Boundary value ploblems for Riemannian symmetric spaces

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§1. Helgason Conjecture

Poisson integral representation of harmonic functions on the unit disk

$$D := \{z \in \mathbb{C}\,;\ |z| < 1\} \simeq SL(2,\mathbb{R})/SO(2):$$

$$(\mathcal{P}f)(z) = \int_0^{2\pi} \left(\frac{1 - |z|^2}{|z - e^{i\theta}|^2} \right) f(e^{i\theta}) \, \frac{d\theta}{2\pi}$$

$$\mathcal{C}(\partial D) \xrightarrow{\sim} \{u \in \mathcal{C}^\infty(D)\,;\ \Delta u = 0\} \cap \mathcal{C}(\overline{D})$$

$$\Delta := \tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}$$

$\mathcal{C}$: can be replaced by $\mathcal{A}$, $\mathcal{C}^\infty$, $\mathcal{C}^m$, $\mathcal{D}'$, $\mathcal{B} = \mathcal{A}'$ etc.

$$\begin{aligned} \mathcal{B}(\partial D) = \mathcal{A}'(\partial D) &\simeq \{u \in C^\infty(D)\,;\ \Delta u = 0\} \cap \mathcal{B}(\overline{D}) \\ &\simeq \{u \in C^\infty(D)\,;\ \Delta u = 0\} \end{aligned}$$

L^p : Hardy space $(1 \le p \le \infty)$

§1. Helgason Conjecture

Poisson integral representation of eigenfunctions on the unit disk

$D := \{z \in \mathbb{C}\,;\ |z| < 1\} \simeq SL(2,\mathbb{R})/SO(2)$:

$$(\mathcal{P}_\lambda f)(z) = \int_0^{2\pi} \left(\frac{1-|z|^2}{|z-e^{i\theta}|^2}\right)^{\lambda+1} f(e^{i\theta})\,\frac{d\theta}{2\pi}$$

$$\mathcal{B}(\partial D) \xrightarrow{\sim} \{u \in C^\infty(D)\,;\ \Delta u = \tfrac{1}{4}\lambda(\lambda+1)u\}$$

$$\Delta := \tfrac{1}{4}(1-|z|^2)^2\left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right)$$

if $\lambda \notin \{-1,-2,-3,\ldots\}$ (Helgason 1970, 1974). $(\lambda \leftrightarrow -1-\lambda)$

$\mathcal{B}$: replaced by $\mathcal{F} = \mathcal{A}$, $\mathcal{C}^\infty$, $\mathcal{C}^m$, $\mathcal{D}'$, $SO(2)$-finite etc., then

$\mathcal{D}'$: $\exists C,\ m > 0$ such that $|u(z)| \le C(1-|z|)^m$

Suppose $\Re\lambda > -\frac{1}{2}$ or $\lambda = -\frac{1}{2}$.

$$\mathcal{F}(\partial D) \xrightarrow{\sim} \{u \in C^\infty(D)\,;\ \Delta u = \tfrac{1}{4}\lambda(\lambda+1)u,\ \exists \lim_{r\to 1-0} \phi_\lambda(re^{i\theta})^{-1}u(re^{i\theta})\}$$

L^p : (weighted) Hardy space

$G = KAN$: Iwasawa decomp. $\supset MAN$: minimal parabolic subgr.

G/K: Riemannian symmetric space of the non-compact type

$\lambda : A \to \mathbb{C}$, $a \mapsto a^{-\lambda}$ (a character, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$)

$\gamma : \mathbb{D}(G/K) \xrightarrow{\sim} S(\mathfrak{a})^W$ (Harish-Chandra isomorphism)

$\mathcal{B}(G/P; L_\lambda) := \{f \in \mathcal{B}(G)\,;\, f(gman) = a^\lambda f(g) \ (man \in MAN)\}$

$\mathcal{A}(G/K; \mathcal{M}_\lambda) := \{u \in \mathcal{A}(G/K)\,;\, \Delta u = \chi_\lambda(D)u \ (D \in \mathbb{D}(G/K)\}$

$\mathcal{P}_\lambda : \mathcal{B}(G/P; L_\lambda) \to \mathcal{A}(G/K; \mathcal{M}_\lambda)$ $\quad$ ($\Leftarrow$ $\xrightarrow{\sim}$: Helgason Conjecture)

$$f \mapsto (\mathcal{P}_\lambda f)(g) := \int_K f(gk)dk = \int_K P_\lambda(k^{-1}g)f(k)dk$$

$P_\lambda(nak) = a^{\lambda+2\rho}$: Poisson kernel for G/K

Theorem. $\mathcal{P}_\lambda$ is a topological G-isomorphism iff $e(\lambda+\rho) \neq 0$

$\Leftarrow \ \Re\langle\lambda+\rho,\alpha\rangle \geq 0 \quad (\alpha > 0)$. $\quad$ ($e(\lambda+\rho)$: denominator of $c(\lambda+\rho)$)

Injectivity & K-finite case: [Helgason 1970, 1976: "A duality ... I, II"]

rank 1 with more conditions: [HaMiOk, Mi, Helgason; 1973–75]

This theorem: solved in 1974, [KO 1977] + [KKOOMT 1978]

Key of the proof: Construction of boundary value map β_λ ($\Rightarrow \mathcal{P}_\lambda^{-1}$)

K-finite and $\Re\lambda + \rho \in \mathcal{C}_+$ (= positive Weyl Chamber):

$$(\beta_\lambda u)(x) = \lim_{G/K \ni z \mapsto x \in G/P} \phi_\lambda(z)^{-1} u(z) = \lim_{r \to 1-0} (1-r)^\lambda u(re^i\theta)$$

General case (upper half plane $\simeq$ unit disk) and $\lambda \notin \{-\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots\}$

$$\exists_1 \tilde{u}_\lambda \in \mathcal{B}(\mathbb{C}) \text{ such that } yQ_\lambda \tilde{u}_\lambda = 0 \text{ and } \tilde{u}_\lambda(z) = \begin{cases} y^\lambda u(z) & (y > 0) \\ 0 & (y < 0) \end{cases}$$

$$\begin{aligned} yQ_\lambda &:= y^\lambda \left(y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{\lambda(\lambda+1)}{4}\right) \circ y^{-\lambda} \\ &= y\left(\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y} - 2\lambda - 1\right) + y\frac{\partial^2}{\partial x^2}\right) \end{aligned} \Rightarrow Q_\lambda \tilde{u}_\lambda = \left((\beta_\lambda u)(x)\right)\delta(y)$$

$$\beta_\lambda \circ \mathcal{P}_\lambda = C_\lambda \cdot id = (2\lambda+1)c(\lambda+\rho)id$$

Injectivity of β_λ: May assume $u(e) = 1$ $\Rightarrow \int_K u(kg)dk = \phi_\lambda(g)$ and $\beta_\lambda \phi_\lambda \neq 0$ if $c(\lambda+\rho) \neq 0$ $\Rightarrow \beta_\lambda u \neq 0$.

§2. Related Problems

Other function spaces: 1977 [OS 1980] $\rightarrow$ [Ben Said-O-Shimeno 2003]; [Lewis 1978], [van den Ban-Schlihitkrull 1987],...

Compactification of G/K: 1976 [O 1978,...], [OS 1980] ; [Satake 1960], [Fulstenberg 1963] ; [Sato 1980], [De Contini-Processi 1983]

Other (compact) boundaries: 1976 $\rightarrow$ [O 1996, 2005], [Oda-O 2006] ; [Johnson 1984]

Shilov boundaries of Hermitian symmetric spaces: Hua, Korányi, Sekigichi, Johnson, [Berline-Vergne 1981], Shimeno etc.,

Line bundles over G/K : 1978 ; Schlichtkrull, Shimeno,...

Semisimple symmetric spaces: 1976 [OS 1980],... ;

$+$ Flensted-Jensen's duality $\Rightarrow$ discrete sereis, c-function,...

Olafsson, van den Ban-Shrichtkrull, Delorme,

§3. Other Boundaries

$\Sigma(\mathfrak{g},\mathfrak{a}) \supset \Sigma^+ \supset \Psi$: fundamental system of the restricted root system

$P_\Theta = M_\Theta A_\Theta N_\Theta$: a parabolic subgr. $\supset P = MAN \quad (\Psi \supset \Theta \;\leftrightarrow\; M_\Theta)$

$$\mathcal{B}(G/P_\Theta\,;\,L_{\Theta,\lambda}) := \{f \in \mathcal{B}(G)\,;\, f(gm_\Theta a_\Theta n_\Theta) = a_\Theta^\lambda f(g)\}$$
$$\subset \mathcal{B}(G/P\,;\,L_\lambda) \qquad (\lambda \in \mathfrak{a}^*_{\Theta,\mathbb{C}} \subset \mathfrak{a}^*_{\mathbb{C}})$$

$$\mathcal{P}_{\Theta,\lambda} : \mathcal{B}(G/P_\Theta\,;\,L_{\Theta,\lambda}) \to \mathcal{A}(G/K;\mathcal{M}_\lambda),\;\; f \mapsto \int_K f(gk)dk$$

Lemma. i) $\operatorname{Im}\mathcal{P}_{\Theta,\lambda} = \operatorname{Im}\mathcal{P}_{\lambda-2\rho_\Theta} \qquad (\rho_\Theta \leftrightarrow M_\Theta)$.

ii) $\operatorname{Im}\mathcal{P}_\mu = \{u \in \mathcal{A}(G/K)\,;\, R_D u = u \quad (D \in I_\mu)\} \qquad (\forall\mu \in \mathfrak{a}^*_{\mathbb{C}})$

$$I_\mu := \{D \in U(\mathfrak{g})\,;\, \bar\gamma(\mathrm{Ad}(k)D)(\mu) = 0 \quad (k \in K)\},$$

$\bar\gamma(D) \in U(\mathfrak{a})$ with $D - \bar\gamma(D) \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{k}$

$M_{\Theta,\lambda} := U(\mathfrak{g})/\sum_{X\in\mathfrak{p}_\Theta} U(\mathfrak{g})\big(X-\lambda(X)\big)$ (generalized Verma module)

Lemma. $I_{\lambda-2\rho_\Theta} = U(\mathfrak{g})\mathfrak{k} + \operatorname{Ann}_{U(\mathfrak{g})}(M_{\Theta,\lambda})$ if

$$e(\lambda+\rho) \neq 0 \quad \text{and} \quad 2\frac{\langle\lambda+\rho_\mathfrak{j},\alpha\rangle}{\langle\alpha,\alpha\rangle} \notin \{-1,-2,\ldots\} \quad (\alpha \in \Sigma(\mathfrak{g},\mathfrak{j})^+)$$

$\mathfrak{a} \subset \mathfrak{j}$: Cartan subalgebra of $\mathfrak{g}$

§4. Minimal Polynomials.

$\pi : \mathfrak{g} \subset \mathfrak{gl}(N) = M(N, \mathbb{C})$: faithful representation

$\mathbb{E} := \big(p(E_{ij})\big) \in M(N, \mathfrak{g}) \subset M(N, U(\mathfrak{g}))$ with the projection p w.r.t. $\operatorname{trace} XY$ for X, $Y \in M(N, \mathbb{C})$.

Definition. The minimal polynomial $p_{\pi,\Theta,\lambda}(t)$ for $M_{\Theta,\lambda}$ is the monic polynomial with the smallest degree such that

$$p_{\pi,\Theta,\lambda}(\mathbb{E})_{ij} M_{\Theta,\lambda} = 0 \quad (1 \leq i \leq N,\ 1 \leq j \leq N).$$

Theorem. $p_{\pi,\Theta,\lambda}(t)$ is explicitly calculated by [Oda-O 2006] and can be used to construct a generator system of $\operatorname{Ann}_{U(\mathfrak{g})}(M_{\Theta,\lambda})$.

Example [O]. i) $\mathfrak{g} = \mathfrak{gl}(n)$, $\pi = id$, $\mathfrak{l}_\Theta = \mathfrak{gl}(n_1) \oplus \cdots \oplus \mathfrak{gl}(n_L)$,

$$p_{\Theta,\lambda}(t) = \prod_{j=1}^{L}(t-\lambda_j-n_1-\cdots-n_{j-1}) \quad (\mathfrak{l}_\Theta = \mathfrak{m}_\Theta+\mathfrak{a}_\Theta,\ \textstyle\sum_{j=1}^{L} n_j = n)$$

ii) $\mathfrak{g} = \mathfrak{sp}(n) \subset \mathfrak{gl}(2n)$, $\mathfrak{l}_\Theta = \mathfrak{gl}(n_1) \oplus \cdots \oplus \mathfrak{gl}(n_L)$, $\sum_{j=1}^{L} n_j = n$,

$$p_{\Theta,\lambda}(t) = \prod_{j=1}^{L}(t-\lambda_j-n_1-\cdots-n_{j-1})(t+\lambda_j-n_{j+1}-\cdots-n_L-n-1)$$

iii) $\mathfrak{g} = \mathfrak{sp}(n) \subset \mathfrak{gl}(2n)$, $\mathfrak{l}_\Theta = \mathfrak{gl}(n_1) \oplus \cdots \oplus \mathfrak{gl}(n_{L-1}) \oplus \mathfrak{gsp}(n_L)$,

$$p_{\Theta,\lambda}(t) = (t-n_1-\cdots-n_{L-1})\prod_{j=1}^{L-1}(t-\lambda_j-n_1-\cdots-n_{j-1})(t+\lambda_j-n_{j+1}-\cdots-n_L-n-1)$$

iv) $p_{\Theta,\lambda}(\mathbb{E})_{ij}$ generate $\mathrm{Ann}_{U(\mathfrak{g})}(M_{\Theta,\lambda}) \mod \mathrm{Ann}_{U(\mathfrak{g})}(M_{\Theta,\lambda}) \cap Z(\mathfrak{g})$
if the infinitesimal character is regular ($Z(\mathfrak{g})$: the center of $U(\mathfrak{g})$)

§5. **Theorem**. Suppose the above iv) is satisfied and $e(\lambda+\rho) \neq 0$.

i) $p_{\Theta,\lambda}(\mathbb{E})_{ij}$ $(1 \leq i,\, j \leq N)$ and $D - \chi_\lambda(D)$ $(D \in \mathbb{D}(G/K))$ define the system of differential equations characterizing the image of $\mathcal{P}_{\Theta,\lambda}$.

ii) Moreover if $\Re(\lambda+\rho) \in \mathcal{C}_+$ $(\Rightarrow e(\lambda+\rho) \neq 0)$,

$$\lim_{G/K \ni z \mapsto x \in G/P_\Theta} \phi_\lambda(z)^{-1}(\mathcal{P}_{\Theta,\lambda} f)(z) = f(x)$$

for $f \in \mathcal{F}(G/P_{\Theta,\lambda})$ $(\mathcal{F} = \mathcal{C}^m$, $\mathcal{A}$, $\mathcal{D}'$, $\mathcal{B}$ etc.$)$.

In particular

$$\begin{aligned}\mathcal{P}_{\Theta,0}\big(\mathcal{C}(G/P_\Theta)\big) &= \{u \in \mathcal{A}(G/K)\,;\, Du = \chi_0(D)u \ (D \in \mathbb{D}(G/K))\} \\ &\qquad \cap\, \mathcal{C}\big(\overline{(G/K)}_\Theta\big) \\ &\subset \{u \in \mathcal{A}(G/K)\,;\, p_{\Theta,0}(\mathbb{E})_{ij} u = 0\}\end{aligned}$$

$\overline{(G/K)}_\Theta$: Satake compactification of G/K with the boundary G/P_Θ.

§6. Boundaries for $U(p,q)/U(p)\times U(q)$ $(p \geq q)$

Shilov boundary

$$p_{\Theta,\lambda}(t) = \begin{cases} (t-\lambda)(t-q)(t+\lambda-p) & (p>q) \\ (t-\lambda)(t+\lambda-p) & (p=q) \end{cases}$$

$$\mathbb{E} = \left(E_{ij}\right) = \begin{pmatrix} K_1 & P \\ Q & K_2 \end{pmatrix} = {}^t\!\left(x_{ij}\right)\left(\frac{\partial}{\partial x_{ij}}\right) \quad (K_1 \in \mathfrak{gl}(p),\ K_2 \in \mathfrak{gl}(q))$$

$$(\mathbb{E}-\lambda)(\mathbb{E}+\lambda-p) \equiv \begin{pmatrix} PQ & 0 \\ (q-p)Q & QP \end{pmatrix} - \lambda(\lambda-p) \mod U(\mathfrak{g})\mathfrak{k}$$

$QP - \lambda(\lambda-p)$: Hua operator ([Berline-Vergne])

General boundaries

$\mathfrak{l}_\Theta \simeq \mathfrak{gl}(n_1,\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(n_L,\mathbb{C}) \oplus \mathfrak{u}(p-q)$: $\sum n_j = q$

$(1,2)$ and $(2,2)$-blocks $\Rightarrow$ order $2L+1 \mapsto 2L$.

Line bundles $t \mapsto t-\mu-\nu$, $q \mapsto q+\mu-\nu$ $(\mu \leftrightarrow U(p),\ \nu \leftrightarrow U(q))$

§6. Boundaries for $U(p,q)/U(p)\times U(q)$ $(p\geq q)$

Shilov boundary

$$p_{\Theta,\lambda}(t)=\begin{cases}(t-\lambda-\mu-\nu)(t-q-2\mu)(t-p+\lambda-\mu-\nu) & (p>q)\\ (t-\lambda-\mu-\nu)(t-p+\lambda-\mu-\nu) & (p=q)\end{cases}$$

$$\mathbb{E}=\begin{pmatrix}K_1 & P\\ Q & K_2\end{pmatrix}={}^t\!\left(x_{ij}\right)\left(\frac{\partial}{\partial x_{ij}}\right)\quad (K_1\in\mathfrak{gl}(p),\ K_2\in\mathfrak{gl}(q))$$

$$(\mathbb{E}-\lambda-\mu-\nu)(\mathbb{E}-p+\lambda-\mu-\nu)\equiv\begin{pmatrix}PQ-2(\mu-\nu)p & 0\\ (q-p)Q & QP\end{pmatrix}$$

$$-(\lambda+\mu-\nu)(\lambda-p-\mu+\nu)\quad \operatorname{mod}\ \textstyle\sum_{X\in\mathfrak{k}}U(\mathfrak{g})(X-\chi_{\mu,\lambda}(X))$$

$QP-(\lambda+\mu-\nu)(\lambda-p-\mu+\nu)$: Hua operator

General boundaries

$\mathfrak{l}_\Theta\simeq\mathfrak{gl}(n_1,\mathbb{C})\oplus\cdots\oplus\mathfrak{gl}(n_L,\mathbb{C})\oplus\mathfrak{u}(p-q)$: $\sum n_j=q$

$$\bar{p}_{\Theta,\lambda}(t)=\textstyle\prod_{j=1}^{L}(t-\lambda_j-n_1-\cdots-n_{j-1}-\mu-\nu)(t+\lambda_j-p-n_{j+1}-\cdots-n_L-\mu-\nu)$$

§7. Shilov boundary for $Sp(n,\mathbb{R})/U(n)$

$$p_{\Theta,\lambda}(t) = (t-\lambda)(t+\lambda-\tfrac{n+1}{2})$$

$$\mathbb{E} = \begin{pmatrix} K & P \\ Q & -{}^tK \end{pmatrix} \quad \text{with} \quad \begin{cases} 2K_{ij} = E_{ij} - E_{i+n,j+n} \\ 2P_{ij} = E_{i,j+n} + E_{j,i+n} \\ 2Q_{ij} = E_{n+i,j} + E_{n+j,i} \end{cases}$$

$$(\mathbb{E}-\lambda)(\mathbb{E}+\lambda-\tfrac{n+1}{2}) \equiv \begin{pmatrix} PQ-(n+1)\nu & 0 \\ 0 & QP \end{pmatrix} - (\lambda-\nu)(\lambda-\tfrac{n+1}{2}+\nu)$$

$$\bmod \sum_{i,j=1}^{n} U(\mathfrak{g})(K_{ij}-\nu\delta_{ij})$$

$QP-(\lambda-\nu)(\lambda-\tfrac{n+1}{2}+\nu)$: Hua operator