy deformation equation of $\mathcal{M}_{\mathbf{A}'}$ $=$ that of $\mathcal{M}_{\mathbf{A}}$

  [Haraoka-Filipuk '06]

$\Rightarrow$ Painlevé VI if $\#\{$accessory parameters$\} = 2$

- Deligne-Simpson problem

  [Simpson '91], [Katz '95], [Kostov '01–'04], [Crawley-Boevey '03]

- $\mathcal{M}_{\mathbf{A}'}$: Okobo normal form (ONF) $\leftarrow \det(\lambda + \sum_{j=1}^{k}(A_j+\mu_j)) \neq 0$

$$(z-T)\frac{du}{dz} = Au \quad (T,\ A \in M(n',\mathbb{C}))$$

$\langle$Katz's addition+middle conv.$\rangle \simeq \langle$Yokoyama's extension+restriction$\rangle$

$$L(m;\lambda) := \quad A_{ij} \ {}_{\substack{1\le i\le N\\ 1\le j\le N}} \in M(n,\mathbb{C}), \quad m=(m_1,\ldots,m_N),\ n=\sum m_j,\ m_1\ge m_2\ge\cdots$$

$$A_{ij} = \begin{cases} \lambda_i I_{m_i} & (j=i) \\ I_{m_i,m_j} := \quad \delta_{\mu,\nu} \ {}_{\substack{1\le\mu\le m_i\\ 1\le\nu\le m_j}} & (j=i+1) \\ 0 & (j\ne i,\ i+1) \end{cases} \quad \in M(m_i,m_j\,;\mathbb{C})$$

$$L(m;\lambda) = \begin{pmatrix} \lambda_1 & & & 1 & & & & & & & & \\ & \ddots & & & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & & & & \\ & & & \lambda_2 & & & 1 & & & & & \\ & & & & \ddots & & & \ddots & & & & \\ & & & & & \lambda_2 & & & & & & \\ & & & & & & \lambda_3 & & & 1 & & \\ & & & & & & & \ddots & & & \ddots & \\ & & & & & & & & \lambda_3 & & & \\ & & & & & & & & & \ddots & & \\ & & & & & & & & & & & \lambda_m \end{pmatrix}$$

$$\dim Z_{M(n,\mathbb{C})}\ L(m;\lambda) \ = m_1^2+m_2^2+\cdots+m_N^2 \quad \text{(independent of } \lambda\text{)}$$

$\mathbf{A} = (A_0, A_1, \dots, A_k) \in M(n, \mathbb{C})_0^{k+1}$: irreducible (and $n > 1$)

$A_j \sim L(\mathbf{m}_j; \lambda_j) \quad (\mathbf{m}_j = (m_{j,1}, m_{j,2}, \dots),\ n = m_{j,1} + m_{j,2} + \cdots)$

$\operatorname{spt} \mathbf{A} := \mathbf{m} = (\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_k)$: spectral type of $\mathbf{A}$

We may assume $\quad$ ($m_{j,0}$ may be 0)

$$\begin{cases} \lambda_{0,1} = \lambda,\ \lambda_{i,0} = 0 & (i = 1, \dots, k), \\ \lambda_{j,\nu} = \lambda_{j,0} \ \Rightarrow \ m_{j,\nu} \le m_{j,0} & (\nu = 1, \dots, n_j,\ j = 0, \dots, k). \end{cases}$$

$$\{\lambda_{\mathbf{m}}\} := \begin{Bmatrix} z = \infty & z = z_1 & \cdots & z = z_k \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{k,1}]_{(m_{k,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{1,2}]_{(m_{1,2})} & \cdots & [\lambda_{k,2}]_{(m_{k,2})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{k,n_k}]_{(m_{k,n_k})} \end{Bmatrix}, \quad [\mu]_N := \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \in \mathbb{C}^N$$

$\mathbf{A} = (A_0, A_1, \ldots, A_k) \in M(n, \mathbb{C})_0^{k+1}$: irreducible (and $n > 1$)

$A_j \sim L(\mathbf{m}_j; \lambda_j) \quad (\mathbf{m}_j = (m_{j,1}, m_{j,2}, \ldots),\ n = m_{j,1} + m_{j,2} + \cdots)$

$\operatorname{spt} \mathbf{A} := \mathbf{m} = (\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_k)$: spectral type of $\mathbf{A}$

We may assume $\quad$ ($m_{j,0}$ may be 0)

$$\begin{cases} \lambda_{0,1} = \lambda,\ \lambda_{i,0} = 0 & (i = 1, \ldots, k), \\ \lambda_{j,\nu} = \lambda_{j,0} \ \Rightarrow\ m_{j,\nu} \le m_{j,0} & (\nu = 1, \ldots, n_j,\ j = 0, \ldots, k). \end{cases}$$

$$\{\lambda_{\mathbf{m}}\} := \begin{Bmatrix} z = \infty & z = z_1 & \cdots & z = z_k \\ [\lambda]_{(m_{0,1})} & [0]_{(m_{1,1})} & \cdots & [0]_{(m_{k,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{1,2}]_{(m_{1,2})} & \cdots & [\lambda_{k,2}]_{(m_{k,2})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{k,n_k}]_{(m_{k,n_k})} \end{Bmatrix}, \quad [\mu]_N := \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \in \mathbb{C}^N$$

$$\xrightarrow{mc_\lambda} \begin{Bmatrix} z = \infty & z = z_1 & \cdots & z = z_k \\ [-\lambda]_{(m_{0,1} - d(\mathbf{m}))} & [0]_{(m_{1,1} - d(\mathbf{m}))} & \cdots & [0]_{(m_{p,1} - d(\mathbf{m}))} \\ [\lambda_{0,2} - \lambda]_{(m_{0,2})} & [\lambda_{1,2} + \lambda]_{(m_{1,2})} & \cdots & [\lambda_{k,2} + \lambda]_{(m_{k,2})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0} - \lambda]_{(m_{0,n_0})} & [\lambda_{1,n_1} + \lambda]_{(m_{1,n_1})} & \cdots & [\lambda_{k,n_k} + \lambda]_{(m_{k,n_k})} \end{Bmatrix}$$

$$\operatorname{idx} \mathbf{m} = \sum m_{j,\nu}^2 - (k-1)n^2,\ d(\mathbf{m}) := \sum m_{j,1} - (k-1)n,\ \partial_{\mathbf{1}}(\mathbf{m}) := \left(m_{j,\nu} - d(\mathbf{m})\delta_{1,\nu} \right)_{j,\nu}$$

$\mathcal{P}_{k+1}^{(n)}$: totality of $(k+1)$-tuples $\mathbf{m} = (m_{j,\nu})_{\substack{j=0,1,\ldots \\ \nu=1,2,\ldots}}$ of partitions of n

$\operatorname{ord}\mathbf{m} := n = m_{j,1} + \cdots + m_{j,n_j} \quad (j = 0,\ldots,k)$

$m_{j,\nu} = 0$ if $\nu > n_j$ and $n_j = 1$, $m_{j,1} = n$ if $j > k$

$\mathcal{P}_{k+1} := \bigcup_{n=1}^{\infty} \mathcal{P}_{k+1}^{(n)}, \quad \mathcal{P} := \bigcup_{k=0}^{\infty} \mathcal{P}_{k+1}$

$\mathbf{m} \in \mathcal{P}$: monotone $\overset{\text{def}}{\Longleftrightarrow} m_{j,1} \ge m_{j,2} \ge \cdots \quad (j = 0,1,\ldots)$

$\mathbf{m} \in \mathcal{P}$: indivisible $\overset{\text{def}}{\Longleftrightarrow} GCD$ of $\{m_{j,\nu}\} = 1$

$s(\mathbf{m}) := (m_{j,\sigma_j(\nu)})$ monotone with suitable $\sigma_j \in \mathfrak{S}_\infty$

$\mathbf{1}_\ell := (m_{j,\nu} = \delta_{\nu,\ell_j}) \in \mathcal{P}^{(1)}$ with $\ell = (\ell_0, \ell_1, \ldots)$ $(\ell_j = 1$ for $j \gg 1)$

$\mathbf{1} := (m_{j,\nu} = \delta_{\nu,1}) = (100\cdots, 100\cdots, \cdots) \in \mathcal{P}^{(1)}$

$$\operatorname{idx}(\mathbf{m}', \mathbf{m}'') := \boxed{\sum_{j=0}^{k} \sum_{\nu=1}^{\infty} m'_{j,\nu} m''_{j,\nu} - (k-1)\operatorname{ord}\mathbf{m}' \cdot \operatorname{ord}\mathbf{m}''} \quad (k \gg 1)$$

$\operatorname{idx}\mathbf{m} = \operatorname{idx}(\mathbf{m},\mathbf{m}), \quad \operatorname{idx}(\mathbf{m},\mathbf{1}) = \sum m_{j,1} - (k-1)\operatorname{ord}\mathbf{m} = d(\mathbf{m})$

$$\partial_\ell(\mathbf{m}) := \bigl(m_{j,\nu} - \delta_{\nu,\ell_j}\operatorname{idx}(\mathbf{m},\mathbf{1}_\ell)\bigr)_{\substack{j=0,1,\ldots \\ \nu=1,2,\ldots}} \quad \partial(\mathbf{m}) := s\partial_{\mathbf{1}} s(\mathbf{m})$$

Theorem [O '08]. (Classification of spectral types under mc_μ)

$\mathcal{K} := \{\mathbf{m} \in \mathcal{P}\,;\ \mathbf{m} : \text{basic} \overset{\text{def}}{\Leftrightarrow} \text{monotone, indivisible, } d(\mathbf{m}) \le 0\}$

$\mathcal{K}(p) := \{\mathbf{m} \in \mathcal{K}\,;\ \operatorname{idx}\mathbf{m} = p\} \qquad (p \in 2\mathbb{Z})$

$\Rightarrow\ \#\mathcal{K}(p) < \infty$ and $\mathcal{K}(p) = \emptyset$ if $p > 0$

$$\mathcal{K}(0) = \{11,11,11,11 \quad 111,111,111 \quad 22,1111,1111 \quad 33,222,111111\}$$

$$\begin{aligned}\mathcal{K}(-2) = \{&11,11,11,11,11 \quad 21,21,111,111 \quad 31,22,22,1111 \quad 22,22,22,211\\ &211,1111,1111 \quad 221,221,11111 \quad 32,11111,11111 \quad 222,222,2211\\ &33,2211,111111 \quad 44,2222,22211 \quad 44,332,11111111 \quad 55,3331,22222\\ &66,444,2222211\}\end{aligned}$$

$\#\mathcal{K}(-4) = 36,\ \#\mathcal{K}(-6) = 67,\ \#\mathcal{K}(-8) = 90,\ \#\mathcal{K}(-10) = 162, \ldots$

$$\partial:\ 411,411,42,33 \xrightarrow{15-2\cdot 6=3} 111,111,21 \xrightarrow{4-3=1} 11,11,11 \xrightarrow{3-2=1} 1,1,1$$

Theorem [Katz '95]. Any irreducible rigid tuple $\mathbf{A} \in M(n,\mathbb{C})_0^{k+1}$ is obtained by a finite iteration of middle convolutions mc_λ and additions M_μ from the trivial tuple $(0,\ldots,0) \in M(1,\mathbb{C})_0^{k+1}$.

Rigid tuples := spectral types of irreducible Rigid systems

ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$	ord	$\#\bar{\mathcal{R}}_3$	$\#\bar{\mathcal{R}}$
2	1	1	7	20	44	12	421	857	17	3276	6128
3	1	2	8	45	96	13	588	1177	18	5186	9790
4	3	6	9	74	157	14	1004	2032	19	6954	12595
5	5	11	10	142	306	15	1481	2841	20	10517	19269
6	13	28	11	212	441	16	2388	4644	40	1704287	2554015

2:11,11,11	3:111,111,21	3:21,21,21,21	4:211,211,211
4:1111,211,22	4:1111,1111,31	4:211,22,31,31	4:22,22,22,31
4:31,31,31,31,31	5:2111,221,311	5:2111,2111,32	5:221,221,221
5:11111,221,32	5:11111,11111,41	5:221,221,41,41	5:221,32,32,41
5:311,311,32,41	5:32,32,32,32	5:32,32,41,41,41	5:41,41,41,41,41,41
6:3111,3111,321	6:2211,2211,411	6:2211,321,321	6:222,3111,321
6:21111,222,411	6:21111,2211,42	6:21111,3111,33	6:2211,2211,33
6:222,222,321	6:21111,222,33	6:111111,321,33	6:111111,222,42
6:111111,111111,51	6:2211,222,51,51	6:2211,33,42,51	6:222,33,33,51
6:222,33,411,51	6:3111,33,411,51	6:321,321,42,51	6:321,42,42,42
6:33,33,33,42	6:33,33,411,42	6:33,411,411,42	6:411,411,411,42
6:33,42,42,51,51	6:321,33,51,51,51	6:411,42,42,51,51	6:51,51,51,51,51,51,51

$$d(\mathbf{m}) = \boxed{m_{0,1} + \cdots + m_{k,1} - (k-1)n} \quad (\mathbf{m}: \text{monotone})$$

$$\partial\mathbf{m} = s(\mathbf{m}') \in \mathcal{P}_{k+1}^{(n-d)} \quad \text{with} \quad \boxed{m'_{j,\nu} = m_{j,\nu} - d(\mathbf{m}) \cdot \delta_{\nu,1}}$$

$$411, 411, 42, 33 \xrightarrow{15-2\cdot 6=3} 111, 111, 21 \xrightarrow{4-3=1} 11, 11, 11 \xrightarrow{3-2=1} 1, 1, 1 = 0$$

$$21, 21, 21, 111 \xrightarrow{7-2\cdot 3=1} 11, 11, 11, 11 \xrightarrow{4-2\cdot 2=0} 11, 11, 11, 11 \circlearrowright \text{(Heun)}$$

$$22, 22, 1111 \xrightarrow{5-4=1} 21, 21, 111 \xrightarrow{5-3=2} \times$$

$\mathbf{m}$: rigid $\overset{\text{def}}{\Leftrightarrow}$ spt (an irreducible rigid system)

$\mathbf{m}$: irreducible realizable $\overset{\text{def}}{\Leftrightarrow}$ spt (an irreducible system)

$\mathbf{m}$: fundamental $\overset{\text{def}}{\Leftrightarrow}$ monotone, irreducibly realizable and $d(\mathbf{m}) \le 0$

Theorem [Kostov '03, Crawley-Boevey '03, (Takemura)].

$$\{\mathbf{m}\,;\ \text{fundamental}\} = \mathcal{K}(0) \cup \bigcup_{p=1}^{\infty} \{q\mathbf{m}\,;\ \mathbf{m} \in \mathcal{K}(-2p),\ q = 1, 2, \ldots\}$$

§4. Kac-Moody root system (Crawley-Boevey '03)

$\mathfrak{h}$: real vector space with the base

$$\Pi = \{\alpha_0, \alpha_{j,\nu}\,;\, j = 0, 1, 2, \ldots,\ \nu = 1, 2, \ldots\}$$

$$\operatorname{supp}\gamma := \{\alpha \in \Pi\,;\, c_\alpha \neq 0,\ \gamma = \textstyle\sum_{\alpha\in\Pi} c_\alpha \alpha\} \qquad (\gamma \in \mathfrak{h})$$

$$Q := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha \supset Q_+ := \sum_{\alpha\in\Pi} \mathbb{Z}_{\geq 0}\alpha$$

$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi), \quad (\alpha_0|\alpha_{j,\nu}) = \begin{cases} 0 & (\nu > 1) \\ -1 & (\nu = 1) \end{cases}$$

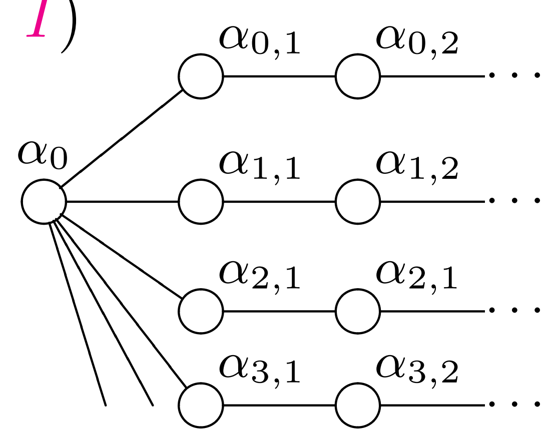
$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \neq j \ \text{ or } \ |\mu - \nu| > 1) \\ -1 & (i = j \ \text{ and } \ |\mu - \nu| = 1) \end{cases}$$

$\mathfrak{g}(C)$: Kac-Moody Lie algebra with the Cartan matrix

$$C := \left(\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}\right)_{i,j\in I} \qquad I := \{0,\ (j, \nu)\,;\, j = 0, 1, \ldots,\ \nu = 1, 2, \ldots\}$$

$W(C)$: Weyl group generated by the reflections

$$r_i(x) = x - 2\frac{(x|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i \qquad (x \in \mathfrak{h},\ i \in I)$$



For $\mathbf{m} = \big(m_{j,\nu}\big)_{j\ge 0,\ \nu\ge 1} \in \mathcal{P}$

$$n_{j,\nu} := m_{j,\nu+1} + m_{j,\nu+2} + \cdots$$

$$\alpha_{\mathbf{m}} := n\alpha_0 + \textstyle\sum_{j,\,\nu} n_{j,\nu}\alpha_{j,\nu} \in Q_+$$

Proposition. i) $\qquad \mathrm{idx}(\mathbf{m},\mathbf{m}') = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'})$

ii) $\alpha_{\mathbf{m}'} := r_i(\alpha_{\mathbf{m}}) \qquad (i \in I)$

$$\mathbf{m}' = \begin{cases} \partial_{\mathbf{1}}\mathbf{m} & (i = 0) \\ (m_{0,1}\cdots, \overset{1}{\smile}{m_{j,1}}\cdots \overset{\nu}{\smile}{m_{j,\nu+1}}\overset{\nu+1}{\smile}{m_{j,\nu}}\cdots,\cdots) & (i = (j,\nu)) \end{cases}$$

$$\alpha_{\partial_\ell(\mathbf{m})} = \alpha_{\mathbf{m}} - 2\frac{(\alpha_\ell|\alpha_{\mathbf{m}})}{(\alpha_\ell|\alpha_\ell)}\alpha_\ell, \qquad \alpha_\ell := \alpha_0 + \sum_{j=0}^{\infty}\sum_{\nu=1}^{\ell_j-1}\alpha_{j,\nu}$$

$\Delta \quad$: {roots of $\mathfrak{g}(C)$} $= \Delta^{re} \cup \Delta^{im}$

Δ^{re} : {real roots} $= W(C)\Pi$

Δ_+^{im} : {positive imaginary roots} $= W(C)K \subset Q_+$

$K := \{\beta \in Q_+ \,;\, \mathrm{supp}\,\beta$ is connected and $(\beta,\alpha) \le 0 \quad (\forall \alpha \in \Pi)\}$

$\Delta_+^{re} = \Delta^{re} \cap Q_+$, $\Delta_+ := \Delta_+^{re} \cup \Delta_+^{im}$ and $\Delta = \Delta_+ \cup -\Delta_+$

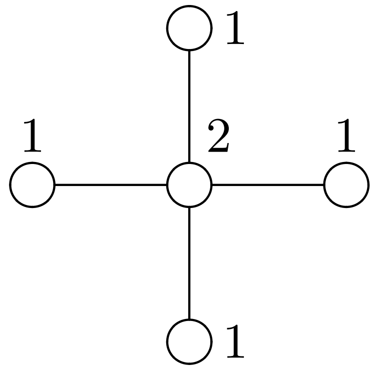
$\mathcal{P}$ (spectral type)	Kac-Moody root system
$\mathrm{idx}(\mathbf{m},\mathbf{m}')$	$(\alpha_{\mathbf{m}}\vert\alpha_{\mathbf{m}'})$
middle convolution ∂_ℓ	reflection w.r.t. α_ℓ
rigid	$\alpha \in \Delta_+^{re}\,;\, \mathrm{supp}\,\alpha \ni \alpha_0$
fundamental	$\alpha \in K\,;\, (\alpha\vert\alpha) < 0$ or indivisible
irreducibly realizable	$\alpha \in \Delta^+\,;\, \mathrm{supp}\,\alpha \ni \alpha_0,\ (\alpha\vert\alpha) < 0$ or indivisible

idx	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
$\#\mathcal{K}(p)$	4	13	36	67	90	162	243	305	420	565	720
# triplets	3	9	24	44	56	97	144	163	223	291	342
# 4-tuples	1	3	9	17	24	45	68	95	128	169	239

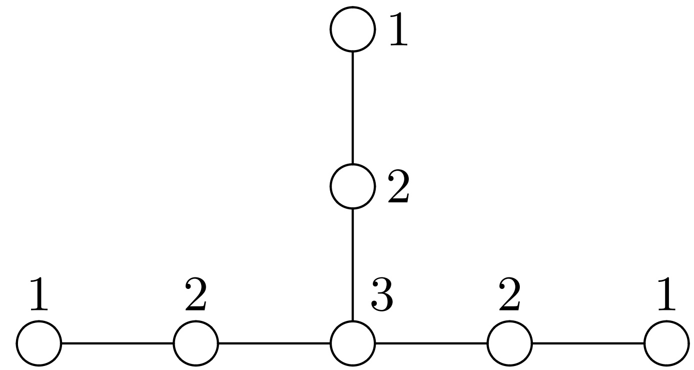
Theorem. $\mathbf{m}$: fundamental $\Rightarrow$ $\mathrm{ord}\,\mathbf{m} \le 6 - 3\cdot \mathrm{idx}\,\mathbf{m}$ $\quad (\le 2 - \mathrm{idx}\,\mathbf{m} \Leftarrow \mathbf{m} \notin \mathcal{P}_3)$

Basic tuples: $\mathrm{idx} = 0$ ($\Rightarrow$ affine)

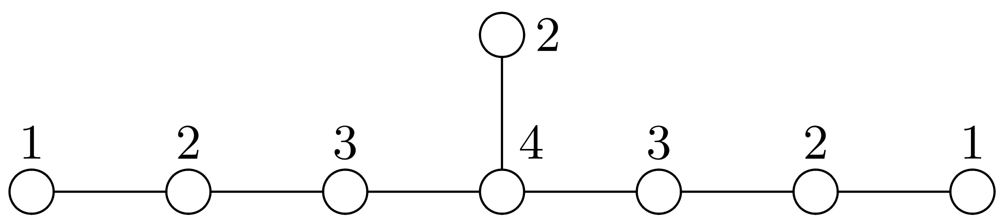
Heun :



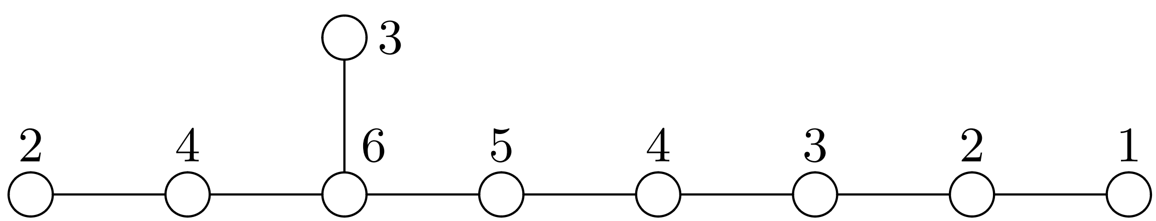
$\tilde{D}_4 : 11, 11, 11, 11$



$\tilde{E}_6 : 111, 111, 111$

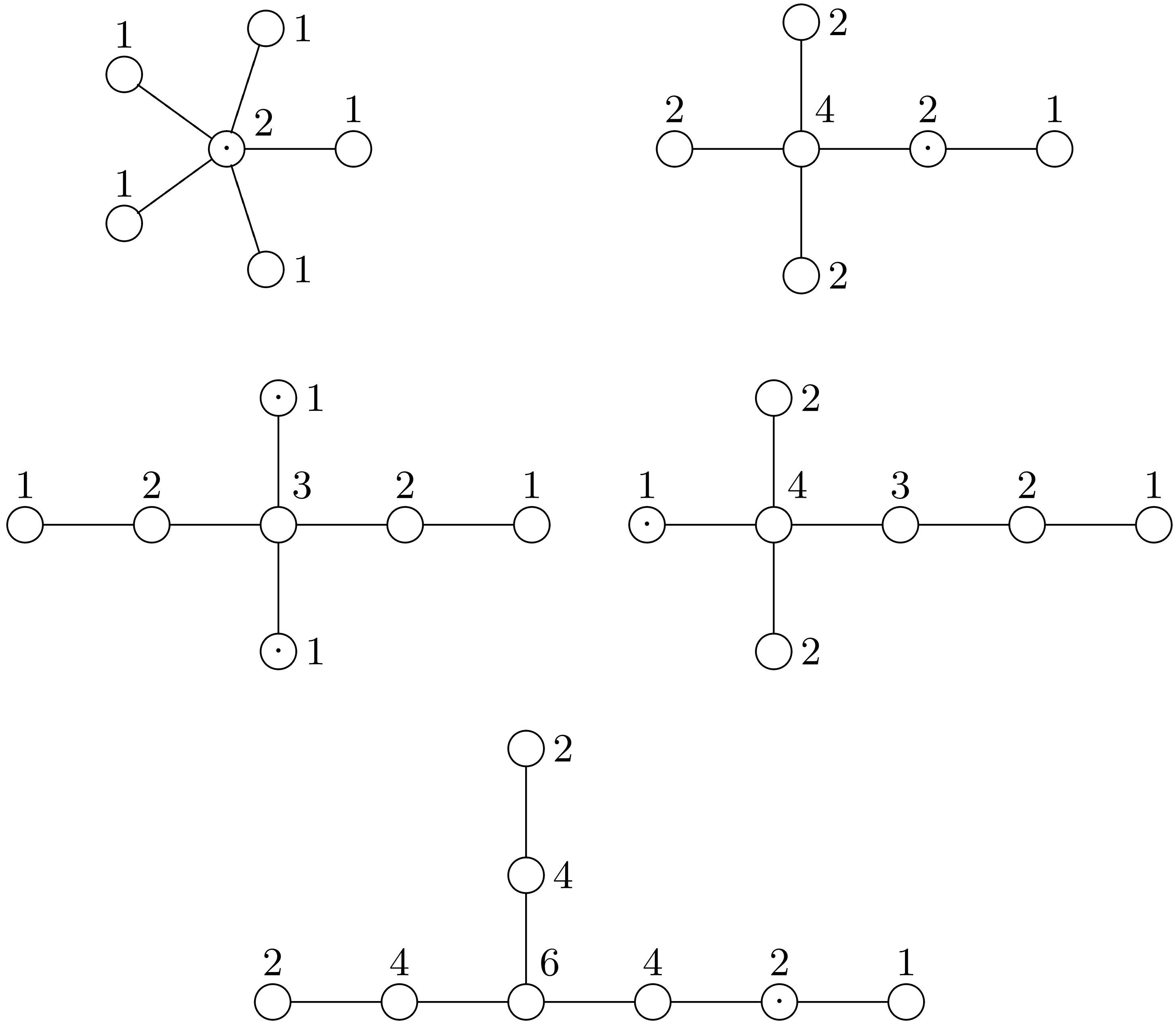


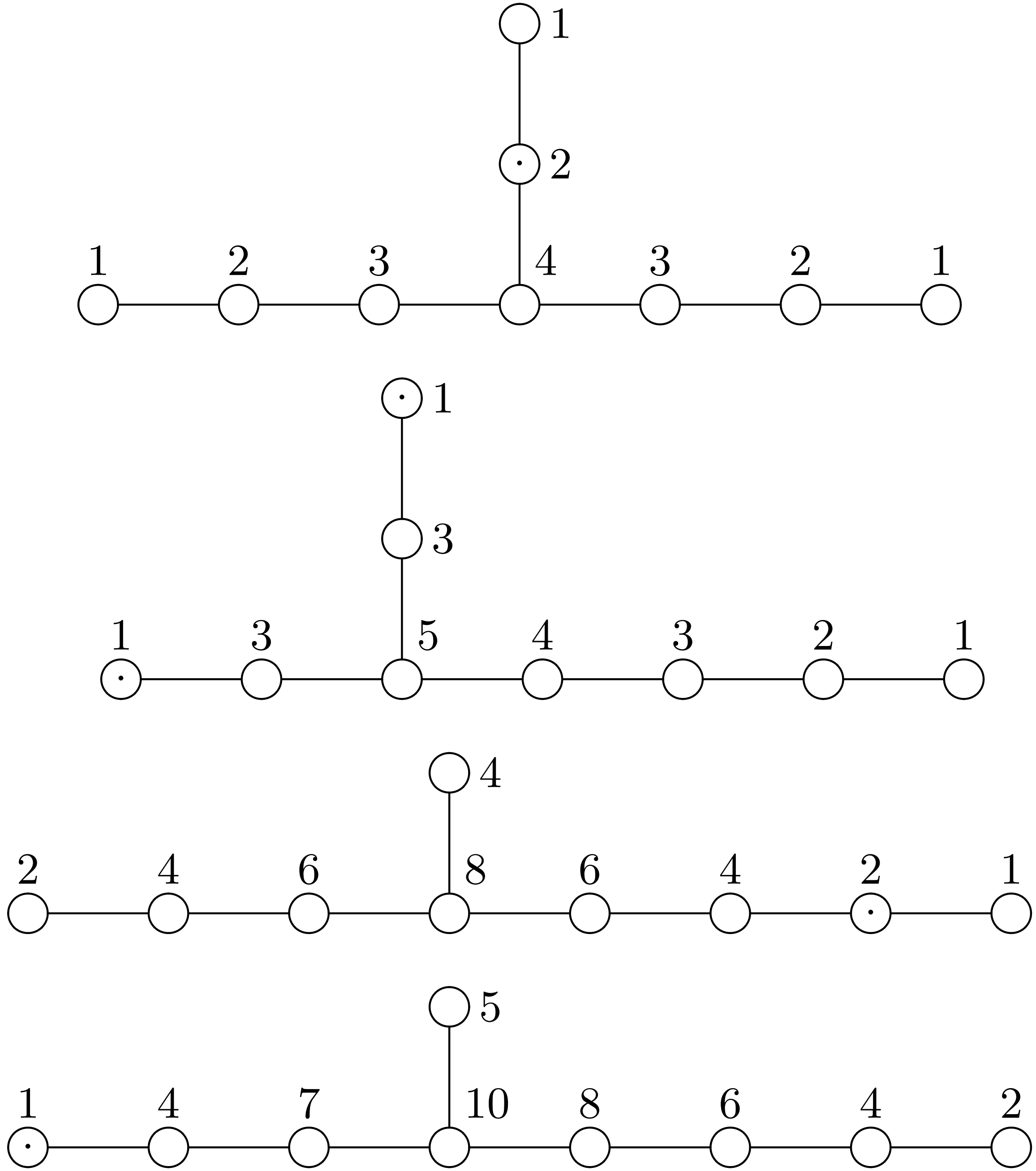
$\tilde{E}_7 : 22, 1111, 1111$



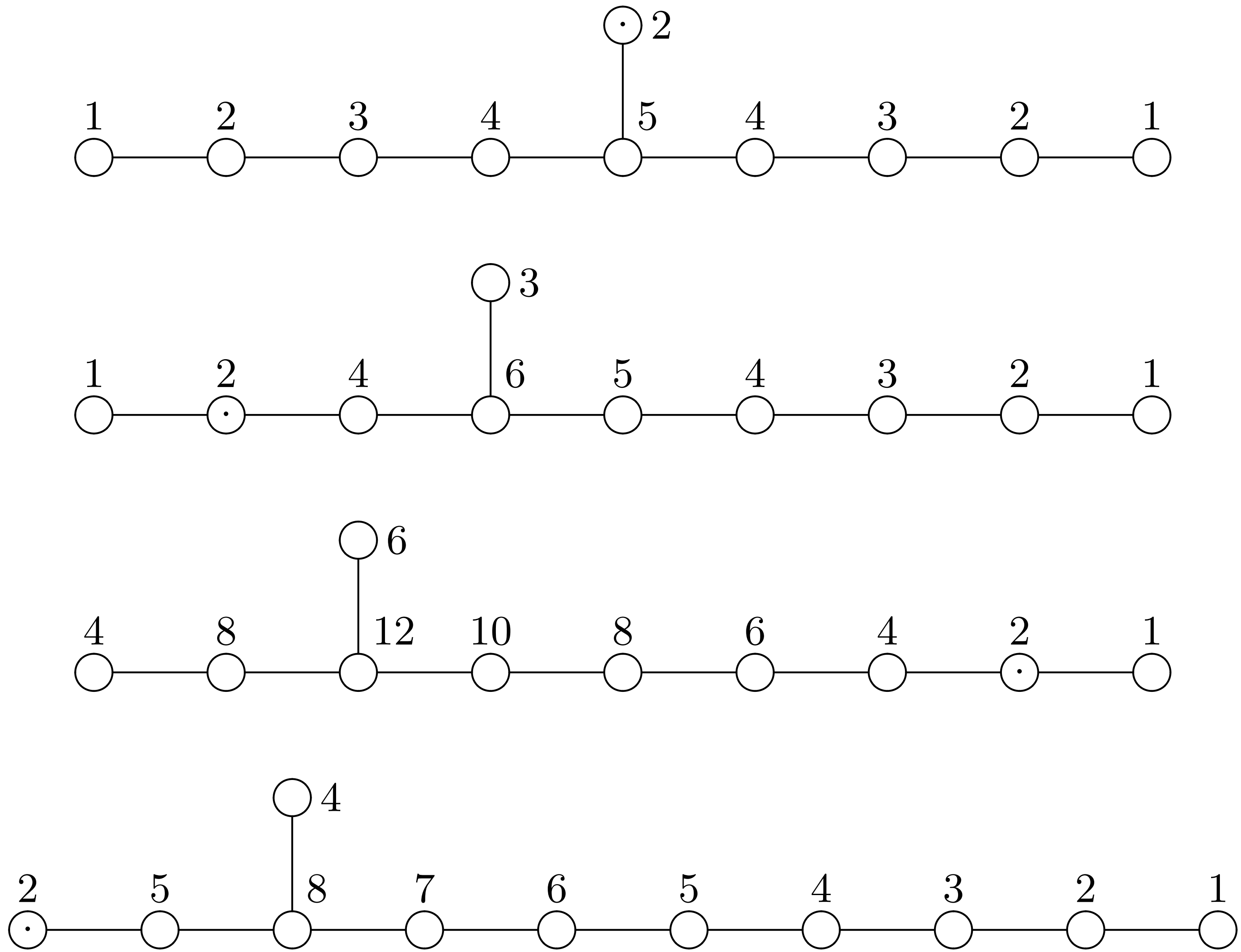
$\tilde{E}_8 : 33, 222, 111111$

Basic tuples: $\mathrm{idx} = -2$ ($\Rightarrow$ Lorentzian)





1
2
1 2 3 4 3 2 1
1
3
1 3 5 4 3 2 1
4
2 4 6 8 6 4 2 1
5
1 4 7 10 8 6 4 2



2
1 2 3 4 5 4 3 2 1
3
1 2 4 6 5 4 3 2 1
6
4 8 12 10 8 6 4 2 1
4
2 5 8 7 6 5 4 3 2 1

§5. Single ODE

$Pu = 0$: single Fuchsian ODE of order n

with regular singularities $z = z_j$ $(j = 0, \dots, k)$: $(z_0 = \infty)$

$P = \left(\prod_{j=1}^{k}(z - z_j)^n\right)\frac{d^n}{dz^n} + a_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \cdots + a_0(z)$

$\deg a_\nu(z) \le kn - (n-\nu)$, $a_\nu(z)$ has zero of order $n - \nu$ at $z = z_j$

Example: Gauss Hypergeometric Equation

$$P = z(1-z)\Big(z(1-z)\frac{d^2}{dz^2} + (\gamma - (\alpha+\beta-1)z)\frac{d}{dz} - \alpha\beta\Big)$$

$$F(\alpha,\beta,\gamma;z) = \sum_{\nu=0}^{\infty}\frac{(\alpha)_\nu(\beta)_\nu}{(\gamma)_\nu}\frac{z^\nu}{\nu!} \qquad (\alpha)_\nu := \alpha(\alpha+1)\cdots(\alpha+\nu-1)$$

$$\in P\begin{Bmatrix} z=0 & 1 & \infty & \\ 0 & 0 & \alpha & ;\ z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix} \quad \text{(Riemann scheme)}$$

Example: Generalized hypergeometric equations

$$ {}_nF_{n-1}(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1};z)=\sum_{\nu=0}^{\infty}\frac{(\alpha_1)_\nu\cdots(\alpha_n)_\nu}{(\beta_1)_\nu\cdots(\beta_{n-1})_\nu}\frac{z^\nu}{\nu!} $$

$$ P=z(z-1)^{n-1}\Big(\prod_{j=1}^{n}(z\tfrac{d}{dz}+\alpha_j)-\prod_{j=1}^{n-1}(z\tfrac{d}{dz}+\beta_j)\cdot\tfrac{d}{dz}\Big) $$

$$ \begin{Bmatrix} z=0 & 1 & \infty & \\ 1-\beta_1 & [0]_{(n-1)} & \alpha_1 & \\ \vdots & & \vdots & ;\ z \\ 1-\beta_{n-1} & & \alpha_{n-1} & \\ 0 & -\beta_n & \alpha_n & \end{Bmatrix} \quad \text{with} \quad \sum_{\nu=1}^{n}\alpha_\nu=\sum_{\nu=1}^{n}\beta_\nu $$

$$ [\lambda]_{(m)}:=\begin{pmatrix}\lambda\\ \lambda+1\\ \vdots\\ \lambda+m-1\end{pmatrix} $$

Theorem [O]. $\mathbf{m}$: rigid $\Leftrightarrow$ $\exists_1$ $Pu=0$ with $\{\lambda_{\mathbf{m}}\}$ $(|\{\lambda_{\mathbf{m}}\}|=0)$

$\{\lambda_{\mathbf{m}}\}:=\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}_{\substack{0\le j\le k\\ 1\le\nu\le n_j}}$: Riemann scheme, $|\{\lambda_{\mathbf{m}}\}|:=\sum_{j,\nu}m_{j,\nu}\lambda_{j,\nu}-\operatorname{ord}\mathbf{m}+1$

§6. Connection problem

$Pu = 0$: rigid Fuchsian ODE with 3 singular points

Riemann Scheme : $\left\{\begin{matrix} [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & [\lambda_{2,1}]_{(m_{2,1})} \\ \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & [\lambda_{2,n_2}]_{(m_{2,n_2})} \end{matrix}\right\}$

Theorem [O]. $\mathbf{m} \in \mathcal{P}_3^{(n)}$: rigid and $m_{0,n_0} = m_{0,n_1} = 1$

$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1})$: connection coefficient of normalized local sol. w.r.t λ_{0,n_0} to that w.r.t. λ_{0,n_1} (Fuchs condition: $|\{\lambda_{\mathbf{m}}\}| = 0$)

$$c(\lambda_{0,n_0} \rightsquigarrow \lambda_{1,n_1}) = \frac{\prod_{\nu=1}^{n_0-1} \Gamma\big(\lambda_{0,n_0} - \lambda_{0,\nu} + 1\big) \cdot \prod_{\nu=1}^{n_1-1} \Gamma\big(\lambda_{1,\nu} - \lambda_{1,n_1}\big)}{\prod_{\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}'',\ m'_{0,n_0}=m''_{1,n_1}=1} \Gamma\big(|\{\lambda_{\mathbf{m}'}\}|\big)}$$

$$\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}'' \overset{\text{def}}{\Longleftrightarrow} \mathbf{m} = \mathbf{m}' + \mathbf{m}'' \text{: rigid} \Rightarrow (\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -1$$

$$\#\{\mathbf{m}';\ \mathbf{m}= \mathbf{m}' \oplus \mathbf{m}''\} = n_0 + n_1 - 2$$

Example (H_n): ${}_{n-1}F_n(\alpha,\beta;z)$ (unique known case)

$$P\begin{Bmatrix} \lambda_{0,1} & [\lambda_{1,1}]_{(n-1)} & \lambda_{2,1} \\ \vdots & & \vdots \\ \lambda_{0,n-1} & & \lambda_{2,n-1} \\ \lambda_{0,n} & \lambda_{1,2} & \lambda_{2,n} \end{Bmatrix} = P\begin{Bmatrix} 1-\beta_1 & [0]_{(n-1)} & \alpha_1 \\ \vdots & & \vdots \\ 1-\beta_{n-1} & & \alpha_{n-1} \\ 0 & -\beta_n & \alpha_n \end{Bmatrix}$$

$$1\ldots1\overline{1};n-1\underline{1};1\ldots1 = 0\ldots0\overline{1};\quad 1\quad \underline{0};0\ldots0\overset{\nu}{1}0\ldots0$$
$$\oplus\, 1\ldots1\overline{0};n-2\underline{1};1\ldots101\ldots1$$

$$c(\lambda_{0,n}\rightsquigarrow\lambda_{1,2}) = \frac{\prod_{\nu=1}^{n-1}\Gamma(\lambda_{0,n}-\lambda_{0,\nu}+1)\cdot\Gamma(\lambda_{1,1}-\lambda_{1,2})}{\prod_{\nu=1}^{n}\Gamma(\lambda_{0,n}+\lambda_{1,1}+\lambda_{2,\nu})} = \prod_{\nu=1}^{n}\frac{\Gamma(\beta_\nu)}{\Gamma(\alpha_\nu)}$$
$$= \lim_{x\to1-0}(1-x)^{\beta_n}{}_nF_{n-1}(\alpha,\beta;x) \qquad (\mathrm{Re}\,\beta_n>0)$$

$$c(\lambda_{0,n}\rightsquigarrow\lambda_{2,n}) = \prod_{\nu=1}^{n-1}\frac{\Gamma(\beta_\nu)\Gamma(\alpha_\nu-\alpha_n)}{\Gamma(\alpha_\nu)\Gamma(\beta_\nu-\beta_n)} \qquad (n=2 : \text{Gauss Hyp. Geom.})$$

Example (EO_{2m}) : $(\mathbf{m} = (1^{2m}, mm-11, mm)$: even family)

$$P\left\{\begin{matrix} \lambda_{0,1} & [\lambda_{1,1}]_{(m)} & [\lambda_{2,1}]_{(m)} \\ \vdots & [\lambda_{1,2}]_{(m-1)} & [\lambda_{2,2}]_{(m)} \\ \lambda_{0,2m} & \lambda_{1,3} & \end{matrix}\right\}$$

$$EO_{2m} = H_1 \oplus EO_{2m-1} = H_2 \oplus EO_{2m-2}$$

reducible $\iff \exists \mu \neq \mu'$, ν s.t. $\lambda_{0,\mu} + \lambda_{1,1} + \lambda_{2,\nu} \in \mathbb{Z}$

or $\lambda_{0,\mu} + \lambda_{0,\mu'} + \lambda_{1,1} + \lambda_{1,2} + \lambda_{2,1} + \lambda_{2,2} \in \mathbb{Z}$

$$c(\lambda_{0,2m} \rightsquigarrow \lambda_{1,3}) = \prod_{k=1}^{2} \frac{\Gamma\big(\lambda_{1,k} - \lambda_{1,3}\big)}{\Gamma\Big(\Big|\Big\{\begin{matrix}\lambda_{0,2m} & \lambda_{1,1} & \lambda_{2,k}\end{matrix}\Big\}\Big|\Big)}$$

$$\cdot \prod_{k=1}^{2m-1} \frac{\Gamma\big(\lambda_{0,2m} - \lambda_{0,k} + 1\big)}{\Gamma\left(\left|\left\{\begin{matrix}\lambda_{0,k} & \lambda_{1,1} & \lambda_{2,1} \\ \lambda_{0,2m} & \lambda_{1,2} & \lambda_{2,2}\end{matrix}\right\}\right|\right)}$$

Example (EO_{2m}) : $(\mathbf{m} = (1^{2m}, mm-11, mm)$: even family)

$$P\left\{\begin{matrix} \lambda_{0,1} & [\lambda_{1,1}]_{(m)} & [\lambda_{2,1}]_{(m)} \\ \vdots & [\lambda_{1,2}]_{(m-1)} & [\lambda_{2,2}]_{(m)} \\ \lambda_{0,2m} & \lambda_{1,3} & \end{matrix}\right\}$$

$$EO_{2m} = H_1 \oplus EO_{2m-1} = H_2 \oplus EO_{2m-2}$$

reducible $\iff \exists \mu \neq \mu',\ \nu$ s.t. $\lambda_{0,\mu} + \lambda_{1,1} + \lambda_{2,\nu} \in \mathbb{Z}$

or $\lambda_{0,\mu} + \lambda_{0,\mu'} + \lambda_{1,1} + \lambda_{1,2} + \lambda_{2,1} + \lambda_{2,2} \in \mathbb{Z}$

$$c(\lambda_{0,2m} \rightsquigarrow \lambda_{1,3}) = \prod_{k=1}^{2} \frac{\Gamma\bigl(\lambda_{1,k} - \lambda_{1,3}\bigr)}{\Gamma\Bigl(\lambda_{0,2m} + \lambda_{1,1} + \lambda_{2,k}\Bigr)}$$

$$\cdot \prod_{k=1}^{2m-1} \frac{\Gamma\bigl(\lambda_{0,2m} - \lambda_{0,k} + 1\bigr)}{\Gamma\left(\begin{matrix} \lambda_{0,k} + \lambda_{1,1} + \lambda_{2,1} + \\ \lambda_{0,2m} + \lambda_{1,2} + \lambda_{2,2} - 1 \end{matrix}\right)}$$

Example (EO_{2m+1}): $(\mathbf{m} = (1^{2m+1}, mm1, m+1m)$: odd family)

$$P\begin{Bmatrix} \lambda_{0,1} & [\lambda_{1,1}]_{(m)} & [\lambda_{2,1}]_{(m+1)} \\ \vdots & [\lambda_{1,2}]_{(m)} & [\lambda_{2,2}]_{(m)} \\ \lambda_{0,2m+1} & \lambda_{1,3} & \end{Bmatrix}$$

$$EO_n = H_1 \oplus EO_{n-1} = H_2 \oplus EO_{n-2} \quad (n = 2m+1)$$

$$c(\lambda_{0,n} \rightsquigarrow \lambda_{1,3}) = \prod_{k=1}^{2} \frac{\Gamma\bigl(\lambda_{1,k} - \lambda_{1,3}\bigr)}{\Gamma\Bigl(\lambda_{0,n} + \lambda_{1,1} + \lambda_{2,k}\Bigr)}$$

$$\cdot \prod_{k=1}^{n-1} \frac{\Gamma\bigl(\lambda_{0,n} - \lambda_{0,k} + 1\bigr)}{\Gamma\begin{pmatrix} \lambda_{0,k} + \lambda_{1,1} + \lambda_{2,1} + \\ \lambda_{0,n} + \lambda_{1,2} + \lambda_{2,2} - 1 \end{pmatrix}}$$

Simpson's list (rigid and $\mathbf{m}_0 = 1^n = 1 \cdots 1$)

H_n and EO_n and Extra case: $111111, 222, 44$

§7. Harmonic analysis on a symmetric space

Zonal spherical function ($\leftarrow$ Riemannian symmetric space)

$\rightsquigarrow$ Heckman-Opdam hypergeometric function

($\leftarrow$ a root system and parameter k)

$\phi_{k,\lambda}$: An eigenfunction of commuting PDOs containing (ex. BC_n)

$$L(k)_{BC_n} := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{k=1}^{n} (2k_3 \coth x_k + 4k_2 \coth 2x_k) \frac{\partial}{\partial x_k}$$

$$+ \sum_{1 \le i < j \le n,\ \epsilon \in \{1,-1\}} 2k_1 \coth(x_i + \epsilon x_j) \Big(\frac{\partial}{\partial x_i} + \epsilon \frac{\partial}{\partial x_j} \Big)$$

1. Uniquely determined by $\phi_{k,\lambda}(0) = 1$
2. $\phi_{k,\lambda}(x) = \sum_{w \in W} c(w\lambda, k)(e^{\langle w\lambda - \rho(k), x \rangle} +$ higer) $\quad (x \longrightarrow \infty)$

Theorem [O-Shimeno]. $\phi_{k,\lambda}|_{x_2 = \cdots = x_n = 0}$ satisfies ODE of EO_{2n}

Corollary. We have above 1 and 2 and the explicit formula of $c(\lambda, k)$

References

http://akagi.ms.u-tokyo.ac.jp/~oshima

arXiv:0811.2916 : classification/connection formula

arXiv:0812.1135 : {middle conv. by Katz} $\overset{\sim}{\leftrightarrow}$ {extensions by Yokoyama}

ftp://akagi.ms.u-tokyo.ac.jp/pub/math/okubo/okubo.zip

okobo.exe: a computer program giving connection formula, rigid tuples, basic tuples, reduction of spectral types by Katz's middle convolutions and Yokoyama's extensions/restrictions

$\Rightarrow$ Connection formula: 4,111,704 independent cases for order ≤ 40

Thank you! End!