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COMPLETELY INTEGRABLE SYSTEMS WITH A SYMMETRY IN COORDINATES*

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Abstract. We explicitly construct the integrals of completely integrable quantum or classical systems whose potential functions are invariant under the action of a classical Weyl group. Our potential functions and integrals are expressed by the Weierstrass elliptic function.

1. Introduction. Many completely integrable quantum or classical dynamical systems have been constructed in connection with root systems (cf. [OP1], [OP2], [In]). Consequently most of them are invariant under the action of the corresponding Weyl groups. Our study is to determine all the completely integrable systems with this invariant property.

Let W be the Weyl group of type A_{n-1} with $n \geq 3$ or of type B_n with $n \geq 2$ or of type D_n with $n \geq 4$. We identify W with the group of the coordinate transformations

$$(x_1, \dots, x_n) \mapsto (\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)})$$

of $\mathbb{R}^n$, where σ are the elements of the n-th permutation group $\mathfrak{S}_n$ and

$$\begin{cases} \varepsilon_1 = \cdots = \varepsilon_n = 1 & \text{if } W \text{ is of type } A_{n-1}, \\ \varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1 & \text{if } W \text{ is of type } B_n, \\ \varepsilon_1 = \pm 1, \cdots, \varepsilon_n = \pm 1 \text{ and } \#\{i\,;\, \varepsilon_i = -1\} \text{ is even} & \text{if } W \text{ is of type } D_n. \end{cases}$$

We study the Schrödinger operator

$$P = -\frac{1}{2} \sum_{1 \leq j \leq n} \frac{\partial^2}{\partial x_j^2} + R(x) \tag{1.1}$$

on $\mathbb{R}^n$ with a W-invariant potential function $R(x)$ which has enough W-invariant commuting differential operators assuring the complete integrability of P. To be precise we assume that there exist W-invariant differential operators $P_1, \dots, P_n$ with

$$[P_i, P_j] = 0 \quad \text{for} \quad 1 \leq i < j \leq n \tag{1.2}$$

and

$$P \in \mathbb{C}[P_1, \dots, P_n] \tag{1.3}$$

such that

$$P_j = \sum_{1 \leq i_1 < \cdots < i_j \leq n} \partial_{i_1} \cdots \partial_{i_j} + R_j \quad \text{with} \quad \operatorname{ord} R_j < j \quad \text{for } 1 \leq j \leq n \tag{1.4}$$

or

$$P_j = \sum_{1 \leq i_1 < \cdots < i_j \leq n} \partial_{i_1}^2 \cdots \partial_{i_j}^2 + R_j \quad \text{with} \quad \operatorname{ord} R_j < 2j \quad \text{for } 1 \leq j \leq n \tag{1.5}$$

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or

$$\begin{cases} P_n = \partial_1 \cdots \partial_n + R_n \quad \text{with} \quad \operatorname{ord} R_n < n, \\ P_j = \displaystyle\sum_{1 \le i_1 < \cdots < i_j \le n} \partial_{i_1}^2 \cdots \partial_{i_j}^2 + R_j \quad \text{with} \quad \operatorname{ord} R_j < 2j \quad \text{for } 1 \le j < n \end{cases} \tag{1.6}$$

if the type of W is A_{n-1} or B_n or D_n, respectively. Here $\mathbb{C}[P_1, \ldots, P_n]$ is the commutative algebra generated by $P_1, \ldots, P_n$, $\operatorname{ord} R_j$ denote the orders of differential operators R_j and for simplicity we put $\partial_i = \frac{\partial}{\partial x_i}$.

We assume that the coefficients of the differential operators are extended to holomorphic functions on a Zariski open subset of an open connected neighborhood of the origin of the complexification $\mathbb{C}^n$ of $\mathbb{R}^n$.

The main result of our previous paper [OS] is the following:

If W is of type A_{n-1} with $n \ge 3$, then

$$R(x) = \sum_{1 \le i < j \le n} u(x_i - x_j) \tag{1.7}$$

with

$$u(t) = C_1 \wp(t) + C_2. \tag{1.8}$$

If W is of type B_n with $n \ge 2$, then

$$R(x) = \sum_{1 \le i < j \le n} \Big(u(x_i - x_j) + u(x_i + x_j) \Big) + \sum_{1 \le j \le n} v(x_j). \tag{1.9}$$

Here if $n \ge 3$, we have

$$\begin{cases} u(t) = C_1 \wp(t) + C_2, \\ v(t) = \dfrac{C_3 \wp(t)^4 + C_4 \wp(t)^3 + C_5 \wp(t)^2 + C_6 \wp(t) + C_7}{\wp'(t)^2} \end{cases} \tag{1.10}$$

or

$$u(t) = C_1 t^{-2} + C_2 t^2 + C_3 \quad \text{and} \quad v(t) = C_4 t^{-2} + C_5 t^2 + C_6 \tag{1.11}$$

or

$$u(t) = C_1 \quad \text{and} \quad v(t) \ \text{ is any even function.} \tag{1.12}$$

If W is of type D_n with $n \ge 4$, then (1.9) *holds with $v = 0$ and u is given by* (1.10) *or* (1.11).

Here C_1, $C_2, \ldots$ are complex numbers and $\wp(t)$ is the Weierstrass elliptic function $\wp(t|2\omega_1, 2\omega_2)$ with primitive half-periods ω_1 and ω_2, which are allowed to be infinity.

The purpose of this paper is to construct the operators $P_1, \ldots, P_n$ mentioned above when u or (u, v) is given by (1.8) or (1.10) for any complex numbers C_1, $C_2, \ldots$ and for any periods of the elliptic function (cf. Theorem 7.2, 7.3 and 7.5), which was announced in [OOS]. Hence we shall have the complete integrability of the corresponding Schrödinger operator (1.1). We remark that if W is of type A_{n-1}, the complete integrability and the operators $P_1, \ldots, P_n$ are already known (cf. [Ca], [Su], [OP2], [OS], [Et], Theorem 3.2 in this paper).

Taking the "classical limit", we shall also obtain the integrals of the Hamiltonian corresponding to the Schrödinger operator (1.1) because of our simple expression of the operators $P_1, \ldots, P_n$.

When W is of type B_2, our argument in this paper is valid but there exist other potentials which assure the complete integrability. This is caused by a symmetry between u and v. We shall treat this case in another paper (cf. [OOS], [OO], [Oc]).

If u or (u,v) is given by (1.11), the operators $P_1,\dots,P_n$ do not exist in general and then we need W-invariant operators of higher orders (cf. [OP2]), which will be discussed in future.

If (u,v) is given by (1.12), the algebra $\mathbb{C}[P_1,\dots,P_n]$ equals the totality of $\mathfrak{S}_n$-invariants of $\mathbb{C}[-\frac{1}{2}\partial_1^2+v(x_1),\dots,-\frac{1}{2}\partial_n^2+v(x_n)]$.

We note that if $2\omega_1=\sqrt{-1}\pi$ and $\omega_2=\infty$, then (1.10) is reduced to

$$\begin{cases} u(t) = C_1'\sinh^{-2}t + C_2', \\ v(t) = C_3'\sinh^{-2}t + C_4'\sinh^{-2}2t + C_5'\sinh^2 t + C_6'\sinh^2 2t + C_7' \end{cases} \tag{1.13}$$

with complex numbers $C_1',\dots,C_7'$. The system studied by Heckman-Opdam ([He1], [He2], [HO], [Op1] and [Op2]) corresponds to this trigonometric case with $C_5'=C_6'=0$ and they proved its complete integrability. When $C_5'=C_6'=0$, an explicit form of $P_1,\dots,P_n$ is given by [De].

Moreover if $\omega_1=\omega_2=\infty$, then (1.10) is reduced to

$$\begin{cases} u(t) = C_1't^{-2} + C_2', \\ v(t) = C_3't^{-2} + C_4't^2 + C_5't^4 + C_6't^6 + C_7'. \end{cases} \tag{1.14}$$

Here we quote a result in [OS] for the operator which commutes with the Shorödinger operator P:

If there exists a nonzero constant ω such that the W-invariant differential operators $P_1,\dots,P_n$ are invariant by the parallel translation $x_1\mapsto x_1+\omega$, then any W-invariant differential operator Q that is also invariant by the same parallel translation is contained in $\mathbb{C}[P_1,\dots,P_n]$ if $[P,Q]=0$.

After this paper [Os] was written, [Ch] proved the completely integrability of the Shrödinger operator (1.1) with the elliptic potential function attached to the root system. If the root system is of type B_n in our situation, the potential considered in [Ch] corresponds to the case where $v(t)=C_3\wp(t)+C_4$ or $v(t)=C_3\wp(2t)+C_4$ in (1.10). The method is quite interesting but different form this note constructing explicitly all the integrals.

Lastly we give a brief overview of the following sections.

In §2 preliminary remarks are made and two results employed thoughout are established.

In §3 the two fundamental operators Δ and Δ_n for A_n and D_n are introduced and their commutativity is proved by the results in §2. An expansion of Δ_n gives the commuting differential operators for A_n.

In §4 the Scrödinger operator is allowed to have a term in the potential only depending on the particle position through a given function v. A functional differential equation (4.4) is established that will ensure the commutativity of the fundamental operators P and P_n for B_n given by (4.2).

In §5, using the lemmas in §2, we establish solutions of the functional differential equation with the assumption $u=w$ which corresponds to the form (1.9).

In §6 we look at various rational and trigonometric degenerations of the solutions of the functional differential equation.

In §7 we bring the results of the previous sections together and establish the commuting differential operators $P_1,\dots,P_n$ for B_n and D_n.

2. Preliminaries.

First we introduce some notation used in this paper. For an element w of the permutation group $\mathfrak{S}_n$ of the set of indices $\{1,\dots,n\}$, we define $w(i)=i$ for any $i\in\mathbb{Z}$ satisfying $i<1$ or $i>n$ and we identify $\mathfrak{S}_n$ with a subgroup of the group of bijective transformations of $\mathbb{Z}$. Then we have naturally $\mathfrak{S}_k\subset\mathfrak{S}_n$ if $k<n$.

When we distinguish the Weyl group that we are looking at, we denote it by $W(A_{n-1})$, $W(B_n)$ or $W(D_n)$ according to its type. Then $W(A_{n-1})\simeq\mathfrak{S}_n$ and $W(A_{n-1})\subset W(D_n)\subset W(B_n)$. We define a homomorphism ε of $W(B_n)$ to $\{\pm1\}$ by

$$\varepsilon(w)=\begin{cases}1 & \text{if } w\in W(D_n),\\ -1 & \text{if } w\notin W(D_n).\end{cases}\tag{2.1}$$

For the coordinate system $(x_1,\dots,x_n)$ of $\mathbb{R}^n$ we put

$$\partial_i=\frac{\partial}{\partial x_i},\ \partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}\quad\text{and}\quad|\alpha|=\alpha_1+\cdots+\alpha_n.$$

Here $\alpha=(\alpha_1,\dots,\alpha_n)$ with non-negative integers α_i.

Let $P=\sum p_\alpha(x)\partial^\alpha$ be a differential operator. Then we put

$${}^tP=\sum(-1)^{|\alpha|}\partial^\alpha p_\alpha(x)\tag{2.2}$$

and we say that P is *self-adjoint* if ${}^tP=P$ and *skew self-adjoint* if ${}^tP=-P$. For $w\in W$ and a differential operator P, we denote by $w(P)$ the differential operator corresponding to P under the coordinate transformation w of $\mathbb{R}^n$. In particular we define $P^-=w^-(P)$ by $w^-\in W(B_n)$ with $w^-(x_1,\dots,x_n)=(-x_1,\dots,-x_n)$ and we call P has an *even parity* if $P^-=P$ and an *odd parity* if $P^-=-P$. Then we note that

$${}^t({}^tP)=(P^-)^-=P,\quad {}^t(P^-)=({}^tP)^-,\quad {}^t(PQ)={}^tQ\,{}^tP,\quad (PQ)^-=P^-Q^-.$$

In general the suffix $\{1,\dots,k\}$ of a function or an operator (eg. $Q_{\{1,\dots,k\}}$) means that it is a function or an operator of the variables $x_1,\dots,x_k$ invariant under $w\in\mathfrak{S}_k$. And for a function or an operator $Q_{\{1,\dots,k\}}$ and a subset I of $\{1,\dots,n\}$, we define $Q_I=w(Q_{\{1,\dots,k\}})$ if there exist $w\in W(A_{n-1})\simeq\mathfrak{S}_n$ with $w(\{1,\dots,k\})=I$.

Now we review the Weierstrass elliptic function (cf. [WW]), which is

$$\wp(z|2\omega_1,2\omega_2)=\frac{1}{z^2}+\sum_{\omega\neq0}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right),\tag{2.3}$$

where the sum ranges over all periods $\omega=2m_1\omega_1+2m_2\omega_2$ $(m_1,\ m_2\in\mathbb{Z})$ except 0. We define

$$\omega_3=-(\omega_1+\omega_2)\quad\text{and}\quad\omega_4=0.\tag{2.4}$$

The Weierstrass elliptic function $\wp(t)$ satisfies the differential equation

$$\begin{aligned}(\wp')^2&=4\wp^3-g_2\wp-g_3\\&=4(\wp-e_1)(\wp-e_2)(\wp-e_3).\end{aligned}\tag{2.5}$$

Here

$$
\begin{aligned}
&g_2 = 60\sum_{\omega\neq 0}\omega^{-4},\ g_3 = 140\sum_{\omega\neq 0}\omega^{-6},\\
&\wp(\omega_\nu) = e_\nu \quad \text{for } \nu = 1,2,3, \qquad (2.6)\\
&e_1+e_2+e_3 = 0,\ e_1e_2+e_2e_3+e_3e_1 = -\frac{g_2}{4},\ e_1e_2e_3 = \frac{g_3}{4}.
\end{aligned}
$$

Moreover we have important formulas

$$
\begin{vmatrix} \wp(x) & \wp'(x) & 1\\ \wp(y) & \wp'(y) & 1\\ \wp(z) & \wp'(z) & 1 \end{vmatrix} = 0 \quad \text{if} \ \ x+y+z=0, \tag{2.7}
$$

$$
\wp(z+\omega_i) = e_i + \frac{(e_i-e_j)(e_i-e_k)}{\wp(z)-e_i} \quad \text{if} \ \ \{i,j,k\} = \{1,2,3\} \tag{2.8}
$$

and

$$
\wp(2z) = \frac{\bigl(12\wp(z)^2-g_2\bigr)^2}{16\wp'(t)^2} - 2\wp(z). \tag{2.9}
$$

In this paper the periods are allowed to be infinity and hence g_1 and g_2 or e_1 and e_2 take any complex numbers. Then the condition

$$
(e_1-e_2)(e_2-e_3)(e_3-e_1) \neq 0 \tag{2.10}
$$

holds if and only if the both periods are finite. On the other hand, if $e_1 = e_2 = \frac{1}{3}\lambda^2$ and $e_3 = -\frac{2}{3}\lambda^2$ with $\lambda\in\mathbb{C}$, then

$$
\wp(z|\sqrt{-1}\lambda^{-1}\pi,\infty) = \lambda^2\sinh^{-2}\lambda z + \tfrac{1}{3}\lambda^2. \tag{2.11}
$$

In particular, if $e_1 = e_2 = e_3 = 0$, we have

$$
\wp(z|\infty,\infty) = z^{-2}. \tag{2.12}
$$

We note that if (2.10) holds, the function $v(t)$ given by (1.10) is rewritten into

$$
v(t) = C_5' + \sum_{j=1}^{4} C_j'\wp(t+\omega_j) \tag{2.13}
$$

with suitable complex numbers $C_1',\ldots,C_5'$ (cf. (2.5) and (2.8)). Moreover for any complex numbers C_1'', C_2'' and C_3'', it follows from (2.9) that

$$
v(t) = C_1''\wp(t) + C_2''\wp(2t) + C_3'' \tag{2.14}
$$

is a special case of (1.10) and the complete integrability of the corresponding Schrödinger operator was a question in [OP2].

Now we prepare

Lemma 2.1. *Let $\tilde v_k(t)$, $\tilde u_{ij}(t)$ and $\tilde w_{ij}(t)$ be functions with a single variable for $1\le i<j\le 3$ and $1\le k\le 3$. Put $v_k = \tilde v_k(x_k)$, $v_k' = \tilde v_k'(x_k)$, $u_{ij} = \tilde u_{ij}(x_i-x_j)$, $u_{ij}' = \tilde u_{ij}'(x_i-x_j)$, $w_{ij} = \tilde w_{ij}(x_i+x_j)$ and $w_{ij}' = \tilde w_{ij}'(x_i+x_j)$ for $1\le i<j\le 3$ and*

$1 \le k \le 3$. *Then we have*

(2.15)

$$\partial_1\big((u_{12}-w_{12})(u_{13}-w_{13})\big)+\partial_2\big((u_{12}-w_{12})(u_{23}-w_{23})\big)+\partial_3\big((u_{13}-w_{13})(u_{23}-w_{23})\big)$$
$$=\begin{vmatrix} u_{12} & u'_{12} & 1\\ u_{23} & u'_{23} & 1\\ u_{13} & -u'_{13} & 1\end{vmatrix}+\begin{vmatrix} u_{12} & u'_{12} & 1\\ w_{13} & -w'_{13} & 1\\ w_{23} & w'_{23} & 1\end{vmatrix}+\begin{vmatrix} u_{23} & u'_{23} & 1\\ w_{12} & -w'_{12} & 1\\ w_{13} & w'_{13} & 1\end{vmatrix}+\begin{vmatrix} u_{13} & -u'_{13} & 1\\ w_{23} & -w'_{23} & 1\\ w_{12} & w'_{12} & 1\end{vmatrix},$$

(2.16)

$$\partial_1\big((u_{12}+w_{12})(u_{13}-w_{13})\big)+\partial_2\big((u_{12}+w_{12})(u_{23}-w_{23})\big)+\partial_3\big((u_{13}+w_{13})(u_{23}+w_{23})\big)$$
$$=\begin{vmatrix} u_{12} & u'_{12} & 1\\ u_{23} & u'_{23} & 1\\ u_{13} & -u'_{13} & 1\end{vmatrix}+\begin{vmatrix} u_{12} & u'_{12} & 1\\ w_{13} & -w'_{13} & 1\\ w_{23} & w'_{23} & 1\end{vmatrix}-\begin{vmatrix} u_{23} & u'_{23} & 1\\ w_{12} & -w'_{12} & 1\\ w_{13} & w'_{13} & 1\end{vmatrix}-\begin{vmatrix} u_{13} & -u'_{13} & 1\\ w_{23} & -w'_{23} & 1\\ w_{12} & w'_{12} & 1\end{vmatrix}$$

and

$$v'_1(u_{12}-w_{12})+2v_1(u'_{12}-w'_{12})+\partial_2\Big((v_1+v_2)(u_{12}+w_{12})-2v_1v_2\Big)$$
$$=\begin{vmatrix} v_1 & v'_1 & 1\\ v_2 & -v'_2 & 1\\ u_{12} & -u'_{12} & 1\end{vmatrix}+\begin{vmatrix} v_1 & -v'_1 & 1\\ v_2 & -v'_2 & 1\\ w_{12} & w'_{12} & 1\end{vmatrix}. \tag{2.17}$$

Proof. If we note that $u'_{ij}=\partial_i u_{ij}=-\partial_j u_{ij}$ and $w'_{ij}=\partial_i w_{ij}=\partial_j w_{ij}$, then equalities (2.15), (2.16) and (2.17) are clear by direct calculations. □

In the case when $u(t)=C_1\wp(t)+C_2$, the function $u(t)$ is even and satisfies

$$\begin{vmatrix} u(x) & u'(x) & 1\\ u(y) & u'(y) & 1\\ u(z) & u'(z) & 1\end{vmatrix}=0 \quad\text{for}\quad x+y+z=0, \tag{2.18}$$

which is clear from (2.7). Hence we have

Corollary 2.2. *For given even functions $u(t)$, $v(t)$ and $w(t)$, put*

$$\begin{aligned}\phi_{ij}&=u(x_i-x_j)+w(x_i+x_j),\\ \psi_{ij}&=u(x_i-x_j)-w(x_i+x_j),\\ v_k&=v(x_k).\end{aligned} \tag{2.19}$$

Then clearly

$$\phi_{ji}=\phi_{ij},\ \psi_{ji}=\psi_{ij}\quad\textit{and}\quad \partial_i\phi_{ij}+\partial_j\psi_{ij}=0. \tag{2.20}$$

i) *If*

$$u(t)=w(t)=C_1\wp(t)+C_2 \tag{2.21}$$

or

$$u(t)=C_1\wp(t)+C_2\quad\textit{and}\quad w(t)=C_3, \tag{2.22}$$

then

$$\partial_i(\psi_{ij}\psi_{ik})+\partial_j(\psi_{ij}\psi_{jk})+\partial_k(\psi_{ik}\psi_{jk})=0 \tag{2.23}$$

and

$$\partial_i(\phi_{ij}\psi_{ik})+\partial_j(\phi_{ij}\psi_{jk})+\partial_k(\phi_{ik}\phi_{jk})=0. \tag{2.24}$$

ii) *If*

$$u(t) = v(t) = w(t) = C_1\wp(t) + C_2 \tag{2.25}$$

or

$$u(t) = w(t) = C_1\wp(t) + C_2 \quad \text{and} \quad v(t) = C_3, \tag{2.26}$$

then

$$(\partial_i v_i)\psi_{ij} + 2v_i(\partial_i\psi_{ij}) + \partial_j\big((v_i+v_j)\phi_{ij} - 2v_iv_j\big) = 0. \tag{2.27}$$

Here the indices i, j and k are mutually different and C_1, C_2 and C_3 are any complex numbers.

Remark 2.3. The equation (2.18) for u and its generalization are studied by [BP], [BBy], [OS] in connection with integrable systems and equations similar to those in Corollary 2.2 are discussed in [BBu]

3. A fundamental integral of type A_{n-1} and D_n.

In this section we use the notation in Corollary 2.2. Put

$$\psi_{\{1,\dots,2k\}} = \frac{1}{2^k k!}\sum_{w\in\mathfrak{S}_{2k}} w(\psi_{12}\psi_{34}\psi_{56}\cdots\psi_{2k-1,2k}). \tag{3.1}$$

We sometimes denote by $\psi_{i,j}$ in place of ψ_{ij} to distinguish the suffices.

Define

$$\Delta = -\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 + \sum_{1\le i<j\le n}\phi_{ij} \tag{3.2}$$

and

$$\Delta_n = \sum_{0\le\nu\le[\frac{n}{2}]}\frac{1}{(2\nu)!(n-2\nu)!}\sum_{w\in\mathfrak{S}_n} w(\psi_{\{1,\dots,2\nu\}}\partial_{2\nu+1}\cdots\partial_n). \tag{3.3}$$

Let $\bar\Delta$ and $\bar\Delta_n$ be the functions of (x,ξ) obtained by replacing ∂_i by ξ_i for $i=1,\dots,n$ in (3.2) and (3.3), respectively.

The Poisson bracket of functions $f(x,\xi)$ and $g(x,\xi)$ is defined by

$$\{f,g\} = \sum_{j=1}^{n}\Big(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial\xi_j}\frac{\partial f}{\partial x_j}\Big). \tag{3.4}$$

Then we have

PROPOSITION 3.1. *Suppose u and w are given by* (2.21) *or* (2.22). *Then*

$$[\Delta_n, \Delta] = \{\bar\Delta_n, \bar\Delta\} = 0.$$

Proof. Put $Q = [\Delta_n,\Delta]$ and suppose $Q\ne 0$. Since ${}^t\Delta_n = (-1)^n\Delta_n$ and ${}^t\Delta = \Delta$, ${}^tQ = {}^t[\Delta_n,\Delta] = -[{}^t\Delta_n, {}^t\Delta] = (-1)^{n+1}Q$. Hence the order of Q equals $n-2m-1$ with a suitable positive integer m. Then by using (2.20) and (2.23), the coefficient of $\partial_{2m+2}\cdots\partial_n$ in the $\mathfrak{S}_n$-invariant operator Q equals

$$\sum_{i=1}^{2m+1}\psi_{\{1,\dots,2m+1\}\setminus\{i\}}\partial_i\sum_{\mu<\nu}\phi_{\mu\nu} + \sum_{j=2m+2}^{n}\partial_j\psi_{\{1,\dots,2m+1,j\}}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2m+1} \sum_{\substack{j\neq i\\ 1\le j\le n}} \psi_{\{1,\dots,2m+1\}\setminus\{i\}}\partial_i\phi_{ij} + \sum_{j=2m+2}^{n}\sum_{i=1}^{2m+1}\psi_{\{1,\dots,2m+1\}\setminus\{i\}}\partial_j\psi_{ij}\\
&= -\sum_{i=1}^{2m+1} \sum_{\substack{j\neq i\\ 1\le j\le 2m+1}} \psi_{\{1,\dots,2m+1\}\setminus\{i\}}\partial_j\psi_{ij}\\
&= -\sum_{i=1}^{2m+1} \sum_{\substack{j\neq i\\ 1\le j\le 2m+1}} \sum_{\substack{k\neq i,\ k\neq j\\ 1\le k\le 2m+1}} \psi_{\{1,\dots,2m+1\}\setminus\{i,j,k\}}\psi_{jk}\partial_j\psi_{ij}\\
&= -\sum_{1\le i<j<k\le 2m+1} \psi_{\{1,\dots,2m+1\}\setminus\{i,j,k\}}\bigl(\partial_i(\psi_{ij}\psi_{ik}) + \partial_j(\psi_{ij}\psi_{jk}) + \partial_k(\psi_{ik}\psi_{jk})\bigr)\\
&= 0,
\end{aligned}
$$

which contradicts to the fact that the order of Q equals $n-2m-1$. Thus we have $[\Delta_n, \Delta] = 0$ and by the same calculation we have also $\{\bar{\Delta}_n, \bar{\Delta}\} = 0$. □

The following theorem is known but we repeat it here for the completeness.

Theorem 3.2 (Type A_{n-1}. [OP2], [OS, Theorem 5.2 and Remark 5.3]). *Put*

$$u(t) = C_1\wp(t) + C \quad \text{and} \quad w(t) = 0.$$

Regard Δ_n as a polynomial function of C, denote it by $P_n(C)$ and put $P_{n-1}(C) = [P_n(C), x_1 + \cdots + x_n]$. Then

$$\bigl[P_n(C), P_n(C')\bigr] = \bigl[P_n(C), P_{n-1}(C')\bigr] = \bigl[P_{n-1}(C), P_{n-1}(C')\bigr] = 0 \tag{3.5}$$

for any C, $C' \in \mathbb{C}$.

Defining P_k by $P_n(C) = \sum_{0\le\nu\le[\frac{n}{2}]} P_{n-2\nu}C^\nu$ and $P_{n-1}(C) = \sum_{0\le\nu\le[\frac{n-1}{2}]}(2\nu+1)P_{n-2\nu-1}C^\nu$, we have

$$
\begin{aligned}
P_k = \sum_{0\le j\le[\frac{k}{2}]} \frac{C_1^j}{2^j j!(k-2j)!} \sum_{w\in\mathfrak{S}_n} w\bigl(&\wp(x_1-x_2)\wp(x_3-x_4)\cdots\\
&\cdot\wp(x_{2j-1}-x_{2j})\partial_{2j+1}\cdots\partial_k\bigr)
\end{aligned} \tag{3.6}
$$

and $P_1, \dots, P_n$ are the required operators for the Schrödinger operator (1.1) *with* (1.7) *and* (1.8) *when W is of type A_{n-1}.*

By replacing ∂_i and $[\ ,\ ]$ by ξ_i and $\{\ ,\ \}$, respectively, we have the same claim for the corresponding Hamiltonian system.

Proof. Put $Q = [P_n(C), P_n(C')]$ and suppose $Q \neq 0$. Since ${}^tP_n(C) = P_n(C)^-$, we have $-{}^tQ = Q^-$. By Jacobi's identity for $[\ ,\ ]$, we have $[Q, \Delta] = 0$, which implies that the coefficients of the terms of highest order in Q are polynomial functions of x (cf. [Be, Lemma 2.5] or [OS, Lemma 3.1 and Lemma 3.5]). Hence if ω_1 is finite, the coefficients are constant because of their periodicity. Moreover by the analytic continuation we can conclude that the coefficients are constant even if $\omega_1 = \infty$. This contradicts to $-{}^tQ = Q^- \neq 0$. Hence we have $[P_n(C), P_n(C')] = 0$.

Note that $[P_{n-1}(C), \Delta] = -[\partial_1 + \cdots + \partial_n, P_n(C)] + [[P_n(C), \Delta], x_1 + \cdots + x_n] = 0$. Hence the same argument as above shows (3.5) by replacing $(P_n(C), P_n(C'))$ by $(P_n(C), P_{n-1}(C'))$ or $(P_{n-1}(C), P_{n-1}(C'))$.

The remaining part of the theorem is clear from the definition of $P_n(C)$. □

4. A functional differential equation. Retain the notation in Corollary 2.2 and the previous section and put

$$\Delta_{\{1,\dots,k\}} = \sum_{0\le\nu\le[\frac{k}{2}]} \frac{1}{(2\nu)!(k-2\nu)!} \sum_{w\in\mathfrak{S}_k} w(\psi_{\{1,\dots,2\nu\}}\partial_{2\nu+1}\cdots\partial_k) \tag{4.1}$$

for $k=1,\dots,n$ (cf. (3.3)). Then we have easily

LEMMA 4.1.

$$\begin{aligned} \big[\Delta_{\{1,\dots,k\}}, x_k\big] &= \Delta_{\{1,\dots,k-1\}}, \\ \Delta_{\{1,\dots,k\}} &= \Delta_{\{1,\dots,k-1\}}\partial_k + \sum_{1\le\nu\le k-1} \psi_{\nu k}\Delta_{\{1,\dots,k-1\}\setminus\{\nu\}}. \end{aligned}$$

Let $q_{\{1,\dots,k\}}$ be suitable symmetric functions of $(x_1,\dots,x_k)$ for $k=1,\dots,n$ and put $q_\varnothing = 1$. For even functions $v_j = v(x_j)$, we examine the condition such that the operators

$$\begin{aligned} P &= -\frac{1}{2}\sum_{j=1}^n \partial_j^2 + \sum_{1\le i<j\le n} \phi_{ij} + \sum_{j=1}^n v_j, \\ P_n &= \sum_{k=0}^n \frac{1}{k!(n-k)!} \sum_{w\in\mathfrak{S}_n} w(q_{\{1,\dots,k\}}\Delta^2_{\{k+1,\dots,n\}}) \end{aligned} \tag{4.2}$$

satisfy $[P_n, P] = 0$. We denote by $\bar\Delta_{\{1,\dots,k\}}$ and $\bar P$ and $\bar P_n$ the functions of (x,ξ) obtained by replacing ∂_i by ξ_i in the above definition of the corresponding operators. We introduce symmetric functions $T_{\{1,\dots,k\}}$ of $(x_1,\dots,x_k)$ such that

$$q_{\{1,\dots,k\}} = \sum_{I_1\amalg\cdots\amalg I_\nu = \{1,\dots,k\}} T_{I_1}\cdots T_{I_\nu}, \tag{4.3}$$

where the sum runs over all different partitions of $\{1,\dots,k\}$. For example

$$\begin{aligned} q_\varnothing = T_\varnothing = 1,\quad q_{\{1\}} = T_{\{1\}},\quad q_{\{1,2\}} = T_{\{1\}}T_{\{2\}} + T_{\{1,2\}}, \\ q_{\{1,2,3\}} = T_{\{1\}}T_{\{2\}}T_{\{3\}} + T_{\{1\}}T_{\{2,3\}} + T_{\{2\}}T_{\{3,1\}} + T_{\{3\}}T_{\{1,2\}} + T_{\{1,2,3\}}. \end{aligned}$$

THEOREM 4.2. *Retain the above notation. Suppose*

$$\begin{cases} T_{\{1\}} &= -2v_1, \\ \partial_k T_{\{1,\dots,k\}} &= \displaystyle\sum_{j=1}^{k-1}\Big(2T_{\{1,\dots,k-1\}}(\partial_j\psi_{jk}) + (\partial_j T_{\{1,\dots,k-1\}})\psi_{jk}\Big) \\ & \qquad \textit{for}\quad k=2,\dots,n. \end{cases} \tag{4.4}$$

Then $\big[P_n, P\big] = \big\{\bar P_n, \bar P\big\} = 0.$

Proof. It follows from Proposition 3.1 that

$$\begin{aligned} \big[P_n, P\big] &= \Big[\sum_{k=0}^n \frac{1}{k!(n-k)!}\sum_{w\in\mathfrak{S}_n} w\big(q_{\{1,\dots,k\}}\Delta^2_{\{k+1,\dots,n\}}\big), P\Big] \\ &= \sum_{k=0}^n \frac{1}{k!(n-k)!}\sum_{w\in\mathfrak{S}_n} w\Big(\Big[q_{\{1,\dots,k\}}\Delta^2_{\{k+1,\dots,n\}}, \end{aligned}$$

$$
\begin{aligned}
&\quad - \tfrac{1}{2}\partial_1^2 - \cdots - \tfrac{1}{2}\partial_k^2 + \sum_{\nu=k+1}^{n} \bigl(v_\nu + \sum_{\mu=1}^{k} \phi_{\mu\nu}\bigr)\Bigr]\Bigr)\\
&= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{w\in\mathfrak{S}_n} w\Bigl(q_{\{1,\dots,k\}}\Bigl[\Delta^2_{\{k+1,\dots,n\}}, \sum_{\nu=k+1}^{n}\bigl(v_\nu + \sum_{\mu=1}^{k}\phi_{\mu\nu}\bigr)\Bigr]\\
&\quad + \tfrac{1}{2}\Bigl[\partial_1^2+\cdots+\partial_k^2, q_{\{1,\dots,k\}}\Bigr]\Delta^2_{\{k+1,\dots,n\}}\Bigr).
\end{aligned}
$$

Hence by Lemma 4.1 and (2.20) we have

$$
\begin{aligned}
\{\bar P_n, \bar P\} &= \sum_{k=0}^{n-1} \frac{1}{k!(n-k-1)!} \sum_{w\in\mathfrak{S}_n} w\Bigl(2q_{\{1,\dots,k\}}\bigl(v'_{k+1} + \sum_{j=1}^{k}\partial_{k+1}\phi_{j,k+1}\bigr)\\
&\qquad \cdot \bar\Delta_{\{k+2,\dots,n\}}\bar\Delta_{\{k+1,\dots,n\}}\Bigr)\\
&\quad + \sum_{k=1}^{n} \frac{1}{(k-1)!(n-k)!} \sum_{w\in\mathfrak{S}_n} w\Bigl((\partial_k q_{\{1,\dots,k\}})\bar\Delta^2_{\{k+1,\dots,n\}}\xi_k\Bigr)\\
&= \sum_{k=1}^{n} \frac{1}{(k-1)!(n-k)!} \sum_{w\in\mathfrak{S}_n} w\Bigl(2q_{\{1,\dots,k-1\}}\bigl(v'_k + \sum_{j=1}^{k-1}\partial_k\phi_{jk}\bigr)\\
&\qquad \cdot \bar\Delta_{\{k+1,\dots,n\}}\bar\Delta_{\{k,\dots,n\}}\Bigr)\\
&\quad + \sum_{k=1}^{n} \frac{1}{(k-1)!(n-k)!} \sum_{w\in\mathfrak{S}_n} w\Bigl((\partial_k q_{\{1,\dots,k\}})\bigl(\bar\Delta_{\{k,\dots,n\}}\\
&\quad - \sum_{j=k+1}^{n}\psi_{kj}\bar\Delta_{\{k+1,\dots,n\}\setminus\{j\}}\bigr)\bar\Delta_{\{k+1,\dots,n\}}\Bigr)\\
&= \sum_{k=1}^{n} \frac{1}{(k-1)!(n-k)!} \sum_{w\in\mathfrak{S}_n} w\Bigl(\Bigl(2q_{\{1,\dots,k-1\}}v'_k\\
&\quad - \sum_{j=1}^{k-1}\bigl(2q_{\{1,\dots,k-1\}}\partial_j\psi_{jk} + (\partial_j q_{\{1,\dots,k-1\}})\psi_{jk}\bigr)\\
&\quad + \partial_k q_{\{1,\dots,k\}}\Bigr)\bar\Delta_{\{k+1,\dots,n\}}\bar\Delta_{\{k,\dots,n\}}\Bigr).
\end{aligned}
$$

Here the last equality follows from

$$
\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{(k-1)!(n-k)!} \sum_{w\in\mathfrak{S}_n} w\Bigl((\partial_k q_{\{1,\dots,k\}})\sum_{j=k+1}^{n}\psi_{kj}\bar\Delta_{\{k+1,\dots,n\}\setminus\{j\}}\bar\Delta_{\{k+1,\dots,n\}}\Bigr)\\
&\quad = \sum_{\ell=1}^{n} \frac{1}{(\ell-1)!(n-\ell)!} \sum_{w\in\mathfrak{S}_n} w\Bigl(\sum_{i=1}^{\ell-1}(\partial_i q_{\{1,\dots,\ell-1\}})\psi_{i\ell}\bar\Delta_{\{\ell+1,\dots,n\}}\bar\Delta_{\{\ell,\dots,n\}}\Bigr).
\end{aligned}
$$

Hence if $q_{\{1,\dots,k\}}$ satisfy

$$
\begin{aligned}
\partial_k q_{\{1,\dots,k\}} &= -2q_{\{1,\dots,k-1\}}v'_k\\
&\qquad + \sum_{j=1}^{k-1}\Bigl(2q_{\{1,\dots,k-1\}}(\partial_j\psi_{jk}) + (\partial_j q_{\{1,\dots,k-1\}})\psi_{jk}\Bigr)
\end{aligned}
\tag{4.5}
$$

for $k = 1, \dots, n$, then $\{\bar{P}_n, \bar{P}\} = 0$. Under the assumption of the theorem, the right hand side of (4.5) equals

$$\begin{aligned}
&\sum_{I_1 \amalg \cdots \amalg I_\nu = \{1,\dots,k-1\}} \Big(T_{I_1} \cdots T_{I_\nu} \partial_k T_{\{k\}} \\
&\quad + \sum_{j=1}^{k-1} \Big(2T_{I_1} \cdots T_{I_\nu} (\partial_j \psi_{jk}) + \big(\partial_j (T_{I_1} \cdots T_{I_\nu})\big) \psi_{jk} \Big) \Big) \\
&= \sum_{I_1 \amalg \cdots \amalg I_\nu = \{1,\dots,k-1\}} \partial_k \Big(T_{I_1} \cdots T_{I_\nu} T_{\{k\}} + \sum_{\mu=1}^{\nu} T_{I_1} \cdots T_{I_\mu \cup \{k\}} \cdots T_{I_\nu} \Big),
\end{aligned}$$

which equals the left hand side of (4.5) and hence we have $\{\bar{P}_n, \bar{P}\} = 0$.

Thus we have

$$\begin{aligned}
[P_n, P] = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{w \in \mathfrak{S}_n} w \Big(& q_{\{1,\dots,k\}} \sum_{\nu=k+1}^{n} \big(\partial_\nu^2 v_\nu + \sum_{1 \le \mu \le k} \partial_\nu^2 \phi_{\mu\nu}\big) \\
& \cdot \Delta^2_{\{k+1,\dots,n\} \setminus \{\nu\}} + \frac{1}{2} \sum_{\nu=1}^{k} (\partial_\nu^2 q_{\{1,\dots,k\}}) \Delta^2_{\{k+1,\dots,n\}} \Big)
\end{aligned}$$

and therefore $[P_n, P]$ is clearly self-adjoint. Since P and P_n are self-adjoint, $[P_n, P]$ is skew self-adjoint. Hence we can conclude $[P_n, P] = 0$. □

REMARK 4.3. *Let* $T^1_{\{1,\dots,k\}}$ *and* $T^2_{\{1,\dots,k\}}$ *be solutions of* (4.4) for $(u, v, w) = (f, g_1, h)$ and (f, g_2, h), respectively. Then $C^{k-1} C' T^1_{\{1,\dots,k\}} + C^{k-1} C'' T^2_{\{1,\dots,k\}}$ are solutions for $(u, v, w) = (Cf, C'g_1 + C''g_2, Ch)$. Here C, C' and C'' are any complex numbers and $k = 1, \dots, n$.

5. Solutions of the functional differential equation.

In this section we shall construct elliptic solutions of (4.4) in the case when

$$u(t) = w(t) = C\wp(t) + C' \tag{5.1}$$

with C, $C' \in \mathbb{C}$ (cf. (2.19)).

Retain the notation in the previous section and assume (5.1).

LEMMA 5.1. *Under the notation in Corollary 2.2, the functions*

$$\begin{cases} \Phi_0 = 1, \\ \Phi_n = (-1)^n \displaystyle\sum_{w \in \mathfrak{S}_n} w\big(\phi_{01}\phi_{12} \cdots \phi_{n-1,n}\big) \quad \textit{for} \quad n \ge 1 \end{cases}$$

satisfy

$$\partial_n \Phi_n = \partial_0 (\Phi_{n-1} \psi_{0n}) + \sum_{j=1}^{n-1} \Big(2\Phi_{n-1} (\partial_j \psi_{jn}) + (\partial_j \Phi_{n-1}) \psi_{jn} \Big). \tag{5.2}$$

Proof. For $j = 1, \dots, n-1$

$$
\begin{aligned}
&(-1)^{n-1}\Big(2\Phi_{n-1}(\partial_j\psi_{jn}) + (\partial_j\Phi_{n-1})\psi_{jn}\Big)\\
&\qquad = \sum_{\substack{w\in\mathfrak{S}_{n-1}\\ w(n-1)=j}} (\partial_j\psi_{jn})w\big(\phi_{01}\phi_{12}\cdots\phi_{n-2,n-1}\big)\\
&\qquad\quad + \sum_{\substack{w\in\mathfrak{S}_{n-1}\\ w(n-1)=j}} \big(\partial_j(\phi_{w(n-2)j}\psi_{jn})\big)w\big(\phi_{01}\phi_{12}\cdots\phi_{n-3,n-2}\big)\\
&\qquad\quad + \sum_{\substack{w\in\mathfrak{S}_{n-1}\\ w(n-1)\neq j}} \big(\partial_j(\phi_{w(w^{-1}(j)-1)j}\psi_{jn})\big) \prod_{\substack{i\neq w(j)\\ 1\le i\le n-1}} w(\phi_{i-1,i})\\
&\qquad\quad + \sum_{\substack{w\in\mathfrak{S}_{n-1}\\ w(n-1)\neq j}} \big(\partial_j(\phi_{jw(w^{-1}(j)+1)}\psi_{jn})\big) \prod_{\substack{i\neq w(j)\\ 1\le i\le n-1}} w(\phi_{i,i+1}).
\end{aligned}
$$

Hence it follows from (2.20) and (2.24) that the right hand side of (5.2) equals

$$
\begin{aligned}
&(-1)^n\Bigg(\sum_{j=1}^{n-1}\sum_{\substack{w\in\mathfrak{S}_{n-1}\\ w(n-1)=j}} \partial_n\Big(\phi_{jn}\prod_{i=1}^{n-1}w(\phi_{i-1,i})\Big)\\
&\quad - \sum_{w\in\mathfrak{S}_{n-1}}\sum_{k=1}^{n-1} w\Big(\partial_k\big(\phi_{k-1,k}\psi_{k,n}\big) + \partial_{k-1}\big(\phi_{k-1,k}\psi_{k-1,n}\big)\Big) \prod_{\substack{i\neq k\\ 1\le i\le n-1}} w(\phi_{i-1,i})\Bigg)\\
&= (-1)^n\Bigg(\sum_{\substack{w\in\mathfrak{S}_n\\ w(n)=n}} \partial_n\Big(\prod_{i=1}^{n} w(\phi_{i-1,i})\Big)\\
&\quad + \sum_{w\in\mathfrak{S}_{n-1}}\sum_{k=1}^{n-1} w\Big(\partial_n\big(\phi_{k-1,n}\phi_{n,k}\big) \prod_{\substack{i\neq k\\ 1\le i\le n-1}} \phi_{i-1,i}\Big)\Bigg)\\
&= \partial_n\Phi_n.
\end{aligned}
$$

Thus we have the lemma. □

LEMMA 5.2. *Suppose there exist a symmetric function $g(s,t)$ of (s,t) such that*

$$2v_1(\partial_1\psi_{12}) + (\partial_1 v_1)\psi_{12} = \partial_2\Big(2g_{12} - (v_1+v_2)\phi_{12}\Big) \tag{5.3}$$

by denoting $g_{ij} = g(x_i, x_j)$. Put

$$
\begin{aligned}
S^o_{\{1\}} &= -2v_1,\\
S^o_{\{1,\dots,k\}} &= 2(-1)^k \sum_{w\in\mathfrak{S}_k} w\big(v_1\phi_{12}\phi_{23}\cdots\phi_{k-1,k}\big) \quad \textit{for} \quad k\ge 1.
\end{aligned}
\tag{5.4}
$$

Then

$$\partial_k\bigl(S^o_{\{1,\dots,k\}}-(-1)^k\sum_{w\in\mathfrak{S}_{k-1}}4w(g_{1k}\phi_{12}\phi_{23}\cdots\phi_{k-2,k-1})\bigr) = \sum_{j=1}^{k-1}\Bigl(2S^o_{\{1,\dots,k-1\}}(\partial_j\psi_{jk})+(\partial_j S^o_{\{1,\dots,k-1\}})\psi_{jk}\Bigr). \tag{5.5}$$

i) *If*

$$2v_1(\partial_1\psi_{12})+(\partial_1 v_1)\psi_{12}=\partial_2\bigl(2\lambda v_1v_2-(v_1+v_2)\phi_{12}\bigr) \tag{5.6}$$

with a complex number λ, then

$$\begin{cases} S_{\{1\}} &= -2v_1,\\ \partial_k S_{\{1,\dots,k\}} &= \displaystyle\sum_{j=1}^{k-1}\Bigl(2S_{\{1,\dots,k-1\}}(\partial_j\psi_{jk})+(\partial_j S_{\{1,\dots,k-1\}})\psi_{jk}\Bigr)\end{cases} \tag{5.7}$$

by putting

$$S_{\{1,\dots,k\}}=\sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k\}}(-\lambda)^{\nu-1}(\nu-1)!S^o_{I_1}\cdots S^o_{I_\nu}\quad \textit{for}\quad k\ge 1. \tag{5.8}$$

ii) *If there exist even functions $f(t)$ and $h(t)$ and complex numbers λ, λ' and λ'' such that*

$$\begin{cases} 2v_1(\partial_1\psi_{12})+(\partial_1 v_1)\psi_{12} &= \partial_2\bigl(2\lambda' f_1f_2+2\lambda''(h_1+h_2)-(v_1+v_2)\phi_{12}\bigr),\\ 2f_1(\partial_1\psi_{12})+(\partial_1 f_1)\psi_{12} &= \partial_2\bigl(-(f_1+f_2)\phi_{12}\bigr),\\ 2h_1(\partial_1\psi_{12})+(\partial_1 h_1)\psi_{12} &= \partial_2\bigl(2\lambda(f_1+f_2)-(h_1+h_2)\phi_{12}\bigr),\end{cases} \tag{5.9}$$

the following functions $S_{\{1,\dots,k\}}$ satisfy (5.7).

$$\begin{aligned} S_{\{1,\dots,k\}} &= S^o_{\{1,\dots,k\}}-\sum_{I_1\amalg I_2=\{1,\dots,k\}}(\lambda' S'_{I_1}S'_{I_2}+\lambda'' S''_{I_1}D_{I_2}+\lambda'' D_{I_1}S''_{I_2})\\ &\qquad+\sum_{I_1\amalg I_2\amalg I_3=\{1,\dots,k\}}2\lambda\lambda''(S'_{I_1}D_{I_2}D_{I_3}+D_{I_1}S'_{I_2}D_{I_3}+D_{I_1}D_{I_2}S'_{I_3}),\\ S'_{\{1,\dots,k\}} &= 2(-1)^k\sum_{w\in\mathfrak{S}_k}w\bigl(f_1\phi_{12}\phi_{23}\cdots\phi_{k-1,k}\bigr),\\ S''_{\{1,\dots,k\}} &= 2(-1)^k\sum_{w\in\mathfrak{S}_k}w\bigl(h_1\phi_{12}\phi_{23}\cdots\phi_{k-1,k}\bigr),\\ D_{\{1,\dots,k\}} &= 2(-1)^k\sum_{w\in\mathfrak{S}_k}w\bigl(\phi_{12}\phi_{23}\cdots\phi_{k-1,k}\bigr).\end{aligned}$$

Here we put $f_j=f(x_j)$ and $h_j=h(x_j)$ for $j\ge 1$ and $S'_{\{1\}}=-2f_1$, $S''_{\{1\}}=-2h_1$ and $D_{\{1\}}=-2$.

Proof. Owing to (5.3) and (2.20) and Lemma 5.1, the right hand side of (5.5) equals

$$2(-1)^{k-1}\sum_{w\in\mathfrak{S}_{k-1}}w\Bigl(2\sum_{j=1}^{k-1}v_1\phi_{12}\cdots\phi_{k-2,k-1}(\partial_j\psi_{jk})$$

$$
\begin{aligned}
&+(\partial_1 v_1)\phi_{12}\cdots\phi_{k-2,k-1}\psi_{1k}+\sum_{j=1}^{k-1}v_1\bigl(\partial_j(\phi_{12}\cdots\phi_{k-2,k-1})\bigr)\psi_{jk}\Bigr)\\
&=2(-1)^k\sum_{w\in\mathfrak{S}_{k-1}}w\Bigl(\partial_k(v_k\phi_{k1}\phi_{12}\cdots\phi_{k-2,k-1}-2g_{1k}\phi_{12}\cdots\phi_{k-2,k-1})\\
&\quad-v_1\phi_{12}\cdots\phi_{k-2,k-1}(\partial_1\psi_{1k})-2\sum_{j=2}^{k-1}v_1\phi_{12}\cdots\phi_{k-2,k-1}(\partial_j\psi_{jk})\\
&\quad-\sum_{j=1}^{k-1}v_1\bigl(\partial_j(\phi_{12}\cdots\phi_{k-2,k-1})\bigr)\psi_{jk}\Bigr)\\
&=2(-1)^k\sum_{w\in\mathfrak{S}_{k-1}}w\Bigl(\partial_k\Bigl(v_k\phi_{k1}\phi_{12}\cdots\phi_{k-2,k-1}-2g_{1k}\phi_{12}\cdots\phi_{k-2,k-1}\\
&\quad+\frac{1}{(k-1)!}\sum_{\substack{w'\in\mathfrak{S}_k\\ w'(1)=1}}w'(v_1\phi_{12}\phi_{23}\cdots\phi_{k-1,k})\Bigr)\Bigr)\\
&=\partial_k\bigl(S^o_{\{1,\dots,k\}}-(-1)^k\sum_{w\in\mathfrak{S}_{k-1}}4w(g_{1k}\phi_{12}\phi_{23}\cdots\phi_{k-2,k-1})\bigr).
\end{aligned}
$$

Hence we have (5.5) and if (5.6) holds, we have

$$
(-1)^k\sum_{w\in\mathfrak{S}_{k-1}}4w(g_{1k}\phi_{12}\phi_{23}\cdots\phi_{k-2,k-1})=\lambda S^o_{\{1,\dots,k-1\}}S^o_{\{k\}}
$$

and therefore the right hand side of the second equation of (5.7) equals

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k-1\}}(-\lambda)^{\nu-1}(\nu-1)!\Bigl(2S^o_{I_1}\cdots S^o_{I_\nu}(\partial_j\psi_{jk})\\
&\quad+(\partial_j S^o_{I_1}\cdots S^o_{I_\nu})\psi_{jk}\Bigr)\\
&=\sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k-1\}}(-\lambda)^{\nu-1}(\nu-1)!\sum_{\mu=1}^{\nu}\partial_k\Bigl(S^o_{I_1}\cdots S^o_{I_\mu\cup\{k\}}\cdots S^o_{I_\nu}\\
&\quad-\lambda S^o_{I_1}\cdots S^o_{I_\mu}\cdots S^o_{I_\nu}S^o_{\{k\}}\Bigr)\\
&=\partial_k\Bigl(\sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k\}}(-\lambda)^{\nu-1}(\nu-1)!S^o_{I_1}\cdots S^o_{I_\nu}\Bigr)\\
&=\partial_k S_{\{1,\dots,k\}}.
\end{aligned}
$$

Now suppose (5.9). Then by denoting

$$
\begin{aligned}
S^1_{\{1,\dots,k\}}&=S^o_{\{1,\dots,k\}},\\
S^2_{\{1,\dots,k\}}&=\sum_{I_1\amalg I_2=\{1,\dots,k\}}S'_{I_1}S'_{I_2},\\
S^3_{\{1,\dots,k\}}&=\sum_{I_1\amalg I_2=\{1,\dots,k\}}(S''_{I_1}D_{I_2}+D_{I_1}S''_{I_2}),
\end{aligned}
$$

$$S^4_{\{1,\dots,k\}} = \sum_{I_1 \amalg I_2 \amalg I_3 = \{1,\dots,k\}} (S'_{I_1} D_{I_2} D_{I_3} + D_{I_1} S'_{I_2} D_{I_3} + D_{I_1} D_{I_2} S'_{I_3})$$

and

$$F^\nu_k = (-1)^k \Big(\partial_k S^\nu_{\{1,\dots,k\}} - \sum_{j=1}^{k-1} \big(2S^\nu_{\{1,\dots,k-1\}} \partial_j \psi_{jk} + (\partial_j S^\nu_{\{1,\dots,k-1\}}) \psi_{jk} \big) \Big)$$

for $\nu = 1, \dots, 4$, it follows from (5.5) that

$$\begin{aligned}
F^1_k &= \partial_k(\lambda' S'_{\{1,\dots,k-1\}} S'_{\{k\}} + \lambda'' S''_{\{1,\dots,k-1\}} D_{\{k\}} + \lambda'' D_{\{1,\dots,k-1\}} S''_{\{k\}}),\\
F^2_k &= \partial_k(S'_{\{1,\dots,k-1\}} S'_{\{k\}}),\\
F^3_k &= \partial_k \Big(S''_{\{1,\dots,k-1\}} D_{\{k\}} + D_{\{1,\dots,k-1\}} S''_{\{k\}}\\
&\quad + \lambda \sum_{I_1 \amalg I_2 = \{1,\dots,k-1\}} \big((S'_{I_1} D_{\{k\}} + D_{I_1} S'_{\{k\}}) D_{I_2} + D_{I_1}(S'_{I_2} D_{\{k\}} + D_{I_2} S'_{\{k\}}) \big) \Big),\\
F^4_k &= \partial_k(\sum_{I_1 \amalg I_2 = \{1,\dots,k-1\}} (S'_{I_1} D_{I_2} D_{\{k\}} + D_{I_1} S'_{I_2} D_{\{k\}} + D_{I_1} D_{I_2} S'_{\{k\}}).
\end{aligned}$$

Since $\partial_k\big((S'_{I_1} D_{I_2} + D_{I_1} S'_{I_2}) D_{\{k\}}\big) = 0$ in the above, we have

$$F^1_k - \lambda' F^2_k - \lambda'' F^3_k + 2\lambda\lambda'' F^4_k = 0,$$

which implies (5.7). Thus we have completed the proof of the lemma. □

Definition 5.3. For given even functions f and g of t, we define

$$\begin{aligned}
\Phi_{\{1,\dots,k\}}(f,g) &= \sum_{w \in W(B_k)} w\big(f(x_1) g(x_1 - x_2) g(x_2 - x_3) \cdots g(x_{k-1} - x_k)\big),\\
\Theta_{\{1,\dots,k\}}(f,g) &= \sum_{I_1 \amalg \cdots \amalg I_\nu = \{1,\dots,k\}} (-1)^{\nu-1} (\nu-1)! \Phi_{I_1}(f,g) \cdots \Phi_{I_\nu}(f,g)
\end{aligned}$$

for $k \geq 1$. Here we note that $\Theta_{\{1\}}(f,g) = \Phi_{\{1\}}(f,g) = 2f(x_1)$ and $\Theta_\varnothing(f,g) = \Phi_\varnothing(f,g) = 0$.

Proposition 5.4. *Suppose*

$$u(t) = w(t) = C_5 \wp(t), \quad v(t) = \sum_{j=1}^4 C_j \wp(t + \omega_j) - \frac{C_0}{2}$$

with $C_0, \dots, C_5 \in \mathbb{C}$. *Then* (4.4) *holds by putting*

$$T_{\{1,\dots,k\}} = (-C_5)^{k-1} \Big(\frac{C_0}{2} \Theta_{\{1,\dots,k\}}(1, \wp(t)) - \sum_{j=1}^4 C_j \Theta_{\{1,\dots,k\}}(\wp(t+\omega_j), \wp(t)) \Big).$$

Proof. Suppose $C_5 = 1$. If $v(t) = \wp(t + \omega_\nu)$ with $\nu = 1, \dots, 4$, the assumption (5.3) in Lemma 5.2 holds with $g_{12} = v_1 v_2$. In fact (2.27) means (5.3) if $v(t) = \wp(t)$ and then the coordinate transformation $x_j \mapsto x_j + \omega_\nu$ for $j = 1, \dots, n$ implies the case when $v(t) = \wp(t + \omega_\nu)$. If v is constant, then (5.3) is also valid with $g_{12} = v_1 v_2$. Hence the proposition follows from Lemma 5.2 and Remark 4.3. □

Remark 5.5. Since we may put $g_{12} = 0$ in the above proof when v is constant, we may replace $\Theta_{\{1,\dots,k\}}(1, \wp(t))$ by $\Phi_{\{1,\dots,k\}}(1, \wp(t))$ in Proposition 5.4.

6. Degenerate solutions of the functional differential equation. We give trigonometric and rational solutions of (4.4):

PROPOSITION 6.1. *For complex numbers λ, $C_0, \dots, C_5$ with $\lambda \neq 0$, put*

$$\begin{aligned}
u(t) &= w(t) = C_5 \sinh^{-2} \lambda t, \\
v(t) &= C_1 \sinh^{-2} \lambda t + C_2 \cosh^{-2} \lambda t + C_3 \sinh^2 \lambda t + \frac{C_4}{4} \sinh^2 2\lambda t - \frac{C_0}{2}, \\
g(s,t) &= C_5 \Big(C_1 \sinh^{-2} \lambda s \cdot \sinh^{-2} \lambda t - C_2 \cosh^{-2} \lambda s \cdot \cosh^{-2} \lambda t \\
&\qquad + C_4 (\sinh^2 \lambda s + \sinh^2 \lambda t + 2 \sinh^2 \lambda s \cdot \sinh^2 \lambda t) \Big).
\end{aligned}$$

Then (5.3) *holds. Moreover we have* (4.4) *with*

$$\begin{aligned}
T_I = (-C_5)^{\# I - 1} \Big(& \frac{C_0}{2} T_I^o(1) - C_1 T_I^o(\sinh^{-2} \lambda t) - C_2 T_I^o(\cosh^{-2} \lambda t) \\
& - C_3 T_I^o(\sinh^2 \lambda t) - C_4 T_I^o(\tfrac{1}{4} \sinh^2 2\lambda t) \Big)
\end{aligned}$$

by putting

$$\begin{aligned}
T_I^o(1) &= \Phi_I(1, \rho), \\
T_I^o(\sinh^{-2} \lambda t) &= \Theta_I(\sinh^{-2} \lambda t, \rho), \\
T_I^o(\cosh^{-2} \lambda t) &= -\Theta_I(-\cosh^{-2} \lambda t, \rho), \\
T_I^o(\sinh^2 \lambda t) &= \Phi_I(\sinh^2 \lambda t, \rho), \\
T_I^o(\tfrac{1}{4} \sinh^2 2\lambda t) &= \Phi_I(\tfrac{1}{4} \sinh^2 2\lambda t, \rho) - \sum_{I_1 \amalg I_2 = I} \Big(2\Phi_{I_1}(\sinh^2 \lambda t, \rho) \cdot \Phi_{I_2}(\sinh^2 \lambda t, \rho) \\
&\qquad + \Phi_{I_1}(\sinh^2 \lambda t, \rho) \cdot \Phi_{I_2}(1, \rho) + \Phi_{I_1}(1, \rho) \cdot \Phi_{I_2}(\sinh^2 \lambda t, \rho) \Big),
\end{aligned}$$

where $I \subset \{1, \dots, n\}$, $\rho = \sinh^{-2} \lambda t$ and the last sum runs over different partitions.

Proof. We can prove (5.3) by direct calculations but here we do it in a different way. First note that we may assume that C_5 and one of the numbers $C_1, \dots, C_4$ equal 1 and that the other 4 numbers are 0. Also by a simple change of coordinates we may assume $\lambda = 1$.

Now put $u(t) = \wp(t) - e_1$. Then if $v(t) = \wp(t + \omega_j) - e_1$ we have (5.3) with $g_{12} = v_1 v_2$ for $j = 1, \dots, 4$. If $e_1 = e_2 = \frac{1}{3}$ and $e_3 = -\frac{2}{3}$, then $\wp(t) - e_1 = \sinh^{-2} t$ and

$$\begin{aligned}
\wp(t + \omega_3) - e_1 &= e_3 - e_1 + \frac{(e_3 - e_1)(e_3 - e_2)}{\wp(t) - e_3} \\
&= -1 + \frac{1}{\sinh^{-2} t + 1} = -\cosh^{-2} t.
\end{aligned}$$

Hence if $u(t) = \sinh^{-2} t$ and $v(t) = \sinh^{-2} t$ or $-\cosh^{-2} t$, we have (5.3) with $g_{12} = v_1 v_2$.

Put $e_1 = \frac{1}{3}$, $e_2 = \frac{1}{3} - \varepsilon$, $e_3 = -\frac{2}{3} + \varepsilon$ with $0 < |\varepsilon| << 1$. Then it follows from (2.8) that

$$\wp(t + \omega_1) - e_1 = \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(t) - e_1} = \varepsilon \sinh^2 t + o(\varepsilon).$$

Hence putting $v(t)=\wp(t+\omega_1)-e_1$, the coefficients of ε in (5.3) proves (5.3) for $(u,v)=(\sinh^{-2}t,\sinh^2 t)$ with $g(s,t)=0$.

Next suppose $v(t)=(\wp(t+\omega_1)-e_1)+(\wp(t+\omega_2)-e_1)+(e_1-e_2)$. Then

$$\begin{aligned}v(t)&=\varepsilon\Big(\frac{e_1-e_3}{\wp(t)-e_1}-\frac{e_2-e_3}{\wp(t)-e_2}\Big)=\varepsilon^2\frac{\wp(t)-e_3}{(\wp(t)-e_1)(\wp(t)-e_2)}\\&=\varepsilon^2\frac{\sinh^{-2}t+1}{\sinh^{-4}t}+o(\varepsilon^2)=\frac{\varepsilon^2}{4}\sinh^2 2t+o(\varepsilon^2)\end{aligned}$$

and (5.3) holds with

$$\begin{aligned}g(s,t)&=(\wp(s+\omega_1)-e_1)(\wp(t+\omega_1)-e_1)+(\wp(s+\omega_2)-e_1)(\wp(t+\omega_2)-e_1)\\&=(\varepsilon\sinh^2 s)(\varepsilon\sinh^2 t)+(-\varepsilon\sinh^2 s-\varepsilon)(-\varepsilon\sinh^2 t-\varepsilon)+o(\varepsilon^2)\\&=\varepsilon^2(1+\sinh^2 s+\sinh^2 t+2\sinh^2 s\cdot\sinh^2 t)+o(\varepsilon^2).\end{aligned}$$

Hence we have (5.3) if

$$\begin{cases}g(s,t)=\sinh^2 s+\sinh^2 t+2\sinh^2 s\cdot\sinh^2 t,\\(u,v)\ =(\sinh^{-2}t,\frac14\sinh^2 2t)\end{cases}$$

The remaining part of the proposition is clear from Lemma 5.2 and Remark 4.3. We can also get it from Proposition 5.4 by considering the limit as above. □

Remark 6.2. Since

$$\begin{aligned}\sinh^2\lambda s+&\sinh^2\lambda t+2\sinh^2\lambda s\cdot\sinh^2\lambda t\\&=\sinh^2\lambda s\cdot\sinh^2\lambda t+\cosh^2\lambda s\cdot\cosh^2\lambda t-1,\end{aligned}$$

we may put

$$\begin{aligned}T_I^o(\tfrac14\sinh^2 2\lambda t)&=\Phi_I(\tfrac14\sinh^2 2\lambda t,\rho)\\&-\sum_{I_1\amalg I_2=I}\Big(\Phi_{I_1}(\sinh^2\lambda t,\rho)\cdot\Phi_{I_2}(\sinh^2\lambda t,\rho)+\Phi_{I_1}(\cosh^2\lambda t,\rho)\cdot\Phi_{I_2}(\cosh^2\lambda t,\rho)\Big)\end{aligned}$$

in Proposition 6.1.

PROPOSITION 6.3. *For complex numbers $C_0,\dots,C_5$, put*

$$\begin{aligned}u(t)=w(t)&=C_5t^{-2},\\v(t)&=C_1t^{-2}+C_2t^2+C_3t^4+C_4t^6-\frac{C_0}{2},\\g(s,t)&=C_5\big(C_1s^{-2}t^{-2}+C_3(s^2+t^2)+C_4(s^4+t^4+3s^2t^2)\big).\end{aligned}$$

Then (5.3) holds. Moreover we have (4.4) with

$$T_I=(-C_5)^{\#I-1}\Big(\frac{C_0}{2}T_I^o(1)-C_1T_I^o(t^{-2})-C_2T_I^o(t^2)-C_3T_I^o(t^4)-C_4T_I^o(t^6)\Big)$$

by putting

$$\begin{aligned}T_I^o(1)&=\Phi_I(1,\rho),\\T_I^o(t^{-2})&=\Theta_I(t^{-2},\rho),\\T_I^o(t^2)&=\Phi_I(t^2,\rho),\\T_I^o(t^4)&=\Phi_I(t^4,\rho)-\sum_{I_1\amalg I_2=\{1,\dots,k\}}\big(\Phi_{I_1}(t^2,\rho)\cdot\Phi_{I_2}(1,\rho)+\Phi_{I_1}(1,\rho)\cdot\Phi_{I_2}(t^2,\rho)\big),\end{aligned}$$

$$
\begin{aligned}
T_I^o(t^6) = \Phi_I(t^6,\rho) - &\sum_{I_1 \amalg I_2 = \{1,\dots,k\}} \bigl(3\Phi_{I_1}(t^2,\rho)\cdot\Phi_{I_2}(t^2,\rho)\\
&\quad + \Phi_{I_1}(t^4,\rho)\cdot\Phi_{I_2}(1,\rho) + \Phi_{I_1}(1,\rho)\cdot\Phi_{I_2}(t^4,\rho)\bigr)\\
+ &\sum_{I_1 \amalg I_2 \amalg I_3 = \{1,\dots,k\}} 6\bigl(\Phi_{I_1}(t^2,\rho)\cdot\Phi_{I_2}(1,\rho)\cdot\Phi_{I_3}(1,\rho)\\
&\quad + \Phi_{I_1}(1,\rho)\cdot\Phi_{I_2}(t^2,\rho)\cdot\Phi_{I_3}(1,\rho) + \Phi_{I_1}(1,\rho)\cdot\Phi_{I_2}(1,\rho)\cdot\Phi_{I_3}(t^2,\rho)\bigr),
\end{aligned}
$$

where $I \subset \{1,\dots,n\}$, $\rho = t^{-2}$ and the sums run over different partitions.

Proof. Note that the proof proceeds in the same way as in the proof of Proposition 6.1. Put $u(t) = \lambda^2 \sinh^{-2}\lambda t$. Then for

$$
\begin{cases}
v(t) &= \lambda^2 \sinh^{-2}\lambda t,\\
g(s,t) &= \lambda^4 \sinh^{-2}\lambda s\cdot\sinh^{-2}\lambda t
\end{cases}
$$

or

$$
\begin{cases}
v(t) &= \lambda^{-2}\sinh^2\lambda t,\\
g(s,t) &= 0
\end{cases}
$$

or

$$
\begin{cases}
v(t) &= \lambda^{-4}(\frac{1}{4}\sinh^2 2\lambda t - \sinh^2\lambda t),\\
g(s,t) &= \lambda^{-2}(\sinh^2\lambda s + \sinh^2\lambda t + 2\sinh^2\lambda s\cdot\sinh^2\lambda t)
\end{cases}
$$

or

$$
\begin{cases}
v(t) &= \lambda^{-6}(1 - 2\sinh^2\lambda t + \frac{1}{4}\sinh^2 2\lambda t - \cosh^{-2}\lambda t),\\
g(s,t) &= \lambda^{-4}(\cosh^2\lambda s\cdot\cosh^2\lambda t + \sinh^2\lambda s\cdot\sinh^2\lambda t\\
&\qquad + \cosh^{-2}\lambda s\cdot\cosh^{-2}\lambda t - 2),
\end{cases}
$$

we have (5.3). By the analytic continuation of these $u(t)$, $v(t)$ and $g(s,t)$ to $\lambda = 0$, we have (5.3) for $u(t) = t^{-2}$ and $v(t) = t^{-2}$ or t^2 or t^4 or t^6 with $g(s,t) = s^{-2}t^{-2}$ or 0 or s^2+t^2 or $s^4+t^4+3s^2t^2$, respectively. In fact, for example, we have

$$
\begin{aligned}
&\lambda^{-6}(1 - 2\sinh^2\lambda t + \frac{1}{4}\sinh^2 2\lambda t - \cosh^{-2}\lambda t)\\
&\quad = \lambda^{-6}\bigl(1 - \sinh^2\lambda t + \sinh^4\lambda t - (1+\sinh^2\lambda t)^{-1}\bigr)\\
&\quad = \lambda^{-6}\sinh^6\lambda t + o(\lambda) = t^6 + o(\lambda),\\
&\lambda^{-4}(\cosh^2\lambda s\cdot\cosh^2\lambda t + \sinh^2\lambda s\cdot\sinh^2\lambda t + \cosh^{-2}\lambda s\cdot\cosh^{-2}\lambda t - 2)\\
&\quad = \lambda^{-4}\Bigl((1+\sinh^2\lambda s)(1+\sinh^2\lambda t) + (1+\sinh^2\lambda s)^{-1}(1+\sinh^2\lambda t)^{-1}\\
&\qquad + \sinh^2\lambda s\cdot\sinh^2\lambda t - 2\Bigr)\\
&\quad = \lambda^{-4}\Bigl((\sinh^2\lambda s + \sinh^2\lambda t)^2 + \sinh^2\lambda s\cdot\sinh^2\lambda t\Bigr) + o(\lambda)\\
&\quad = s^4 + t^4 + 3s^2t^2 + o(\lambda).
\end{aligned}
$$

The remaining part of the proposition is clear from Lemma 5.2 and Remark 4.3. We can also get it from Proposition 6.1 by taking the limit at $\lambda = 0$. □

7. Integrals of type B_n and D_n.

The argument in the preceding sections gives the integrals when W is of type B_n or D_n.

Definition 7.1. For given even function $u(t)$ and symmetric functions $T_{\{1,\dots,k\}}$ of $(x_1,\dots,x_k)$ for $k=1,\dots,n$, define $W(B_n)$-invariant differential operator

$$P(u,T)=\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\sum_{w\in\mathfrak{S}_n}w\Bigl(q_{\{1,\dots,k\}}\Delta^2_{\{k+1,\dots,n\}}\Bigr)$$

by

$$\begin{aligned}\Delta_{\{1,\dots,k\}}&=\sum_{0\le j\le[\frac{k}{2}]}\frac{1}{2^kj!(k-2j)!}\sum_{w\in W(B_k)}\varepsilon(w)w\Bigl(u(x_1-x_2)u(x_3-x_4)\cdots\\&\qquad\cdot u(x_{2j-1}-x_{2j})\partial_{2j+1}\partial_{2j+2}\cdots\partial_k\Bigr),\\q_{\{1,\dots,k\}}&=\sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k\}}T_{I_1}\cdots T_{I_\nu},\end{aligned}$$

where

$$\begin{aligned}&q_\varnothing=1,\ q_{\{1\}}=T_{\{1\}},\ q_{\{12\}}=T_{\{1\}}T_{\{2\}}+T_{\{1,2\}},\dots\\&T_{w(\{1,\dots,k\})}=w(T_{\{1,\dots,k\}}),\quad\Delta_{w(\{1,\dots,k\})}=w(\Delta_{\{1,\dots,k\}})\quad\text{for}\quad w\in\mathfrak{S}_n.\end{aligned}$$

Replacing ∂_i by ξ_i for $i=1,\dots,n$ in the definition of $\Delta_{\{1,\dots,k\}}$ and $P(u,T)$, we define functions $\bar\Delta_{\{1,\dots,k\}}$ and $\bar P(u,T)$ of (x,ξ), respectively.

Theorem 7.2 (Elliptic Potentials: Generic cases of Type B_n). *Put*

$$\begin{cases}u(t)=C_5\wp(t),\\v(t)=\sum_{j=1}^{4}C_j\wp(t+\omega_j)-\dfrac{C_0}{2}\end{cases}\tag{7.1}$$

and define $P_n(C_0)=P(u,T)$ and $\bar P_n(C_0)=\bar P(u,T)$ by

$$\begin{aligned}T_{\{1,\dots,k\}}&=(-C_5)^{k-1}\Bigl(\frac{C_0}{2}T^o_{\{1,\dots,k\}}(1)-\sum_{j=1}^{4}C_jT^o_{\{1,\dots,k\}}(\wp(t+\omega_j))\Bigr),\\T^o_{\{1,\dots,k\}}(\psi)&=\sum_{I_1\amalg\cdots\amalg I_\nu=\{1,\dots,k\}}(-1)^{\nu-1}(\nu-1)!S_{I_1}(\psi)\cdots S_{I_\nu}(\psi),\\S_{\{1,\dots,k\}}(\psi)&=\sum_{w\in W(B_k)}w\Bigl(\psi(x_1)\wp(x_1-x_2)\wp(x_2-x_3)\cdots\wp(x_{k-1}-x_k)\Bigr).\end{aligned}$$

Then

$$\bigl[P_n(C),P_n(C')\bigr]=\bigl\{\bar P_n(C),\bar P_n(C')\bigr\}=0\tag{7.2}$$

for C, $C'\in\mathbb{C}$.

Let P_j be the coefficient of C_0^{n-j} in $P_n(C_0)$. Then $P_1,\dots,P_n$ are the required commuting differential operators (1.5) *for the Schrödinger operator*

$$P=-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+\sum_{1\le i<j\le n}\bigl(u(x_i-x_j)+u(x_i+x_j)\bigr)+\sum_{k=1}^{n}v(x_k)\tag{7.3}$$

in the case when W is of type B_n.

By using $\bar{P}_n(C_0)$ in place of $P_n(C_0)$, we have integrals $\bar{P}_j$ of the Hamiltonian

$$\bar{P} = -\frac{1}{2}\sum_{j=1}^{n}\xi_j^2 + \sum_{1\le i<j\le n}\bigl(u(x_i-x_j)+u(x_i+x_j)\bigr) + \sum_{k=1}^{n} v(x_k), \tag{7.4}$$

where $\bar{P}_1,\dots,\bar{P}_n$ are functionally independent and satisfy $\{\bar{P}_i,\bar{P}_j\}=0$.

Proof. Theorem 4.2 and Proposition 5.4 imply

$$\bigl[P_n(C_0),P\bigr]=0. \tag{7.5}$$

Fix C, $C'\in\mathbb{C}$ and put $Q=[P_n(C),P_n(C')]$. Then we have $[Q,P]=0$ and $Q^-=-{}^tQ=Q$ and therefore we have $Q=0$ as in the proof of Theorem 3.2.

Since $q_{\{1,\dots,k\}}$ is a monic polynomial of C_0 with degree k, it is clear that P_j for $j=1,\dots,n$ satisfy (1.5). The remaining part of the theorem is also clear. □

Theorem 7.3 (Type D_n). *Suppose W is of type D_n. Then by putting $C_1=C_2=C_3=C_4=0$, the operators $P_1,\dots,P_{n-1}$ in* Theorem 7.2 *and $P_n=\Delta_{\{1,\dots,n\}}$ are the required commuting differential operators* (1.6) *for the Schrödinger operator*

$$P=-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+\sum_{1\le i<j\le n}\bigl(u(x_i-x_j)+u(x_i+x_j)\bigr) \tag{7.6}$$

with the function $u(t)$ given by (7.1)*. Here the periods of $\wp(t)$ are allowed to be infinity.*

Proof. Theorem 7.2 and Proposition 3.1 prove $[P_j,P]=0$ for $j=1,\dots,n$. Then the commutators $Q_j=[P_j,P_n]$ satisfy $Q_j^-=-{}^tQ_j=(-1)^nQ_j$ and $[Q_j,P]=0$ and hence $Q_j=0$ as in the proof of Theorem 3.2. □

Remark 7.4. i) In Theorem 7.2 we have $P_n=P_n(0)$ and

$$\begin{aligned} P_{n-k}=\sum_{i=k}^{n}\sum_{j=i}^{n}\frac{1}{i!(j-i)!(n-j)!}&\sum_{w\in\mathfrak{S}_n}\sum_{I_1\amalg\cdots\amalg I_k=\{1,\dots,i\}}\\ &w\bigl((-C_5)^{i-k}2^{-k}T^o_{I_1}(1)\cdots T^o_{I_k}(1)q_{\{i+1,\dots,j\}}\Delta^2_{\{j+1,\dots,n\}}\bigr)\end{aligned} \tag{7.7}$$

for $k=1,\dots,n-1$, where $q_{\{i+1,\dots,j\}}$ are defined by putting $C_0=0$.

ii) Because of the uniqueness of $\mathbb{C}[P_1,\dots,P_n]$ in terms of (u,v) (cf. [OS, Theorem 6.5]), the existence of the commuting differential operators $P_1,\dots,P_n$ for (1.10) which satisfy (1.5) is guaranteed by the analytic continuation of the parameters g_2 and g_3 of $\wp(t)$ even if ω_1 or ω_2 is infinite. We have explicitly given the analytic continuation. In fact Theorem 4.2, Proposition 6.1, Proposition 6.3 and the proof of Theorem 7.2 imply the following theorem.

Theorem 7.5 (Degenerate cases of Type B_n). *Suppose*

i) Trigonometric Potentials:

$$\begin{cases} u(t) = C_5\sinh^{-2}\lambda t,\\ v(t) = C_1\sinh^{-2}\lambda t + C_2\cosh^{-2}\lambda t + C_3\sinh^2\lambda t + \dfrac{C_4}{4}\sinh^2 2\lambda t - \dfrac{C_0}{2}\end{cases} \tag{7.8}$$

with a non-zero complex number λ or

ii) Rational Potentials:

$$\begin{cases} u(t) = C_5 t^{-2},\\ v(t) = C_1t^{-2}+C_2t^2+C_3t^4+C_4t^6-\dfrac{C_0}{2}.\end{cases} \tag{7.9}$$

Then for the function $T_{\{1,\dots,k\}}$ *defined in* Proposition 6.1 *or* Proposition 6.3*, we have the same statements as in* Theorem 7.2.

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