VERSAL UNFOLDING OF IRREGULAR SINGULARITIES OF A LINEAR DIFFERENTIAL EQUATION ON THE RIEMANN SPHERE

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Abstract. For a linear differential operator P on $\mathbb{P}^1$ with unramified irregular singular points we examine a realization of P as a confluence of singularities of a Fuchsian differential operator $\tilde{P}$ having the same index of rigidity as P, which we call an unfolding of P. We conjecture that this is always possible. For example, if P is rigid, this is true and the unfolding helps us to study the equation $Pu = 0$.

1. Introduction

Gauss hypergeometric function $F(\alpha,\beta,\gamma;x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} x^k$ is a solution to Gauss hypergeometric equation

$$x(1-x)u'' + \big(\gamma - (\alpha+\beta+1)x\big)u' - \alpha\beta u = 0, \tag{1.1}$$

which has regular singularities at 0, 1 and ∞. Here we use the notation $(a)_k = (a)(a+1)\cdots(a+k-1)$. Putting $y = \beta x$ and taking the limit $\beta \to \infty$, two singular points $y = \beta$ and ∞ converge to a confluent irregular singular point ∞. Then Gauss hypergeometric function converges to Kummer function:

$$\begin{aligned} F(\alpha,\beta,\gamma;\tfrac{y}{\beta}) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-\frac{yt}{\beta})^{-\beta}dt \\ &\xrightarrow{\beta\to\infty} {}_1F_1(\alpha,\gamma;y) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k k!} y^k = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{yt}dt. \end{aligned}$$

And the equation (1.1) converges to Kummer equation $y\frac{d^2v}{dy^2} + (\gamma-y)\frac{dv}{dy} - \alpha v = 0$. Similarly, the confluence of three singular points of Gauss hypergeometric equation with suitable limits of parameters gives Hermite equation.

For a given linear differential equation

$$Pu = 0 \tag{1.2}$$

on $\mathbb{P}^1$ with irregular singularities, we want to construct a Fuchsian differential equation $\tilde{P}\tilde{u} = 0$ with singular points as holomorphic parameters such that a confluence of some singular points gives the original equation. Here we call $\tilde{P}$ an unfolding of P. Note that the irregular singularity is more difficult to analyze it than the regular singularity and if it is a confluent singularity, the procedure of the confluence helps to analyze the equation. In the case of Gauss hypergeometric family, a versal unfolding exists, which we call versal Gauss equation. As is given in §6, it is

$$(1-t_1x)(1-t_2x)\tilde{u}'' + (\bar{\lambda}_1 + \bar{\lambda}_2 x)\tilde{u}' + \bar{\mu}\big(\bar{\lambda}_2 - t_1t_2(\bar{\mu}+1)\big)\tilde{u} = 0, \tag{1.3}$$

Key words and phrases. irregular singularity, middle convolution, confluence of singularities.
2010 Mathematics Subject Classification. Primary 34M35; Secondary 34C20, 34M25
Supported by Grant-in-Aid for Scientific Researches (C), No. No.18K03341.

which has the Riemann scheme (6.1). Put $t_1 = 1$ and $y = x - 1$. Then (1.3) gives Gauss hypergeometric equation or Kummer equation according to $t_2 = \frac{1}{2}$ or 0, respectively. When $t_1 = t_2 = 0$, (1.3) gives Hermite equation.

We will assume that the irregular singularities of (1.2) are unramified (cf. §2). In §3 we define a generalized Riemann scheme, a spectral type and the index of rigidity for the equation as in the case of Fuchsian differential equations (cf. [12, Chapter 4]). In §4 we define a versal unfolding of the Riemann scheme attached to the spectral type, which is a generalized Riemann scheme of a Fuchsian differential equation having singular points as parameters and the same index of rigidity. Using the versal addition and the middle convolution of the element of the Weyl algebra, which are defined in [12], in §5 we examine the construction of the versal unfolding $\tilde{P}$ of P with this generalized Riemann scheme which has singular points as holomorphic parameters such that special values of parameters give the original equation.

Since Katz [10] introduced two operations of linear differential equations, namely, the middle convolution and the addition, the study of Fuchsian differential equations on $\mathbb{P}^1$ has been greatly developed. For example, there are an interpretation of these operations in Schlesinger systems by [4], an answer to additive Deligne-Simpson problem by [3] which gives a correspondence between the spectral types of Fuchsian equations and the roots of a Kac-Moody root system and an analysis of the solution to the single equation $Pu = 0$ by [12] which constructs the universal model with a given spectral type and studies the irreducibility condition of the equation, connection coefficients of the solutions etc. and gives many explicit formulas when the equation is rigid.

Assume the equation $Pu = 0$ has no ramified irregular singularity. Then these operations are still quite useful. They keep the index of rigidity. There are classified into only finite types under these operations for any fixed index of rigidity (cf. [7]). In particular, if the equation is rigid, it is constructed by successive applications of these operations from the trivial equation and for example, the local monodromy around an irregular singularity is explicitly calculated (cf. [13]).

This paper will give a link between the study of Fuchsian differential equations and that of equations allowing unramified irregular singularities. The main result in this paper was announced by the author's invited lecture on the annual meeting of Mathematical society of Japan held in March, 2012.

2. Preriminary Results

We explain the notation used in this paper. We denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ the set of integers, the sets of rational numbers and the set complex numbers, respectively. Then we put $\mathbb{Z}_{>0} := \{k \in \mathbb{Z} \mid k > 0\}$, $\mathbb{Z}_{\geq 0} := \{k \in \mathbb{Z} \mid k \geq 0\}$, $\mathbb{Z}_{\leq 0} := \{k \in \mathbb{Z} \mid k \leq 0\}$ etc. We denote by $\mathbb{C}[x]$ the ring of polynomials of one variable x with coefficients in $\mathbb{C}$, by $\mathbb{C}(x)$ the quotient filed of $\mathbb{C}[x]$, by $\mathbb{C}[\![x]\!]$ the ring of formal power series of x and by $\mathbb{C}(\!(x)\!)$ the quotient field of $\mathbb{C}[\![x]\!]$. The ring of convergent power series at $x = c$ is denoted by $\mathcal{O}_c$. Note that $\mathbb{C}[x] \subset \mathcal{O}_0 \subset \mathbb{C}[\![x]\!] \subset \mathbb{C}(\!(x)\!)$.

In this section we review on elementary results on singularities of the equation $Pu = 0$. We denote by $W[x]$ the Weyl algebra of one variable x with coefficients in $\mathbb{C}$, which is generated by x and $\frac{d}{dx}$. Namely, the element of $W[x]$ is a linear ordinary differential operator with polynomial coefficients. We put $\partial = \frac{d}{dx}$ and $\vartheta = x\partial$ for simplicity. Then $[\partial, x] = 1$ is the fundamental relation of $W[x]$. We define $W(x) := \mathbb{C}(x) \otimes W[x]$ and $W(\!(x)\!) := \mathbb{C}(\!(x)\!) \otimes W[x]$.

The degree of a polynomial $a(x)$ is denoted by $\deg_x a(x)$ or $\deg a(x)$. The order of a linear ordinary differential operator

$$P = a_n(x)\,\partial^n + \cdots + a_1(x)\,\partial + a_0(x) \tag{2.1}$$

with $a_n \neq 0$ is n and denoted by $\operatorname{ord} P$. In this paper we study the equation $Pu = 0$ in the cases when $P \in W[x]$, $W(x)$ or $W(\!(x)\!)$.

For a function $\varphi(x)$ we put

$$\operatorname{Ad}(\varphi)P := \varphi(x) \circ P \circ \varphi(x)^{-1}$$

if this expression has a meaning. In particular

$$\operatorname{Ad}(\varphi)\,\partial = \partial - \tfrac{\partial(\varphi)}{\varphi} \quad \text{and} \quad \operatorname{Ad}\bigl(\exp(\varphi)\bigr)\,\partial = \partial - \partial(\varphi).$$

Note that if $u(x)$ satisfies $Pu(x) = 0$, then $v(x) = \varphi(x)u(x)$ satisfies $\operatorname{Ad}(\varphi)(P)v(x) = 0$. If $\frac{\partial(\varphi)}{\varphi} \in \mathbb{C}(x)$, $\operatorname{Ad}(\varphi)$ defines an automorphism of $W(x)$, which we call an *addition*.

Let $P \in W(\!(x)\!)$ be

$$P = \partial^n + a_{n-1}(x)\,\partial^{n-1} + \cdots + a_0(x) \qquad (a_j(x) \in \mathbb{C}(\!(x)\!)). \tag{2.2}$$

Here

$$a_j(x) = \sum_{\nu=m_j}^{\infty} a_{j,\nu}x^{\nu} \quad (a_{j,\nu} \in \mathbb{C},\ a_{j,m_j} \neq 0)$$

with $m_j \in \mathbb{Z} \cup \{\infty\}$. We assume that the origin is a singular point of P, namely, there exists j satisfying $m_j < 0$. Then the number

$$\operatorname{Prk} P := \max\left\{-\frac{m_j}{n-j},\, 1 \,\middle|\, j = 1, \dots, n-1\right\} - 1 \in \mathbb{Q}_{\geq 0}$$

is called the *Poincaré rank* of P. We also define $\operatorname{Prk} \tilde{P} = \operatorname{Prk} P$ for $\tilde{P} = \phi(x)P$ with $\phi(x) \in \mathbb{C}(\!(x)\!) \setminus \{0\}$. The singularity is regular or irregular according to the condition $\operatorname{Prk} \tilde{P} = 0$ or $\operatorname{Prk} \tilde{P} > 0$, respectively. Then

$$\nu - (\operatorname{Prk} P + 1)j \geq -(\operatorname{Prk} P + 1)n \quad \text{if } a_{j,\nu} \neq 0 \quad (\nu \geq m_j,\ j = 0, \dots, n-1) \tag{2.3}$$

and if $\operatorname{Prk} P > 0$ and $n > 0$, there exists $(j, \nu) = (j_0, m_0)$ such that the equality holds in (2.3). For $r \in \mathbb{Q}_{\geq 0}$ and $\tilde{P} = \sum c_{j,\nu}x^{\nu}\,\partial^j \in W(\!(x)\!) \setminus \{0\}$, we put

$$\begin{aligned}
\operatorname{wt}_r(\tilde{P}) &:= \min\{\nu - (r+1)j \mid c_{j,\nu} \neq 0\},\\
\sigma_{(r)}(\tilde{P}) &:= \begin{cases} \displaystyle\sum_{\nu-(r+1)j=\operatorname{wt}_r(\tilde{P})} c_{j,\nu}s^j & (r > 0),\\ \displaystyle\sum_{\nu-(r+1)j=\operatorname{wt}_r(\tilde{P})} c_{j,\nu}s(s-1)\cdots(s-j+1) & (r = 0).\end{cases}
\end{aligned}$$

Then we have easily the following lemma.

Lemma 2.1. *For P, $Q \in W(\!(x)\!)$, $\phi \in \mathbb{C}(\!(x)\!) \setminus \{0\}$, $k \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ and $r \in \mathbb{Q}_{\geq 0}$*

$$\begin{aligned}
\sigma_{(r)}(\phi P) = \sigma(P), \quad \sigma_{(r)}(Px^k)(s) &= \begin{cases} \sigma_{(r)}(P)(s) & (r > 0),\\ \sigma_{(r)}(P)(s-k) & (r = 0),\end{cases}\\
\operatorname{wt}_r(PQ) = \operatorname{wt}_r(P) + \operatorname{wt}_r(Q), &\quad \sigma_{(r)}(PQ) = \sigma_{(r)}(P) \cdot \sigma_{(r)}(Q),\\
\sigma_{(r)}\bigl(\operatorname{Ad}(e^{-\frac{\lambda}{rx^r}})P\bigr)(s) &= \sigma_{(r)}(P)(s-\lambda) \quad (r > 0),\\
\sigma_{(0)}\bigl(\operatorname{Ad}(x^{\lambda})P\bigr)(s) &= \sigma_{(0)}(P)(s-\lambda).
\end{aligned}$$

By this lemma we have the decomposition theorem (for example, see [15]).

Theorem 2.2. *Let $P \in W(\!(x)\!)$ with the expression* (2.2)*. Put $r = \operatorname{Prk} P$. Then*

$$P = \bigl(\partial - x^{-r-1}\varphi_n(x^{\frac{1}{p}})\bigr) \cdots \bigl(\partial - x^{-r-1}\varphi_1(x^{\frac{1}{p}})\bigr) \tag{2.4}$$

with suitable $p \in \mathbb{Z}_{>0}$ and $\varphi_j(x) \in \mathbb{C}[\![x]\!]$ satisfying $pr \in \mathbb{Z}$.

Proof. We can prove the theorem by the induction on $\operatorname{ord} P$. We may assume $\operatorname{ord} P > 1$.

Suppose $r = 0$. Then the origin is a regular singular point of P and the theorem is well known. Let λ be one of the solutions to $\sigma_{(0)}(P)(s) = 0$ such that $\operatorname{Re}\lambda$ is maximal[1]. Put $\tilde P = \operatorname{Ad}(x^{-\lambda})P$. Then $\sigma_0(\tilde P)(0) = 0$ and $\sigma_{(0)}(P)(j) \neq 0$ for $j = 1, 2, \ldots$ and we easily obtain $\psi(x) \in \mathbb{C}[\![x]\!]$ satisfying $\tilde P\psi = 0$. Then $\tilde P = Q(\partial - \partial(\psi)) + g(x)$ with $g(x) \in \mathbb{C}[\![x]\!]$ (cf. [12, §1.4.1]). Since $\tilde P(\psi) = 0$, we have $g = 0$, $\operatorname{Prk} Q = 0$, $\operatorname{ord} Q = \operatorname{ord} P - 1$ and $P = \operatorname{Ad}(x^\lambda)(Q) \cdot \operatorname{Ad}(x^\lambda)(\partial - \partial(\psi))$. Thus we have the theorem by the induction.

Suppose $r > 0$. Note that the polynomial

$$\sigma_{(r)}(P)(s) = s^n + \sum_{\substack{m_j + (n-j)(r+1) = 0\\ 0 \le j \le n-1}} a_{j,m_j} s^j$$

has a non-zero root.

If $r \notin \mathbb{Z}$, the coefficient of s^{n-1} of this polynomial equals 0 and therefore the polynomial has at least two different roots. Putting $x = y^q$, $\operatorname{Prk} P$ in the coordinate y is the q times of that in the coordinate x. Under the coordinate y, we may assume $\operatorname{Prk} P$ is a positive integer and $\sigma_{(r)}(P)(s)$ has different roots.

Suppose $r \in \mathbb{Z}_{>0}$ and $\sigma_{(r)}(P)(s) = 0$ has only one root λ with multiplicity n. Then $P' = \operatorname{Ad}(e^{\frac{\lambda}{rx^r}})(P)$ satisfies $\sigma_{(r)}(P') = s^n$, which means $\operatorname{Prk} P' < r$. Repeating this process, we may assume $\operatorname{Prk} P = 0$ or $\operatorname{Prk} P \notin \mathbb{Z}$ or $\sigma_{(r)}(P)(s) = 0$ has different roots.

Thus we may assume $r \in \mathbb{Z}_{>0}$ and moreover that $\sigma_{(r)}(P)(s) = 0$ has a root λ with multiplicity m such that $1 \le m \le n-1$. By the transformation $\operatorname{Ad}(e^{\frac{\lambda}{rx^r}})$ we have

$$\sigma_{(r)}(P) = s^n + a_{n-1,0}s^{n-1} + \cdots + a_{m,0}s^m \quad (a_{j,0} \in \mathbb{C},\ a_{m,0} \neq 0).$$

Then we have a decomposition

$$\begin{aligned}&x^{n(r+1)}P = QR,\\ &\sigma_{(r)}(Q) = s^{n-m} + a_{n-1,0}s^{n-m-1} + \cdots + a_{m,0}, \quad \sigma_{(r)}(R) = s^m,\end{aligned} \tag{2.5}$$

which is proved as follows. Denoting $\vartheta_r = x^{r+1}\partial$, we put

$$\begin{aligned}
x^{n(r+1)}P &= \vartheta_r^n + a_{n-1}(x)\vartheta_r^{n-1} + \cdots + a_0(x),\\
Q &= \vartheta_r^{n-m} + q_{n-m-1}(x)\vartheta_r^{n-m-1} + \cdots + q_0(x),\\
R &= \vartheta_r^m + r_{m-1}(x)\vartheta_r^{m-1} + \cdots + r_0(x),\\
a_j(x) &= \sum_{\nu=0}^\infty a_{j,\nu}x^\nu, \quad q_j(x) = \sum_{\nu=0}^\infty q_{j,\nu}x^\nu, \quad r_j(x) = \sum_{\nu=0}^\infty r_{j,\nu}x^\nu,\\
&a_{j,0} = 0 \quad (0 \le j \le m-1), \quad a_{m,0} \neq 0,\\
&q_{j,0} = a_{j+m,0} \quad (0 \le j \le n-m-1), \quad r_{j,0} = 0 \quad (0 \le j \le m-1).
\end{aligned}$$

For $S = \sum c_{j,\nu}x^\nu\vartheta_{(r)}^j \in W(\!(x)\!)$ and $N \in \mathbb{Z}_{\ge 0}$, we put $S_N = \sum_{j \le N} c_{j,\nu}x^\nu\vartheta_{(r)}^j$. Since $\operatorname{wt}_r(RS - R_NS_N) > N$, we inductively define $\{r_{j,N},\ q_{j',N} \mid 0 \le j \le m-1,\ 0 \le j' \le n-m-1\}$ for $N = 1, 2, \ldots$ so that $\operatorname{wt}_r(x^{r(n+1)}P - Q_NR_N) > N$, which is

[1] The ordering of $\lambda_j(x)$ given by (2.11) follows this choice of λ.

possible because the system

$$\begin{aligned}(\vartheta_r^{n-m} + a_{n-1,0}\vartheta_r^{n-m-1} + \cdots + a_{m,0})(r_{m-1,N}\vartheta_r^{m-1} + \cdots + r_{0,N})\\ + (q_{n-m-1,N}\vartheta_r^{n-m-1} + \cdots + q_{0,N})\vartheta_r^m = \sum_{j=0}^{n-1} c_{j,N}\vartheta_r^j\end{aligned} \tag{2.6}$$

is solved for any given $c_{j,N}$. In fact we can determine $r_{0,N}, \ldots, r_{m-1,N}$, $q_{0,N}, \ldots,$ $q_{n-m-1,N}$ in this order by the coefficients of ϑ_r^j for $j = 0, \ldots, n-1$. Here $c_{j,N}$ are determined by R_{N-1}, Q_{N-1} and P_N. Thus we have the theorem by the hypothesis of the induction. □

Remark 1. Retain the notation in Theorem 2.2. When $r = 0$, the origin is a non-singular point or a *regular singular point* of P and we can choose $p = 1$. When $r < 0$, then the origin is called an *irregular singular point* of P. If $r < 0$ and we can choose $p = 1$, the origin is called an *unramified* irregular singular point of P.

By Theorem 2.2 we may assume

$$x^n P = \big(\vartheta - \phi_n(x^{\frac{1}{p}})\big) \cdots \big(\vartheta - \phi_1(x^{\frac{1}{p}})\big) \quad (\phi_j \in \mathbb{C}(\!(x)\!)). \tag{2.7}$$

We define $\tilde\lambda_j(x) \in \mathbb{C}[x^{\frac{1}{p}}]$, $\deg \tilde\lambda_j(x)$ and $\lambda_j(x)$ so that

$$\tilde\lambda_j(\tfrac{1}{x}) - \phi_j(x^{\frac{1}{p}}) \in x^{\frac{1}{p}}\mathbb{C}[\![x^{\frac{1}{p}}]\!] \quad (j = 1, \ldots, n), \tag{2.8}$$

$$\deg \tilde\lambda_j(x) = \begin{cases} \frac{1}{p}\deg_y \tilde\lambda_j(y^p) & (\tilde\lambda_j \neq 0),\\ 0 & (\tilde\lambda_j = 0),\end{cases} \tag{2.9}$$

$$\lambda_j(x) = \tilde\lambda_j(x) + \sum_{\nu=1}^{j-1} \deg\big(\tilde\lambda_j(x) - \tilde\lambda_\nu(x)\big). \tag{2.10}$$

We may assume

(2.11)
$$\begin{aligned}&\lambda_i(x) - \lambda_j(x) \in \mathbb{C} \text{ and } i < j \ \Rightarrow\ \lambda_i(x) - \lambda_\nu(x) \in \mathbb{C} && (i < \forall\nu < j),\\ &\lambda_i(x) - \lambda_j(x) \in \mathbb{Z} \text{ and } i < j \ \Rightarrow\ \lambda_{i'}(x) - \lambda_{j'}(x) \in \mathbb{Z}_{\geq 0} && (i \leq \forall i' < \forall j' \leq j).\end{aligned}$$

We call $\{\lambda_1(x), \ldots, \lambda_n(x)\}$ the set of *characteristic exponents* of P at the origin.

Let

$$\mu(x) = \mu_0 + \mu_1 x^{r_1} + \cdots + \mu_{m-1}x^{r_m} \quad (0 < r_1 < \cdots < r_m)$$

be a characteristic exponent. Here $\mu_j \in \mathbb{C}$. We define the *characteristic function* $e_\mu(x)$ with the exponent $\mu(x)$ by

$$e_\mu(x) := x^{-\mu_0}\exp(-\mu_1\tfrac{x^{r_1}}{r_1} - \cdots - \mu_m\tfrac{x^{r_m}}{r_m})$$

and put $\check e_\mu(x) := e_\mu(\frac{1}{x}) = x^{\mu_0}\exp(-\frac{\mu_1}{r_1x^{r_1}} - \cdots - \frac{\mu_m}{r_mx^{r_m}})$. Then

$$\begin{aligned}&\big(\vartheta + \mu(x)\big)e_\mu(x) = 0, \quad \mathrm{Ad}\big(e_\mu(x)\big)\vartheta = \vartheta + \mu(x), \quad e_\mu(x)e_\lambda(x) = e_{\mu+\lambda}(x),\\ &\big(\vartheta - \mu(\tfrac{1}{x})\big)\check e_\mu(x) = 0, \quad \mathrm{Ad}\big(\check e_\mu(x)\big)\vartheta = \vartheta - \mu(\tfrac{1}{x}).\end{aligned}$$

We examine the solution to $Pu = 0$. We may assume that the singularity is unramified by the transformation $x \mapsto x^{\frac{1}{p}}$. Because of the decomposition (2.4) we have only to study the equation

$$(\vartheta - \phi(x))u(x) = f(x) \tag{2.12}$$

for $\phi \in \mathbb{C}(\!(x)\!)$. Let $\mu(x) \in \mathbb{C}[x]$ such that

$$\bar\phi(x) := \phi(x) - \mu(\tfrac{1}{x}) = \sum_{\nu=1}^{\infty} c_\nu x^\nu \in x\mathbb{C}[\![x]\!]. \tag{2.13}$$

Define

$$\textstyle\int x^{-1}\bar\phi(x)dx := \sum_{\nu=1}^{\infty}\frac{c_\nu}{\nu}x^\nu,$$
$$\exp(\textstyle\int x^{-1}\phi(x)dx) := \exp(\int x^{-1}\bar\phi(x)dx)\check e_\mu(x) \in \mathbb C[\![x]\!]\check e_\mu(x).$$

Then we have $\bigl(\vartheta-\phi(x)\bigr)\exp(\int x^{-1}\phi(x)dx)=0$.

Put

$$\mathbb C[\![x]\!]_{(m)} := \bigoplus_{j=0}^{m}\mathbb C[\![x]\!]\log^j x \qquad (0\le m\le\infty).$$

For a non-zero element

$$u(x)=\sum_{j=0}^{m}\sum_{\nu=0}^{\infty}c_{j,\nu}x^\nu\log^j x\cdot\check e_\mu(x)\in\mathbb C[\![x]\!]_{(m)},$$

we define

$$\sigma(u) := c_{j_o,\nu_o}x^{\nu_o}\log^{j_o}x\cdot\check e_\mu(x)$$

and put $\sigma(0)=0$. Here (j_o,ν_o) is defined by the condition that $c_{j_o,\nu_o}\ne0$, $c_{j,\nu_o}=0$ for $j>j_o$ and $c_{j,\nu}=0$ for $\nu<\nu_o$. Note that $\sigma(e^{\int x^{-1}\phi(x)dx})=\check e_\mu(x)$.

We want to solve (2.12) for

$$f(x)\in\mathbb C[\![x]\!]_{(m)}\check e_\lambda(x)\ \text{ with }\ \sigma(f)=\log^m x\cdot\check e_\lambda(x). \tag{2.14}$$

Put $\tilde\phi(x)=\exp(\int x^{-1}\phi(x)dx)$, $\tilde u(x)=\tilde\phi(x)^{-1}u(x)$ and $\tilde f(x)=\tilde\phi(x)^{-1}f(x)$. Then the equation (2.12) becomes

$$\vartheta\tilde u(x)=\tilde f(x)\quad\bigl(\tilde f\in\mathbb C[\![x]\!]_{(m)}\check e_{\lambda-\mu}(x),\ \sigma(\tilde f)=\log^m x\cdot\check e_{\lambda-\mu}(x)\bigr). \tag{2.15}$$

Putting $r=\deg(\lambda-\mu)$, we have

$$\begin{aligned}&\sigma\bigl(\vartheta(x^\nu\log^m x\cdot\check e_{\lambda-\mu+r}(x))\bigr)\\&\quad=\begin{cases}Crx^\nu\log^m x\cdot\check e_{\lambda-\mu}(x) & (r>0),\\(\lambda-\mu+\nu)x^\nu\log^m x\cdot\check e_{\lambda-\mu}(x) & (r=0,\ \lambda-\mu\notin\mathbb Z_{\le0})\end{cases}\end{aligned}$$

for $\nu\in\mathbb Z_{\ge0}$. Here C is the coefficient of the top term of the polynomial $\lambda-\mu$.

Hence if $\lambda-\mu\notin\mathbb Z_{\le0}$, there exists $u_m(x)\in\mathbb C[\![x]\!]\log^m x\cdot\check e_{\mu-\lambda+r}(x)$ such that $f(x)-\vartheta u_m(x)\in\mathbb C[\![x]\!]_{(m-1)}\check e_{\lambda-\mu+r}(x)$. Then we have $\tilde u(x)\in\mathbb C[\![x]\!]_{(m)}\check e_{\lambda-\mu+r}(x)$ satisfying $\vartheta\tilde u=\tilde f$ by the induction on m. Here $\sigma(\tilde u)=\tilde Cx^r\sigma(\tilde f)$ with $\tilde C\in\mathbb C\setminus\{0\}$.

Suppose $\lambda-\mu\in\mathbb Z_{\le0}$. Since $\vartheta(\log^{m+1}x)=(m+1)\log^m x$, we have $\tilde u(x)\in\mathbb C[\![x]\!]\log^m x\cdot\check e_{\lambda-\mu}(x)\oplus\mathbb C\log^{m+1}x$ satisfying $\vartheta\tilde u(x)=\tilde f(x)$.

Thus we solve (2.12) as follows. If $\lambda-\mu\notin\mathbb Z_{\le0}$,

$$\begin{aligned}&u(x)\in\mathbb C[\![x]\!]_{(m)}\check e_{\lambda+r}(x),\\&\sigma(u(x))=C\log^m x\cdot\check e_{\lambda+r}(x).\end{aligned} \tag{2.16}$$

If $\lambda-\mu\in\mathbb Z_{\le0}$,

$$\begin{aligned}&u(x)\in\mathbb C[\![x]\!]_{(m+1)}\check e_\lambda(x),\\&\sigma(u)=\begin{cases}C\log^m x\cdot\check e_\lambda(x) & (\lambda\ne\mu),\\C\log^{m+1}x\cdot\check e_\lambda(x) & (\lambda=\mu)\end{cases}\end{aligned} \tag{2.17}$$

with a non-zero constant C. Thus we have the following theorem (for example, see [15]).

Theorem 2.3. *Retain the notation* (2.7)–(2.10)*. Then we have linearly independent solutions* $u_1(x),\dots,u_n(x)$ *to* (1.2) *such that*

$$\begin{aligned} &u_j(x)\in\mathbb{C}[\![x]\!]_{(m_j)}\check{e}_{\lambda_j}(x), && m_j=\#\{\nu\mid\lambda_j-\lambda_\nu\in\mathbb{Z}_{\le 0},\ 1\le\nu<j\},\\ &\sigma(u_j)=\log^{\bar m_j}x\cdot\check{e}_{\lambda_j}(x), && \bar m_j=\#\{\nu\mid\lambda_j=\lambda_\nu,\ 1\le\nu<j\}.\end{aligned}$$

Proof. Put $u_j^{(j)}(x)=\exp(\int x^{-1}\phi_j(x)dx)\in\mathbb{C}[\![x]\!]\check{e}_{\tilde\lambda_j}(x)$ with $\sigma(u_j^{(j)})=\check{e}_{\tilde\lambda_j}(x)$. We have $u_j^{(j-\nu)}(x)$ satisfying $(\vartheta-\phi_{j-\nu})u_j^{(j-\nu)}=u_j^{(j-\nu+1)}$ for $\nu=1,\dots,j$. Then $Pu_j^{(0)}=0$ and the theorem follows from the argument just before the theorem (cf. (2.10), (2.12)–(2.17)). □

We have the following important theorem related to these formal solutions $u_j(x)$.

Theorem 2.4 ([8, 9])**.** *Retain the notation above. Suppose there exists* $k\in\mathbb{Z}_{\ge0}$ *such that* $x^ka_i(x)\in\mathcal{O}_o$ *for* $i=0,\dots,n$ *in* (2.1)*. Then for any* $\theta_0\in\mathbb{R}$*, there exist* $\theta_1>0$*,* $L>0$ *and holomorphic solutions* $\tilde u_j(x)$ *of* (1.2) *such that the following asymptotic expansion holds.*

$$\tilde u_j(x)\sim u_j(x)\quad (V_{\theta_0,\theta_1,L}\ni x\to0,\ j=1,\dots,n), \tag{2.18}$$

$$V_{\theta_0,\theta_1,L}:=\{re^{i\theta}\in\mathbb{C}\mid\theta\in[\theta_0-\theta_1,\theta_0+\theta_1],\ 0<r<L\}. \tag{2.19}$$

3. Riemann scheme

In this section we study the equation (1.2) with $P\in W[x]$. Let $\{c_0,\dots,c_p\}$ $(\subset\mathbb{C}\cup\{\infty\})$ be the set of singular points of the equation and let $\{\lambda_{j,\nu}\mid\nu=1,\dots,n\}$ be the set of the characteristic exponents of P at $x=c_j$. Note that $\lambda_{j,\nu}\in\mathbb{C}[x^{\frac1q}]$ with a certain positive integer q. We put $c_0=\infty$. Sometimes we allow that some of c_j's are not singular points of P. Here the characteristic exponents are defined when $c_j=0$ in the previous section and in general they are given under the coordinate y

$$y=\begin{cases}x-c & (c\ne\infty),\\ \frac1x & (c=\infty)\end{cases} \tag{3.1}$$

for a singular point c. Then the table

$$\begin{Bmatrix} x=c_0=\infty & \cdots & x=c_p\\ \lambda_{0,1} & \cdots & \lambda_{p,1}\\ \vdots & \vdots & \vdots\\ \lambda_{0,n} & \cdots & \lambda_{p,n}\end{Bmatrix} \tag{3.2}$$

is called the *Riemann scheme* of P (or the equation (1.2)).

Theorem 3.1 ([1, 2])**.** *We have Fuchs-Hukuhara relation*

$$\sum_{j=0}^p\sum_{i=1}^n\Bigl(\lambda_{j,i}(0)-\sum_{\nu=1}^{i-1}\deg(\lambda_{j,i}-\lambda_{j,\nu})\Bigr)=\frac{(p-1)(n-1)n}{2} \tag{3.3}$$

for the Riemann scheme (3.2) *of* $P\in W[x]$.

Proof. We may assume

$$P=\partial^n+b_{n-1}(x)\,\partial^{n-1}+\cdots+b_0(x),$$
$$x^nP=\vartheta^n+a_{n-1}(x)\vartheta^{n-1}+\cdots+a_0(x)$$

with $a_\nu(x)\in\mathbb{C}(x)$. We examine the residue of $a_{n-1}(x)$ at $x=c_j$ for $1\le j\le p$. We may assume $c_j=0$ by the coordinate transformation $x\mapsto x-c_j$. Under the notation (2.7)–(2.10) the residue of $x^{-1}a_{n-1}(x)$ at $x=c_j$ equals $-\tilde\lambda_{j,1}(0)-\cdots-\tilde\lambda_{j,n}(0)$, which equals $-\sum_{i=1}^n\bigl(\lambda_{j,i}(0)-\sum_{\nu=1}^{i-1}\deg(\lambda_{j,i}-\lambda_{j,\nu})\bigr)$. Since $\operatorname{ord}(\vartheta^n-$

$x^n\,\partial^n - \frac{n(n-1)}{2}x^{n-1}\,\partial^{n-1}) < n-1$, we have $xb_{n-1}(x) = a_{n-1}(x) + \frac{n(n-1)}{2}$ and the residue of $b_{n-1}(x)$ at $x = c_j$ equals

$$\frac{n(n-1)}{2} - \sum_{i=1}^{n}\Bigl(\lambda_{j,i}(0) - \sum_{\nu=1}^{i-1}\deg(\lambda_{j,i}-\lambda_{j,\nu})\Bigr).$$

Putting $y = \frac{1}{x}$ and $\vartheta_y = y\frac{d}{dy}$, we have $\vartheta = -\vartheta_y$ and $(-1)^n x^n P = \vartheta_y^n - a_{n-1}(\frac{1}{y})\vartheta_y^{n-1} + \cdots + (-1)^n a_0(\frac{1}{y})$. Hence the residue of $y^{-1}a_{n-1}(\frac{1}{y})$ at $y=0$ equals

$$\sum_{i=1}^{n}\Bigl(\lambda_{0,i}(0) - \sum_{\nu=1}^{i-1}\deg(\lambda_{0,i}-\lambda_{0,\nu})\Bigr).$$

Note that $y^{-1}a_{n-1}(\frac{1}{y}) = y^{-2}b_{n-1}(\frac{1}{y}) - \frac{n(n-1)}{2y}$ and the sum of the residues of $b_{n-1}(x)$ at $x = c_1,\dots,c_p$ equals the residue of $y^{-2}b_{n-1}(\frac{1}{y})$ at $y=0$ by Cauchy's integral formula. Hence we have the theorem. $\square$

Hereafter in this paper we assume that any singularity of the equation (1.2) is a regular singularity or an unramified irregular singularity. We define a generalized Riemann scheme, which we denote by GRS in this paper, and a spectral type as in the case of Fuchsian differential equations defined by [12].

Definition 3.2. Let $x = c$ be a singularity of the equation (1.2). For a polynomial $\lambda \in \mathbb{C}[x]$ and a positive integer m the equation has a *generalized characteristic exponent* $[\lambda]_{(m)}$ if the equation has formal solutions

$$u_\nu(y) = \check e_{\lambda+\nu}(y) + \psi_j(y)\check e_{\lambda+m}(y) \quad (\nu = 0,\dots,m-1)$$

with $\psi_j \in \mathbb{C}[\![x]\!]_{(\infty)}$. Here y is given by (3.1).

A *generalized Riemann scheme*, GRS in short, is the table

$$\begin{Bmatrix} x = c_0 = \infty & \cdots & x = c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots \\ [\lambda_{0,n_1}]_{(m_{0,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix}. \tag{3.4}$$

Here

$$n = m_{j,1} + \cdots + m_{j,n_j} \qquad (j = 0,\dots,p)$$

are $(p+1)$ tuples of partitions of n. The Riemann scheme corresponding to (3.4) is given by putting

$$[\lambda]_{(m)} := \begin{pmatrix} \lambda \\ \lambda+1 \\ \vdots \\ \lambda+m-1 \end{pmatrix} \text{ and } [\lambda]_m := \begin{pmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{pmatrix}. \tag{3.5}$$

Suppose

$$\lambda_{j,\nu'} - \lambda_{j,\nu} \notin \{0,1,\dots,m_{j,\nu}-1\} \quad (1 \le \nu < \nu' \le n_j,\ j = 0,\dots,p). \tag{3.6}$$

Then we define that P has GRS (3.4) if $[\lambda_{j,\nu}]_{(m_{j,\nu})}$ $(\nu = 1,\dots,n_j)$ are generalized characteristic exponents at $x = c_j$ for $j = 0,\dots,p$. (See the definition of GRS in [12] when (3.6) is not valid.)

Remark 2. Suppose

$$\deg(\lambda_{j,\nu} - \lambda_{j,\nu'}) > 0 \text{ or } \lambda_{j,\nu} - \lambda_{j,\nu'} \notin \mathbb{Z} \quad (1 \le \nu < \nu' \le n_j,\ j = 0,\dots,p). \tag{3.7}$$

Then P has GRS (3.4) if and only if P has the Riemann scheme corresponding to (3.4) and (1.2) has linearly independent solutions of the form $\psi(y)\check e_\lambda(y)$ with

$\psi(x) \in \mathbb{C}[\![x]\!]$, namely, they have not any $\log y$ term. Here y is given by (3.1) with $c = c_j$.

Let $\{[\lambda_{j,1}]_{(m_{j,1})}, \ldots, [\lambda_{j,n_j}]_{(m_{j,n_j})}\}$ be the set of generalized exponents of P at the singular point c_j. Then $n = m_{j,1} + \cdots + m_{j,n_j}$ be a partition of $n = \operatorname{ord} P$. For $r \in \mathbb{Z}_{\geq 0}$ we define equivalence relations $\underset{j,r}{\sim}$ between the elements of $I_n := \{1, \ldots, n\}$ as follows. For $i \in I_n$, we put $\nu_{j,i} \in \{1, \ldots, n_j\}$ by

$$m_{j,1} + \cdots + m_{j,\nu_{j,i}-1} < i \leq m_{j,1} + \cdots + m_{j,\nu_{j,i}}$$

and define

$$i \underset{j,r}{\sim} i' \iff \begin{cases} \nu_{j,i} = \nu_{j,i'} & (r = 0), \\ \deg\bigl(\lambda_{j,\nu_{j,i}}(x) - \lambda_{j,\nu_{j,i'}}(x)\bigr) < r & (r \geq 1). \end{cases} \tag{3.8}$$

Let $n_{j,r}$ be the number of equivalence classes under $\underset{j,r}{\sim}$ and let R_j be the Poincaré rank of P at the singular point c_j. Then

$$n_{j,0} = n_j \geq n_{j,1} \geq \cdots \geq n_{j,R_j} \geq n_{j,R_j+1} = 1.$$

By a suitable permutation of the indices $\nu \in \{1, \ldots, n_j\}$ of $m_{j,\nu}$ we may assume

$$i \leq i'' \leq i', \ i \underset{j,r}{\sim} i' \ \Rightarrow \ i \underset{j,r}{\sim} i''.$$

Let

$$n = m_{j,1}^{(r)} + \cdots + m_{j,n_{j,r}}^{(r)} \tag{3.9}$$

be the corresponding partition of n such that $m_{j,\nu}^{(0)} = m_{j,\nu}$ and for $\nu = 1, \ldots, n_{j,r}$

$$I_{j,\nu}^{(r)} := \{\nu \in \mathbb{Z}_{\geq 0} \mid m_{j,1}^{(r)} + \cdots + m_{j,\nu-1}^{(r)} < \nu \leq m_{j,1}^{(r)} + \cdots + m_{j,\nu}^{(r)}\} \tag{3.10}$$

give the equivalence classes under $\underset{j,r}{\sim}$. Note that $\{I_{j,\nu}^{(r)} \mid \nu = 1, \ldots, n_{j,r}\}$ is a refinement of $\{I_{j,\nu}^{(r+1)} \mid \nu = 1, \ldots, n_{j,r+1}\}$. Then we define that the $(R_0 + \cdots + R_p + p + 1)$ tuples of partitions $\mathbf{m} = \bigl(\mathbf{m}_j^{(r)}\bigr)_{\substack{r=0,\ldots,R_j \\ j=0,\ldots,p}} = \bigl(m_{j,\nu}^{(r)}\bigr)_{\substack{\nu=1,\ldots,n_{j,r} \\ r=0,\ldots,R_j \\ j=0,\ldots,p}}$ of n is the *spectral type* of P and that of GRS (3.4). Then the number of full parameters of GRS (3.4) with the spectral type $\bigl(m_{j,\nu}^{(r)}\bigr)_{\substack{\nu=1,\ldots,n_{j,r} \\ r=0,\ldots,R_j \\ j=0,\ldots,p}}$ equals $R = \sum_{j=0}^{p} \sum_{r=0}^{R_j} n_{j,r} - 1$. Here we note that we always impose Fuchs-Hukuhara condition on GRS.

As in the case of Fuchsian differential equation, this spectral type is expressed by writing the numbers $m_{j,\nu}^{(r)}$. The numbers are separated by "," indicating different singular points and by "|" indicating different levels of the equivalence relations:

$$m_{0,1}^{(0)} m_{0,2}^{(0)} \cdots m_{0,n_{0,0}}^{(0)} | \cdots | m_{0,1}^{(R_1)} \cdots m_{0,n_{0,R_1}}^{(R_1)}, m_{1,1}^{(0)} \cdots, \cdots m_{p,n_{p,R_p}}^{(R_p)}$$

Remark 3. Note that for any $(j, r, k) \in \mathbb{Z}^3$ with $0 \leq j \leq p$, $1 \leq r \leq R_j$ and $1 \leq k \leq n_j$ there exists $\ell \in \mathbb{Z}$ such that

$$m_{j,1}^{(r)} + \cdots + m_{j,k}^{(r)} = m_{j,1}^{(r-1)} + \cdots + m_{j,\ell}^{(r-1)}. \tag{3.11}$$

The *index of the rigidity* of GRS (3.4) is defined by that of the tuples of the partitions $\bigl(m_{j,\nu}^{(r)}\bigr)$ (cf. [10, 12]):

$$\operatorname{idx}\{\lambda_{\mathbf{m}}\} := \operatorname{idx} \mathbf{m} = \operatorname{idx}\bigl(m_{j,\nu}^{(r)}\bigr) = 2n^2 - \sum_{j=0}^{p} \sum_{r=0}^{R_j} \Bigl(n^2 - \sum_{\nu=1}^{n_{j,r}} \bigl(m_{j,\nu}^{(r)}\bigr)^2\Bigr). \tag{3.12}$$

Lemma 3.3. i) The index of the rigidity satisfies

$$\mathrm{idx}\,\mathbf{m} = 2n^2 - \sum_{j=0}^{p}\Bigl(n^2 - \sum_{\nu=1}^{n_j} m_{j,\nu}^2\Bigr) - \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{\nu'=1}^{n_j} m_{j,\nu}m_{j,\nu'}\deg\bigl(\lambda_{j,\nu}(x)-\lambda_{j,\nu'}(x)\bigr).$$

ii) Put $\mathrm{ord}\,\mathbf{m} = m_{j,1}+\cdots+m_{j,n_j} = n$. As in the Fuchsian case (cf. [12, Definition 4.17]), Fuchs-Hukuhara relation is given by

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu}(0) = \mathrm{ord}\,\mathbf{m} - \tfrac{1}{2}\mathrm{idx}\,\mathbf{m}. \tag{3.13}$$

Proof. Since

$$n^2 - \sum_{\nu=1}^{n_{j,r}}\bigl(m_{j,\nu}^{(r)}\bigr)^2 = 2\sum_{1\le\nu<\nu'\le n_{j,r}} m_{j,\nu}^{(r)}m_{j,\nu'}^{(r)} = \sum_{\deg(\lambda_{j,\nu}-\lambda_{j,\nu'})\ge r,\ \nu\ne\nu'} m_{j,\nu}m_{j,\nu'},$$

we have

$$\begin{aligned}
\mathrm{idx}\,\mathbf{m} &= 2n^2 - \sum_{j=0}^{p}\sum_{r=0}^{R_j}\Bigl(n^2 - \sum_{\nu=1}^{n_{j,r}}\bigl(m_{j,\nu}^{(r)}\bigr)^2\Bigr)\\
&= 2n^2 - \sum_{j=0}^{p}\Bigl(n^2 - \sum_{\nu=1}^{n_j}\bigl(m_{j,\nu}^{(0)}\bigr)^2\Bigr) - \sum_{j=0}^{p}\sum_{r=1}^{R_j}\sum_{\substack{\deg(\lambda_{j,\nu}-\lambda_{j,\nu'})\ge r\\ \nu,\,\nu'\in\{1,\dots,n_{j,r}\}}} m_{j,\nu}m_{j,\nu'}\\
&= 2n^2 - \sum_{j=0}^{p}\Bigl(n^2 - \sum_{\nu=1}^{n_j} m_{j,\nu}^2\Bigr) - \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{\nu'=1}^{n_j} m_{j,\nu}m_{j,\nu'}\deg\bigl(\lambda_{j,\nu}-\lambda_{j,\nu'}\bigr).
\end{aligned}$$

In the case of GRS (3.4), Fuchs-Hukuhara relation (3.3) says

$$\sum_{j=0}^{p}\sum_{i=1}^{n_j}\sum_{\nu=0}^{m_{j,i}-1}\Bigl(\lambda_{j,i}(0)+\nu-\sum_{\nu=1}^{i-1} m_{j,\nu}\deg(\lambda_{j,i}-\lambda_{j,\nu})\Bigr) = \frac{(p-1)(n-1)n}{2},$$

which means

$$\begin{aligned}
\sum_{j=0}^{p}\sum_{i=1}^{n_j} m_{j,i}\lambda_{j,i}(0) &= -\sum_{j=0}^{p}\sum_{i=1}^{n_j}\frac{(m_{j,i}-1)m_{j,i}}{2} + \sum_{j=0}^{p}\sum_{i=0}^{p}\sum_{\nu=1}^{i-1} m_{j,\nu}m_{j,i}\deg(\lambda_{j,i}-\lambda_{j,\nu})\\
&\quad + \frac{(p-1)(n-1)n}{2}\\
&= -\frac{\mathrm{idx}\,\mathbf{m}}{2} + n^2 - \frac{(p+1)(n-1)n}{2} + \frac{(p-1)(n-1)n}{2} = n - \frac{\mathrm{idx}\,\mathbf{m}}{2}.
\end{aligned}$$

□

Example 3.4. Suppose that the left scheme below is the generalized Riemann scheme of P, which may be written as the right scheme below (cf. (3.5)):

$$\begin{Bmatrix} x=\infty & x=0\\ [a_0+a_1x+a_2x^2]_{(2)} & [c_1]_{(2)}\\ b_0+b_1x & c_2\\ c_0+b_1x & c_3 \end{Bmatrix} = \begin{Bmatrix} x=\infty & (1) & (2) & x=0\\ [a_0]_{(2)} & [a_1]_2 & [a_2]_2 & [c_1]_{(2)}\\ b_0 & [b_1]_2 & [0]_2 & c_2\\ c_0 & & & c_3 \end{Bmatrix}.$$

Then the spectral type is $211|22|22,211$, which equals $\mathbf{m}_0^{(0)}|\mathbf{m}_0^{(1)}|\mathbf{m}_0^{(2)},\mathbf{m}_1^{(0)}$ and the Fuchs-Hukuhara relation is $2a_0+b_0+c_0+2c_1+c_2+c_3=6$.

Here we assume that the complex numbers a_i, b_j, c_k are generic under the Fuchs-Hukuhara relation. The spectral type is kept invariant even if we replace b_0+b_1x and c_0+b_1x by $b_0+b_1x+b_2x^2$ and $c_0+b_1x+b_2x^2$, respectively. If c_0+b_1x is replaced by c_0+b_2x, the spectral type changes into $211|211|22,211$ and

$$\mathrm{idx}(211|22|22,211) = -4 \quad\text{and}\quad \mathrm{idx}(211|211|22,211) = -6.$$

Note that (1.2) has solutions $u(x)$ with the asymptotic behavior

$$\begin{aligned} u(x) &\sim x^{-a_0}(1+o(x^{-1}))e^{-a_1x-\frac12 a_2x^2},\ x^{-a_0-1}e^{-a_1x-\frac12 a_2x^2} && (x\to+\infty),\\ &\sim x^{-b_0}e^{-b_1x},\ x^{-c_0}e^{-b_1x} && (x\to+\infty),\\ &\sim (1+o(x))x^{c_1},\ x^{c_1+1},\ x^{c_2},\ x^{c_3} && (x\to+0). \end{aligned}$$

4. Unfolding of Riemann scheme

Let m be a positive integer. Consider the equation

$$\Bigl(\prod_{i=0}^m (x-c_i)\Bigr)u' = \lambda u \tag{4.1}$$

with holomorphic parameters $c_0,\dots,c_m$, which has a solution $\exp(\int_\infty^x \frac{\lambda ds}{\prod_{0\le i\le m}(s-c_i)})$. If $c_i\ne c_j$ for $0\le i<j\le m$, the equation is Fuchsian and the Riemann scheme equals

$$\begin{Bmatrix} x=c_i\ (i=0,\dots,m)\\ \frac{\lambda}{\prod_{0\le\nu\le m,\,\nu\ne i}(c_i-c_\nu)} \end{Bmatrix}. \tag{4.2}$$

The Riemann scheme of the confluence limit for $\forall c_i\to c$ equals

$$\begin{Bmatrix} x=c\\ \lambda x^m \end{Bmatrix} = \begin{Bmatrix} x=c & (1) & \cdots & (m-1) & (m)\\ 0 & 0 & \cdots & 0 & \lambda \end{Bmatrix}. \tag{4.3}$$

We say that the Riemann scheme (4.2) is an unfolding of (4.3).

Recall the *versal additions* for $m\in\mathbb{Z}_{>0}$ introduced by [12, §2.3]:

$$\begin{aligned} \mathrm{AdV}^0_{(c_0,\dots,c_m)}(\lambda_0,\dots,\lambda_m) &:= \mathrm{Ad}\Bigl((x-c_0)^{\lambda_0}\exp\Bigl(\int_\infty^x\sum_{k=1}^m\frac{\lambda_k ds}{\prod_{\nu=0}^k(s-c_\nu)}\Bigr)\Bigr)\\ &= \mathrm{Ad}\Bigl(\prod_{i=0}^m (x-c_i)^{\sum_{k=i}^m\frac{\lambda_k}{\prod_{1\le\nu\le k,\,\nu\ne i}(c_i-c_\nu)}}\Bigr), \end{aligned} \tag{4.4}$$

$$\mathrm{AdV}^0_{(c_0,\dots,c_m)}(\lambda_0,\dots,\lambda_m)(\partial) = \partial - \sum_{k=0}^m\frac{\lambda_k}{\prod_{0\le\nu\le k}(x-c_\nu)}, \tag{4.5}$$

$$\sum_{k=0}^m\frac{\lambda_k}{\prod_{0\le\nu\le k}(x-c_\nu)} = \sum_{i=0}^m\Bigl(\sum_{k=i}^m\frac{\lambda_k}{\prod_{1\le\nu\le k,\,\nu\ne i}(c_i-c_\nu)}\Bigr)\frac{1}{x-c_i}, \tag{4.6}$$

$$\begin{aligned} \mathrm{AdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_m})}(\lambda_1,\dots,\lambda_m) &:= \mathrm{Ad}\Bigl(\exp\Bigl(-\sum_{k=1}^m\int_0^x\frac{\lambda_k s^{k-1}ds}{\prod_{1\le i\le k}(1-c_is)}\Bigr)\Bigr),\\ &= \mathrm{Ad}\Bigl(\prod_{i=1}^m(1-c_ix)^{\sum_{k=i}^m\frac{\lambda_i}{c_i\prod_{1\le\nu\le k\,\nu\ne i}(c_i-c_\nu)}}\Bigr), \end{aligned} \tag{4.7}$$

$$\mathrm{AdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_m})}(\lambda_1,\dots,\lambda_m)(\partial) = \partial + \sum_{k=1}^m\frac{\lambda_k x^{k-1}}{\prod_{i=1}^k(1-c_ix)}, \tag{4.8}$$

$$\sum_{k=1}^m\frac{\lambda_k x^{k-1}}{\prod_{i=1}^k(1-c_ix)} = -\sum_{i=1}^m\Bigl(\sum_{k=i}^m\frac{\lambda_k}{c_i\prod_{1\le\nu\le k,\,\nu\ne i}(c_i-c_\nu)}\Bigr)\frac{1}{x-\frac{1}{c_i}}. \tag{4.9}$$

Here $(\lambda_0,\dots,\lambda_m),\ (c_0,\dots,c_m)\in\mathbb{C}^{m+1}$. Summing up the residues of the equality (4.6) and putting $c_0=0$, we have the identities

$$\sum_{i=0}^{k}\frac{1}{\prod_{0\le\nu\le k,\,\nu\ne i}(c_i-c_\nu)}=0\quad(1\le k\le m),$$
$$\frac{(-1)^k}{\prod_{\nu=1}^k c_\nu}+\sum_{i=1}^{k}\frac{1}{c_i\prod_{1\le\nu\le k,\ \nu\ne i}(c_i-c_\nu)}=0\quad(1\le k\le m).$$

Definition 4.1 (Unfolding of generalized Riemann scheme)**.** Let

$$\lambda_{j,\nu}(x)=\lambda_{j,\nu,0}+\lambda_{j,\nu,1}x+\cdots+\lambda_{j,\nu,R_j}x^{R_j}\in\mathbb{C}[x] \tag{4.10}$$

be the generalized characteristic exponents $\lambda_{j,\nu}$ of GRS (3.4). Put

$$c_{j,r}=\begin{cases}\frac{1}{t_{0,0}}=\infty & (j=r=0,\ t_{0,0}=0),\\ \frac{1}{t_{0,r}} & (j=0,\ 1\le r\le R_j),\\ c_j+t_{j,r} & (1\le j\le p,\ 0\le r\le R_j).\end{cases}$$

For $j=0,\dots,p$ and $r=0,\dots,R_j$ and $\nu=1,\dots,n_{r,j}$, we choose $\ell_{j,r,\nu}$ by

$$m_{j,1}^{(r)}+\cdots+m_{j,\nu}^{(r)}=m_{j,1}+\cdots+m_{j,\ell_{j,r,\nu}}$$

(cf. (3.11)) and define

$$\lambda_{j,\nu}^{(r)}(t):=\sum_{k=1}^{R_j}\frac{\tilde\lambda_{j,\ell_{j,r,\nu},k}(t)}{\prod_{\substack{0\le s\le k\\ s\ne r}}(t_{j,r}-t_{j,s})}\quad(0\le j\le p,\ 0\le r\le R_j,\ 1\le\nu\le n_{j,r}). \tag{4.11}$$

Here $\tilde\lambda_{j,\nu,k}=\lambda_{j,\nu,k}$ or in general, $\tilde\lambda_{j,\nu}=\tilde\lambda_{j,\nu,0}+\tilde\lambda_{j,\nu,1}x+\cdots+\tilde\lambda_{j,\nu,R_j}x^{R_j}$ and $\tilde\lambda_{j,\nu,k}$ are holomorphic functions of $(t_{j,0},\dots,t_{j,R_j})$ satisfying

$$\tilde\lambda_{j,\nu,k}(0)=\lambda_{j,\nu,k}\quad\text{and}\quad\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}m_{j,\nu}\tilde\lambda_{j,\nu,0}(t)=n-\tfrac12\mathrm{idx}\,\mathbf{m}. \tag{4.12}$$

Then the *unfolding* of RS (3.4) is a generalized Riemann scheme

$$\left\{\begin{matrix}x=c_{j,r}\quad(0\le r\le R_j,\ 0\le j\le p)\\ [\lambda_{j,1}^{(r)}]_{(m_{j,1}^{(r)})}\\ \vdots\\ [\lambda_{j,n_{j,r}}^{(r)}]_{(m_{j,n_{j,r}}^{(r)})}\end{matrix}\right\} \tag{4.13}$$

of a Fuchsian differential equation with $(R_0+\cdots+R_p+p+1)$ regular singular points and GRS (3.4) is called the confluence of GRS (4.13) when $\forall t_{j,\nu}\to0$. We sometimes fix $t_{j,0}=0$ for $j=0,\dots,p$.

Remark 4. i) The index of rigidity and Fuchs-Hukuhara relation of GRS are kept invariant by the above unfolding (cf. (3.12), (3.13)).

ii) Let $P=a_1(x)\,\partial-a_0(x)\in W[x]$. If $a_1(0)\ne0$, there exist $m\in\mathbb{Z}_{\ge0}$, $c_1,\dots,c_m$, $\lambda_1,\dots,\lambda_m\in\mathbb{C}$ such that $P=a_0(x)\,\mathrm{AdV}_{(\frac{1}{c_1},\dots,\frac{1}{c_m})}(\lambda_1,\dots,\lambda_m)\,\partial$.

iii) The unfolding of GRS is essentially unique because of the following result.

The equality (4.6) gives a one-to-one correspondence between the set of holomorphic functions $\{\lambda_0,\dots,\lambda_m\}$ of $c=(c_0,\dots,c_m)$ and the holomorphic function $\Lambda(c,x)$ of c valued in $\mathbb{C}(x)$ such that

$$\Lambda(c,x)\in\sum_{r=0}^{m}\mathbb{C}\frac{1}{x-c_r}\quad\text{for any fixed c satisfying } c_k\ne c_\ell\quad(0\le k<\ell\le m).$$

See [11, Lemma 6.3] for a removable singularity of meromorphic parameters.

Example 4.2. Putting $t_{0,0} = t_{1,0} = 0$, $t_{0,1} = t_1$, $t_{0,2} = t_2$, we have an unfolding

$$\left\{\begin{matrix} x=\infty & x=\frac{1}{t_1} & x=\frac{1}{t_2} & x=0 \\ [a_0-\frac{a_1}{t_1}+\frac{a_2}{t_1t_2}]_{(2)} & [\frac{a_1}{t_1}+\frac{a_2}{t_1(t_1-t_2)}]_{(2)} & [\frac{a_2}{t_2(t_2-t_1)}]_{(2)} & [c_1]_{(2)} \\ b_0-\frac{b_1}{t_1} & [\frac{b_1}{t_1}]_{(2)} & [0]_{(2)} & c_2 \\ c_0-\frac{b_1}{t_1} & & & c_3 \end{matrix}\right\}$$

of GRS in Example 3.4.

Definition 4.3. The spectral type $\mathbf{m} = \left(\mathbf{m}_j^{(r)}\right)_{\substack{r=0,\dots,R_j\\ j=0,\dots,p}} = \left(m_{j,\nu}^{(r)}\right)_{\substack{\nu=1,\dots,n_{j,r}\\ r=0,\dots,R_j\\ j=0,\dots,p}}$ of GRS (3.4) is *irreducibly realizable* if there exist an irreducible equation (1.2) with GRS (3.4) for a generic value of parameters. The spectral type $\mathbf{m}$ is *versally realizable* if there exists an irreducible Fuchsian differential equation $Pu = 0$ with GRS (4.13) which has holomorphic parameters $t_{j,r} \in \mathbb{C}$ in a neighborhood of 0 and $(1-\frac{1}{2}\operatorname{idx}\mathbf{m})$ accessory parameters and moreover the confluence limit P_0 of P at $\forall t_{j,\nu} = 0$ has GRS (1.2) when the parameters $\lambda_{j,\nu}$ are generic. In this case P is called the *versal operator* with the spectral type $\mathbf{m}$, which is a generalization of the universal model in [12, §6.4] when $\mathbf{m}$ is of Fuchsian type.

Conjecture. We conjecture that the following three conditions are equivalent.
1. The spectral type $\mathbf{m}$ is irreducibly realizable.
2. The spectral type $\mathbf{m}$ is versally realizable.
3. The spectral type $\left(m_{j,\nu}^{(r)}\right)$ of the unfolding is irreducibly realizable as a Fuchsian differential equation.

Note that [12, §6] gives the necessary and sufficient condition for $\left(m_{j,\nu}^{(r)}\right)$ to be irreducibly realizable as a Fuchsian differential equation and moreover an algorithm to construct the Fuchsian differential operator P with a given generalized Riemann scheme, which is implemented as a computer program in [14]. Hence the key to solve the conjecture is to show that the operator P with GRS (4.13) has not a pole along any hyperplane defined by $t_{j,\nu} = t_{j,\nu'}$ ($\nu \neq \nu'$). Thus we can check that the conjecture is affirmative if $\operatorname{idx}\mathbf{m} \geq -2$ and $\mathbf{m}$ is indivisible, which will be explained in the following section.

5. Middle convolution

Let $u(x)$ be a solution to the equation (1.2) which has a singularity at the origin. Then for $\mu \in \mathbb{C}$, the middle convolution $\operatorname{mc}_\mu(P)$ of $P \in W[x]$ is characterized by the equation $\operatorname{mc}_\mu(P)I_\mu(u) = 0$ satisfied by the Riemann-Liouville integral

$$(I_\mu(u))(x) := \frac{1}{\Gamma(\mu)}\int_0^x u(t)(x-t)^{\mu-1}dt$$

of $u(x)$. Assume that the coefficients $a_\nu(x) \in \mathbb{C}[x]$ ($\nu = 0,\dots,n$) of $P \in W[x]$ given in (2.1) has no non-trivial common factor. Let N be an integer satisfying $N \geq \deg a_\nu(x)$ for $\nu = 0,\dots,n$. Put

$$P = \sum_{j=0}^{n}\sum_{i=0}^{N} c_{ij}x^i\,\partial^j \tag{5.1}$$

with $c_{ij} \in \mathbb{C}$. Then in [12, §1.3] the middle convolution $\operatorname{mc}_\mu(P)$ of P is defined by

$$\operatorname{mc}_\mu(P) := \partial^{-L}\sum_{j=0}^{n}\Big(\sum_{i=0}^{N} c_{ij}\,\partial^{N-i}(\vartheta+1-\mu)_i\,\partial^j\Big) \in W[x]. \tag{5.2}$$

Here L is the maximal integer under the condition $\operatorname{mc}_\mu(P) \in W[x]$. We note that $\partial^N P = \sum_{j=0}^n\sum_{i=0}^N c_{ij}\,\partial^N x^i\,\partial^j = \sum_{j=0}^n\sum_{i=0}^N c_{ij}\,\partial^{N-i}(\vartheta+1)_i\,\partial^j$.

For $P \in W(x)$, we put $\mathrm{mc}_\mu(P) = \mathrm{mc}_\mu(\phi P)$ with a non-zero suitable function $\phi \in \mathbb{C}(x)$ so that ϕP satisfies the above assumption. In this case $\mathrm{mc}_\mu(P)$ is defined up to a constant multiple.

We have the following lemma which can be applied to the solution to (1.2).

Lemma 5.1. *Let $u(x)$ be a holomorphic function on the domain $V_{\theta_0,\theta_1,L}$ given by* (2.19). *Suppose*

$$u(x) = \Bigl(\sum_{\nu=0}^{k} c_\nu x^\nu + o(x^k)\Bigr) x^\lambda \exp\Bigl(-\sum_{j=0}^{K} \frac{C_j}{x^{m_j}}\Bigr) \quad (V_{\theta_0,\theta_1,L} \ni x \to 0).$$

Here $k \in \mathbb{Z}_{\ge 0}$, $K \in \mathbb{Z}_{>0}$, c_ν, $C_j \in \mathbb{C}$, $m_j \in \mathbb{Z}$ and $m_0 > m_1 > \cdots > m_K > 0$.

Suppose moreover $\mathrm{Re}\,\frac{C_0}{x^{m_0}} > 0$ for $x \in V_{\theta_0,\theta_1,L}$ and $\mathrm{Re}\,\mu > 1$. Then

$$I_\mu(u)(x) = \Bigl(\sum_{\nu=0}^{k} c'_\nu x^\nu + o(x^k)\Bigr) x^{\lambda+(m_0+1)\mu} \exp\Bigl(-\sum_{j=0}^{K} \frac{C_j}{x^{m_j}}\Bigr) \quad (V_{\theta_0,\theta_1,L} \ni x \to 0)$$

with $c'_\nu \in \mathbb{C}$ satisfying $c'_0 = (m_0 C_0)^{-\mu} c_0$.

Proof. We may assume $c_0 = 1$ and $\theta_0 = 0$ and the positive numbers L and θ_1 are sufficiently small for the proof. For simplicity put $C = C_0$, $m = m_0$, $v(x) = x^{-\lambda} \exp(\sum_{j=0}^{K} C_j x^{-m_j}) u(x)$. Then $\lim_{V_{\theta_0,\theta_1,L} \ni x \to 0} v(x) = 1$.

$$\begin{aligned}
I &:= \int_0^x t^{\lambda+1} e^{-\frac{C}{t^m} - \frac{C_j}{t^{m_1}} - \cdots} (x-t)^{\mu-1} u(t) \tfrac{dt}{t} \\
&= x^{\lambda+\mu} \int_1^\infty s^{-\lambda-\mu} e^{-\frac{Cs^m}{x^m} - \frac{C_j s^{m_1}}{x^{m_1}} - \cdots} (s-1)^{\mu-1} v(\tfrac{x}{s}) \tfrac{ds}{s} \qquad (s = \tfrac{x}{t}) \\
&= x^{\lambda+\mu} e^{-\frac{C}{x^m} - \frac{C_j}{x^{m_1}} - \cdots} \int_0^\infty e^{-Cy(1 + \frac{C_j((x^m y+1)^{\frac{m_1}{m}} - 1)}{Cx^{m_1} y} + \cdots)} (x^m y + 1)^{-\frac{\lambda+\mu}{m} - 1} \\
&\quad \Bigl(\frac{(x^m y+1)^{\frac{1}{m}} - 1}{(\frac{x^m y}{m})}\Bigr)^{\mu-1} \Bigl(\frac{x^m y}{m}\Bigr)^{\mu-1} v\bigl(x(x^m y+1)^{-\frac{1}{m}}\bigr) x^m \frac{dy}{m} \qquad (y = \tfrac{s^m - 1}{x^m}).
\end{aligned}$$

Here we note that

$$\begin{aligned}
\frac{(s+1)^r - 1}{s} &= \sum_{\nu=0}^{\infty} \binom{r}{\nu+1} s^\nu = r + \frac{r(r-1)}{2} s + \cdots \quad (|s| < 1), \\
\Bigl|\frac{(s+1)^r - 1}{s}\Bigr| &\le \min\bigl\{r,\ \frac{3}{|s|^{1-r}}\bigr\} \quad (0 \le r \le 1,\ \mathrm{Re}\, s \ge 0).
\end{aligned}$$

Then putting

$$\begin{aligned}
z &= y\Bigl(1 + \sum_{j=1}^{K} \frac{C_j((x^m y+1)^{\frac{m_j}{m}} - 1)}{Cx^m y} x^{m-m_j}\Bigr), \\
I &= x^{\lambda+(m+1)\mu} m^{-\mu} e^{-\sum_{j=0}^{K} \frac{C_j}{x^{m_j}}} \int_0^\infty e^{-Cz} \Bigl(\frac{Cz}{C}\Bigr)^\mu \cdot \tilde{v}(x,z) \frac{dz}{z}.
\end{aligned}$$

Here

$$\begin{aligned}
\tilde{v}(x,z) &= \sum_{j=0}^{k} \Bigl(\sum_{i=0}^{\infty} \tilde{c}_{ij} x^i\Bigr) z^j + o(z^k) \qquad (\mathbb{R}_+ \ni z \to 0,\ x \in V_{\theta_0,\theta_1,L}), \\
|\tilde{v}(x,z)| &\le M(1+|z|)^M \qquad (z \ge r,\ x \in V_{\theta_0,\theta_1,L})
\end{aligned}$$

with certain complex numbers $\tilde{c}_{ij}$, r and M are suitable positive numbers and $\tilde{c}_{00} = c_0$. Thus we have easily the lemma (by Watson's lemma). □

Remark 5. i) Since I_μ is defined for $\mu \in \mathbb{C}$ by the analytic continuation, the assumption $\operatorname{Re}\mu > 1$ is not necessary in Lemma 5.1. Similarly, we have

$$I_\mu x^\lambda = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)} x^{\lambda+\mu} \quad (x > 0).$$

ii) Replacing $V_{\theta_1,\theta_1,L}$ by $V'_{\theta_0,\theta_1,L} := \{\frac{1}{x} \mid x \in V_{-\theta_0,\theta_1,\frac{1}{L}}\}$ in the previous lemma, we have

$$\frac{1}{\Gamma(\mu)}\int_{\infty e^{i\theta_0}}^x u(t)(t-x)^{\mu-1}dt = \Bigl(\sum_{\nu=0}^k c'_\nu x^{-\nu} + o(x^{-k})\Bigr)x^{-\lambda+(m_0+1)\mu}\exp\Bigl(-\sum_{j=0}^K C_j x^{m_j}\Bigr)$$
$$(V'_{\theta_0,\theta_1,L} \ni x \to \infty)$$

with $c'_0 = (m_0C_0)^{-\mu}c_0$ if a holomorphic function on $V'_{\theta_0,\theta_1,L}$ satisfies

$$u(x) = \Bigl(\sum_{\nu=0}^k c_\nu x^{-\nu} + o(x^{-k})\Bigr)x^{-\lambda}\exp\Bigl(-\sum_{j=0}^K C_j x^{m_j}\Bigr) \quad (V'_{\theta_0,\theta_1,L} \ni x \to \infty)$$

and $\operatorname{Re} C_0 x^{m_0} > 0$ for $x \in V'_{\theta_0,\theta_1,L}$.

The generalized Riemann scheme of $\mathrm{mc}_\mu(P)$ is given by the following theorem.

Theorem 5.2 ([12, Theorem 5.2], [5, Theorem 3.2]). *Suppose $P \in W[x]$ has the generalized Riemann scheme $\bigl\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\bigr\}_{\substack{\nu=1,\dots,n_j\\ j=0,\dots,p}}$ given in (3.4) and it is irreducible in $W(x)$. We may assume $\lambda_{j,1} = 0$ for $j = 1,\dots,p$ and $\mu = \lambda_{0,1} - 1$. Here some $m_{j,1}$ are allowed to be zero. If $\{\lambda_{j,\nu}\}$ and μ are generic (cf. [12, 5]) under this assumption, $\mathrm{mc}_\mu(P)$ has GRS $\bigl\{[\lambda'_{j,\nu}]_{(m'_{j,\nu})}\bigr\}_{\substack{\nu=1,\dots,n_j\\ j=0,\dots,p}}$ given by*

$$\begin{aligned}
&d(\mathbf{m}) := 2n - \sum_{j=0}^p\sum_{r=0}^{R_j}\bigl(n - m_{j,1}^{(r)}\bigr),\\
&m'_{j,\nu} = m_{j,\nu} - \delta_{\nu,1}\cdot d(\mathbf{m}) && (1 \le \nu \le n_j,\ 0 \le j \le p),\\
&\lambda'_{j,0} = \delta_{j,0}\cdot(1-\mu) && (j = 0,\dots,p),\\
&\lambda'_{j,\nu} = \lambda_{j,\nu} + (-1)^{\delta_{j,0}}(1+\deg\lambda_{j,\nu})\cdot\mu && (1 \le \nu \le n_j,\ 0 \le j \le p)
\end{aligned}$$

and the index of rigidity and the irreducibility of P are kept under mc_μ.

Remark 6. i) Theorem 5.2 is proved by [12] when P has only regular singularities. It is extended by [5] in the case when P has unramified irregular singularities together with regular singularities.

ii) Note that

$$\begin{aligned}
d(\mathbf{m}) &= 2n - \sum_{j=0}^p\bigl(n - m_{j,1}\bigr) - \sum_{j=0}^p\sum_{r=1}^{R_j}\sum_{\nu=1}^{n_j}\sum_{\deg(\lambda_{j,\nu}-\lambda_{j,1})\ge r} m_{j,\nu}\\
&= 2n - \sum_{j=0}^p\bigl(n - m_{j,1}\bigr) - \sum_{j=0}^p\sum_{\nu=1}^{n_j}\deg\bigl(\lambda_{j,\nu}-\lambda_{j,1}\bigr)m_{j,\nu}.
\end{aligned}$$

Hence if

$$\sum_{r=0}^{R_j} m_{j,1}^{(r)} \ge \sum_{r=0}^{R_j} m_{j,\nu}^{(r)} \quad (1 \le \nu \le n_j,\ j = 0,\dots,p), \tag{5.3}$$

we have

$$m_{j,1} - \sum_{\nu'=1}^{n_j}\deg(\lambda_{j,\nu'} - \lambda_{j,1})m_{j,\nu'} \ge m_{j,\nu} - \sum_{\nu'=1}^{n_j}\deg(\lambda_{j,\nu'} - \lambda_{j,\nu})m_{j,\nu'}$$

and

$$
\begin{aligned}
0 \le {} & \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\Bigl(m_{j,1} - \sum_{\nu'=1}^{n_j}\deg(\lambda_{j,\nu'}-\lambda_{j,1})m_{j,\nu'}\Bigr)m_{j,\nu}\\
& - \Bigl(\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}^2 - \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{\nu'=1}^{n_j}\deg(\lambda_{j,\nu'}-\lambda_{j,\nu})m_{j,\nu'}m_{j,\nu}\Bigr)\\
= {} & n\cdot d(\mathbf{m}) - \operatorname{idx}\mathbf{m},
\end{aligned}
$$

which implies $d(\mathbf{m}) > 0$ when $\operatorname{idx}\mathbf{m} > 0$ in Theorem 5.2.

Suppose $\mathbf{m} = \bigl(m_{j,\nu}^{(r)}\bigr)$ is a realizable spectral type and let $P \in W[x]$ be the versal operator with GRS (3.4) and (4.13). We may assume (5.3) by suitable permutations of indices $\nu \in \{1,\dots,n_j\}$ of $m_{j,\nu}$. Applying suitable versal additions to P corresponding to $-\tilde\lambda_{j,1}$, we may assume $\tilde\lambda_{j,1} = 0$ for $j = 1,\dots,p$ and $\deg\tilde\lambda_{0,1} = 0$ for any t. Then we apply mc_μ in Theorem 5.2 to P and we have $\mathrm{mc}_\mu(P)$ with the spectral type $\mathbf{m}'$ satisfying $\operatorname{ord}\mathbf{m}' = \operatorname{ord}\mathbf{m}' - d(\mathbf{m})$. Here we note that since L in (5.2) is determined for generic values of holomorphic parameters $t_{j,\nu}$ corresponding to the unfolding GRS (4.13), the integer L in (5.2) is invariant for generic values of parameters and $t_{j,\nu}$ are holomorphic parameters of $\mathrm{mc}_\mu(P)$. Then $\mathrm{mc}_\mu(P)$ is a versal operator with the spectral type $\mathbf{m}'$ (cf. Definition 4.1). Since $\mathrm{mc}_{-\mu}\circ\mathrm{mc}_\mu = \mathrm{id}$, the spectral type $\mathbf{m}$ is versally realizable if and only if so is $\mathbf{m}'$. In the same way it is proved that $\mathbf{m}$ is irreducibly realizable if and only if so is $\mathbf{m}'$. We call this procedure Katz's reduction of $\mathbf{m}$ (cf. [10]).

We put $\mathbf{m}' = \partial\,\mathbf{m}$ according to the above procedure, namely, suitable permutations of indices, versal additions and a middle convolution. Let K be the minimal non-negative integer such that one of the following holds

1. $\operatorname{ord}\partial^{i-1}\mathbf{m} > \operatorname{ord}\partial^{i}\mathbf{m}$ for $i = 1,\dots,K$ and $\partial^K\mathbf{m} = \bigl(\tilde m_{j,\nu}^{(r)}\bigr)$ is not tuples of partitions, namely, there exists $\tilde m_{j,\nu}^{(r)}$ with $\tilde m_{j,\nu}^{(r)} < 0$.
2. $\operatorname{ord}\partial^{i-1}\mathbf{m} > \operatorname{ord}\partial^{i}\mathbf{m}$ for $i = 1,\dots,K$ and $\partial^K\mathbf{m}$ is a trivial tuples of partitions with $\operatorname{ord}\partial^K\mathbf{m} = 1$.
3. $\operatorname{ord}\partial^{K-1}\mathbf{m} \le \operatorname{ord}\partial^{K}\mathbf{m}$.

Dividing into these three cases, we can conclude the following in each case:

1. $\mathbf{m}$ is not irreducibly realizable.
2. $\operatorname{idx}\mathbf{m} = 2$ and $\mathbf{m}$ is versally realizable.
3. $\operatorname{idx}\mathbf{m} \le 0$. We conjecture that $\mathbf{m}$ is versally realizable if and only if $\operatorname{idx}\mathbf{m} \ne 0$ or $\mathbf{m}$ is *indivisible*, namely, the greatest common divisor of $\{m_{j,\nu}^{(r)}\}$ equals 1.

Remark 7. In the case of Schlesinger systems the above conjecture is proved by [6] if we replace "versally realizable" by "irreducibly realizable".

Example 5.3. We give some examples of the above procedure. The number $d(\mathbf{m})$ is indicated upon the arrow corresponding to the procedure (cf. [12, Example 5.11]), which equals $2\times$(the sum of numbers in one block) $-$ (the sum of numbers without an underline).

$$\underline{1}111|\underline{2}2|\underline{2}2 \xrightarrow{2\cdot4-7=1} \underline{1}11|\underline{2}1|\underline{2}1 \xrightarrow{2\cdot3-4=2} (-1)11|1|1 \quad \text{(not irreducibly realizable)}$$

$$\underline{1}1|\underline{1}1|\underline{1}1 \xrightarrow{2\cdot2-3=1} 1|1|1 \quad \text{(versally realizable and rigid, versal Gauss hypergeometric)}$$

$$\underline{2}1121|\underline{2}113|\underline{4}3 \xrightarrow{2\cdot7-13=1} 11121|1113|33 \rightsquigarrow \underline{2}1111|\underline{3}111|\underline{3}3 \xrightarrow{2\cdot6-10=2} 1111|1111|13$$

$$\rightsquigarrow \underline{1}111|\underline{1}111|\underline{3}1 \xrightarrow{2\cdot4-7=1} \underline{1}11|\underline{1}11|\underline{2}1 \xrightarrow{2\cdot3-5=1} \underline{1}1|\underline{1}1|\underline{1}1 \xrightarrow{2\cdot2-3=1} 1|1|1 \quad \text{(rigid)}$$

$$\underline{1}11|\underline{2}1|\underline{2}1|\underline{2}1 \xrightarrow{2\cdot3-5=1} \underline{1}1|\underline{1}1|\underline{1}1|\underline{1}1 \xrightarrow{2\cdot2-4=0} 11|11|11|11 \quad \text{(basic, versally realizable)}$$

An irreducibly realizable spectral type $\mathbf{m}$ is called rigid if $\operatorname{idx}\mathbf{m} = 2$. Suppose the spectral type $\mathbf{m}$ satisfies $\operatorname{ord}\partial\mathbf{m} \geq \operatorname{ord}\mathbf{m}$. If $\operatorname{idx}\mathbf{m} \neq 0$ or $\mathbf{m}$ is indivisible, $\mathbf{m}$ is called *basic*. Then the basic spectral type is not rigid (cf. Remark 6 ii)).

We show in [7] that there are only finite number of indivisible basic spectral types with the same index of rigidity. For example, there are the following 15 indivisible basic spectral types $\mathbf{m}$ with $\operatorname{idx}\mathbf{m} = 0$:

$$\begin{aligned}
&\tilde{D}_4: \quad 11|11|11|11 \quad 11|11|11,11 \quad 11|11,11|11 \quad 11|11,11,11 \quad 11,11,11,11\\
&\tilde{E}_6: \quad 111|111|111 \quad 111|111,111 \quad 111,111,111\\
&\tilde{E}_7: \quad 1111|1111|22 \quad 1111|1111,22 \quad 1111,1111|22 \quad 1111,1111,22\\
&\tilde{E}_8: \quad 111111|2222,33 \quad 111111|33,222 \quad 111111,222,33
\end{aligned}$$

The versal operator for the spectral type $11|11|11|11$ contains the operator with the spectral type $11|11|11,11$ as special values of parameters (cf. §6). The basic spectral type which is not obtained as special values of parameters of a versal operator with another spectral type is called *basic confluent spectral type*. Then there are 5 indivisible basic confluent spectral types with the rigidity of index 0. They are

$$11|11|11|11 \quad 111|111|111 \quad 1111|1111|22 \quad 111111|222,33 \quad 111111|33,222.$$

Remark 8. A list of basic confluent spectral types $\mathbf{m}$ with $\operatorname{idx}\mathbf{m} \geq -2$ are obtained in [7] and we can construct versal operators with these spectral types. Hence the conjectures in this paper are valid for $\mathbf{m}$ if $\operatorname{idx}\mathbf{m} \geq -2$ and $\mathbf{m}$ is indivisible. They are confluent spectral types of basic Fuchsian spectral types. But in general there exists a basic confluent spectral type which is not any confluent spectral type of a basic Fuchsian spectral type as is given in [7, Remark 3.33].

6. Examples

As is shown in the previous section, there is one-to-one correspondence between the rigid spectral type of the equation (1.2) with regular or unramified irregular singularities and the rigid Fuchsian spectral type with a suitable confluence which corresponds to the condition stated in Remark 3. A simple and classical example is $11|11|11$ with the following Riemann scheme and the versal Riemann scheme:

$$\begin{Bmatrix} x=\infty \\ 1-\mu \\ \mu+\lambda_1 x+\lambda_2 x^2 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} x=\infty & x=\frac{1}{t_1} & x=\frac{1}{t_2} \\ 1-\tilde{\mu} & 0 & 0 \\ \tilde{\mu}-\frac{\tilde{\lambda}_1}{t_1}+\frac{\tilde{\lambda}_2}{t_1t_2} & \frac{\tilde{\lambda}_1}{t_1}+\frac{\tilde{\lambda}_2}{t_1(t_1-t_2)} & \frac{\tilde{\lambda}_2}{t_2(t_2-t_1)} \end{Bmatrix}.$$

Inverting Katz's reduction in Example 5.3, we have the versal operator

$$\begin{aligned}
\tilde{P} &= \mathrm{mc}_{\mu'} \circ \mathrm{AdV}_{(\frac{1}{t_1},\frac{1}{t_2})}(\lambda_1',\lambda_2')\,\partial \\
&= \mathrm{mc}_{\mu'}\Big(\partial + \frac{\lambda_1'}{1-t_1x} + \frac{\lambda_2' x}{(1-t_1x)(1-t_2x)}\Big) \\
&= \mathrm{mc}_{\mu'}\big(\partial(1-t_1x)(1-t_2x)\,\partial + \partial\big(\lambda_1'(1-t_2x)+\lambda_2' x\big)\big) \\
&= \big((1-t_1x)\,\partial + t_1(\mu'-1)\big)\big((1-t_2x)\,\partial + t_2\mu'\big) \\
&\quad + \lambda_1'\,\partial + (\lambda_2' - \lambda_1' t_2)(x\,\partial + 1 - \mu') \\
&= (1-t_1x)(1-t_2x)\partial^2 + (\bar{\lambda}_1 + \bar{\lambda}_2 x)\partial + \bar{\mu}\big(\bar{\lambda}_2 - t_1t_2(\bar{\mu}+1)\big),
\end{aligned}$$
$$\big(\lambda_1' = \bar{\lambda}_1 + (t_1+t_2)\bar{\mu},\quad \lambda_2' = \bar{\lambda}_2 + \bar{\lambda}_1 t_2 - (t_1-t_2)t_2\bar{\mu},\quad \mu' = 1-\bar{\mu}\big)$$

with the Riemann scheme

$$\left\{\begin{matrix} x=\infty & x=\frac{1}{t_1} & x=\frac{1}{t_2} \\ \bar\mu & 0 & 0 \\ \frac{\bar\lambda_2}{t_1t_2}-\bar\mu-1 & \frac{\bar\lambda_2+t_1\bar\lambda_1}{t_1(t_1-t_2)}+1 & \frac{\bar\lambda_2+t_2\bar\lambda_1}{t_2(t_2-t_1)}+1 \end{matrix}\right\} \qquad \bigl(t_1t_2(t_1-t_2)\neq 0\bigr), \tag{6.1}$$

which we call *versal Gauss hypergeometric operator* (see [12, §2.4]). Here we have

$$\tilde\lambda_1=\bar\lambda_1+t_1+t_2,\ \tilde\lambda_2=\bar\lambda_2+t_2(\bar\lambda_1-t_1+t_2)\ \text{ and }\ \tilde\mu=1-\bar\mu.$$

Special values of parameters corresponding to the confluence of (6.1) give

$$\left\{\begin{matrix} x=\infty & x=\frac{1}{t_1} \\ \bar\mu & 0 \\ -\frac{\bar\lambda_2}{t_1}x-\frac{\bar\lambda_2}{t_1^2}-\frac{\bar\lambda_1}{t_1}-\bar\mu & \frac{\bar\lambda_2}{t_1^2}+\frac{\bar\lambda_1}{t_1}+1 \end{matrix}\right\} \qquad (t_1\neq 0,\ t_2=0),$$

$$\left\{\begin{matrix} x=\infty & x=\frac{1}{t_1} \\ \bar\mu & 0 \\ \frac{\bar\lambda_2}{t_1^2}-\bar\mu-1 & (\frac{\bar\lambda_2}{t_1}+\bar\lambda_1)x-\frac{\bar\lambda_2}{t_1^2}+2 \end{matrix}\right\} \qquad (t_1=t_2\neq 0),$$

$$\left\{\begin{matrix} x=\infty \\ \bar\mu \\ 1-\bar\mu+\bar\lambda_1x+\bar\lambda_2x^2 \end{matrix}\right\}=\left\{\begin{matrix} x=\infty & (1) & (2) \\ \bar\mu & 0 & 0 \\ 1-\bar\mu & \bar\lambda_1 & \bar\lambda_2 \end{matrix}\right\} \qquad (t_1=t_2=0).$$

According to (4.8), we have a solution to $\tilde Pu=0$ with an integral representation

$$\int_{\frac{1}{t_1}}^{x}\exp\Bigl(-\int_0^t\Bigl(\frac{\lambda_1'}{1-t_1s}+\frac{\lambda_2's}{(1-t_1s)(1-t_2s)}\Bigr)ds\Bigr)(t-x)^{\mu'-1}dt.$$

In the same way as above we have solutions

$$\int_c^x\int_c^t\exp\Bigl(-\int_0^s\frac{\lambda_1(1-t_2u)+\lambda_1'u}{(1-t_1u)(1-t_2u)}du\Bigr)(t-s)^{\mu_1-1}(1-t_1t)^{\frac{\lambda_2}{t_1}}(x-t)^{\mu_2-1}ds\,dt$$

of a versal equation $\tilde Pu=0$ for the spectral type $111|111|21$. Here $c=\frac{1}{t_1}$ or $\frac{1}{t_2}$ or ∞ and $\tilde P=\mathrm{mc}_{\mu_2}\circ\mathrm{AdV}_{(\frac{1}{t_1})}(\lambda_2)\circ\mathrm{mc}_{\mu_1}\circ\mathrm{AdV}_{(\frac{1}{t_1},\frac{1}{t_2})}(\lambda_1,\lambda_1')\,\partial$. When $t_1=0$, $(1-t_1t)^{\frac{\lambda_2}{t_1}}=e^{-\lambda_2t}$. Note that $111,111,21$ is the spectral type of the equation satisfied by the generalized hypergeometric function ${}_3F_2$ (cf. [12, §13.5]).

The spectral type $\mathbf{m}=\overbrace{11|11|\cdots|11}^{p+1\text{ copies of }11}$ corresponds to the second order operator P which has only one unramified irregular singularity with Poincaré rank p. In this case $\operatorname{idx}\mathbf{m}=6-2p$ and we have the versal operator

$$\tilde P=\prod_{j=1}^{p}\Bigl(1-t_jx\Bigr)\partial^2+\Bigl(\sum_{j=1}^{p}\lambda_jx^{j-1}\Bigr)\partial+\mu\Bigl(\lambda_p-(-1)^p(\mu+1)\prod_{j=1}^{p-1}t_j\Bigr)x^{p-2}+\sum_{j=0}^{p-3}r_jx^j$$

for the spectral type $\mathbf{m}$. The generalized Riemann scheme of $\tilde P$ equals

$$\left\{\begin{matrix} x=\infty & x=\frac{1}{t_j}\quad(j=1,\dots,p) \\ \mu & 0 \\ \frac{(-1)^p\lambda_p}{t_1\cdots t_p}-\mu-1 & \frac{\sum_{i=1}^{p}t_j^{p-i}\lambda_i}{t_j\prod_{1\le i\le p,\,i\neq j}(t_j-t_i)}+1 \end{matrix}\right\}.$$

Here $r_0,\dots,r_{p-3}$ are accessory parameters and P is the operator with the Riemann scheme

$$\left\{\begin{matrix} x=\infty \\ \mu \\ p-1-\mu+\lambda_1x+\cdots+\lambda_px^p \end{matrix}\right\}=\left\{\begin{matrix} x=\infty & (1) & \cdots & (p) \\ \mu & 0 & \cdots & 0 \\ p-1-\mu & \lambda_1 & \cdots & \lambda_p \end{matrix}\right\}.$$

When $p = 3$, $\tilde{P}$ gives the confluent equations of Huen's equation whose spectral types are $11|11|11|11$, $11|11|11,11$, $11|11,11|11$ and $11|11,11,11$.

A versal Jordan-Pochhammer operator $\tilde{P}$ is an unfolding of the operator with the rigid spectral type $\overbrace{1(p-1)|1(p-1)|\cdots|1(p-1)}^{p+1 \text{ copies of } 1(p-1)}$, which is give in [12, §2.4], namely

$$\tilde{P} = \mathrm{mc}_\mu \circ \mathrm{AdV}_{(\frac{1}{t_1},\ldots,\frac{1}{t_p})}(\lambda_1,\ldots,\lambda_p)\,\partial = \sum_{k=0}^{p} p_k(x)\,\partial^{p-k},$$

$$p_0(x) = \prod_{j=1}^{p}(1-t_jx),\ \ q(x) = \sum_{k=1}^{p}\lambda_k x^{k-1}\prod_{j=k+1}^{p}(1-t_jx),$$

$$p_k(x) = \binom{-\mu+p-1}{k}p_0^{(k)}(x) + \binom{-\mu+p-1}{k-1}q^{(k-1)}(x).$$

The equation $\tilde{P}u = 0$ has GRS

$$\begin{Bmatrix} x=\infty & x=\frac{1}{t_i}\ \ (i=1,\ldots,p) \\ [1-\mu]_{(p-1)} & [0]_{(p-1)} \\ \displaystyle\sum_{i=1}^{p}\frac{(-1)^i\lambda_i}{\prod_{1\le\nu\le i}c_\nu}-\mu & \displaystyle\sum_{k=i}^{p}\frac{\lambda_k}{c_i\prod_{\substack{1\le\nu\le k\\ \nu\ne i}}(c_i-c_\nu)}+\mu \end{Bmatrix},$$

$$\begin{Bmatrix} x=\infty \\ [1-\mu]_{(p-1)} \\ (p-1)\mu+\displaystyle\sum_{i=1}^{p}\lambda_i x^i \end{Bmatrix} \qquad (\forall t_i = 0)$$

and a solution

$$\int_c^x \exp\Bigl(-\sum_{k=1}^{p}\int_0^t\frac{\lambda_k s^{k-1}ds}{\prod_{1\le i\le k}(1-t_is)}\Bigr)(x-t)^{\mu-1}dt \qquad (c=\tfrac{1}{t_i}\ \text{ or }\ \infty).$$

Let L be an integer satisfying $0 < L < m$. Put $t_{L+1} = \cdots = t_m = 0$ for the equation $\tilde{P}u = 0$. Then the point $x = \infty$ is an irregular singular point with Poincaré rank $m - L$. The last example in [13] calculates the local monodromy of the space of local solutions at this singular point. Note that [13] gives an algorithm calculating the local monodromy of any irregular singular point of the rigid equation $Pu = 0$ on $\mathbb{P}_1$ which has no ramified irregular singular point. The algorithm is implemented in [14].

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