Reducibility of hypergeometric equations

Toshio Oshima

Abstract. We study a necessary and sufficient condition so that hypergeometric equations are reducible. Here the hypergeometric equations with one variable mean the rigid Fuchsian linear ordinary differential equations. If the equations with one variable have more than four singular points, they naturally define hypergeometric equations with several variables including Appell's hypergeometric equations. We also study the reducibility of such equations with several variables and we find a new kind of reducibility, which appears, for example, in a decomposition of Appell's F_4.

Keywords. hypergeometric function, monodromy representations, middle convolution, Pfaffian system.

1. Introduction and preliminary results

The Gauss hypergeometric equation

$$x(1-x)u'' + \bigl(c-(a+b+1)x\bigr)u' - abu = 0 \tag{1.1}$$

is reducible if and only if at least one of the numbers

$$a,\ b,\ a-c,\ b-c \tag{1.2}$$

is an integer. An elementary proof of this result using neither an integral representation nor a connection formula of the solution is given in [O2]. Here the linear ordinary differential equation with coefficients in rational functions is said to be reducible if and only if the equation has a non-zero solution satisfying a linear ordinary differential equation with coefficients in rational functions whose order is lower than the original equation. Note that the reducibility of the equation is equivalent to the reducibility of the monodromy group of the solutions if the equation is Fuchsian.

2010 Mathematics Subject Classification. Primary 34M03; Secondary 34A30, 34M35, 33C65.

Supported by Grant-in-Aid for Scientific Researches (B), No. 25287017, Japan Society of Promotion of Science.

The Riemann scheme of the Gauss hypergeometric equation is

$$\begin{Bmatrix} x=\infty & 0 & 1 \\ a & 0 & 0 \\ b & 1-c & c-a-b \end{Bmatrix}, \tag{1.3}$$

which is a table of singular points of (1.1) and the characteristic exponents at each singular point. The characteristic exponents 0 and $1-c$ at $x=0$ mean that (1.1) has solutions $u(x)$ satisfying $u(x)\sim x^0$ and $u(x)\sim x^{1-c}$, respectively, when $x\to 0$. If the parameters a, b and c are generic, the differential equation with this Riemann scheme is (1.1). Since the coefficients of the equation are polynomial functions of the parameters a, b and c, the equation (1.1) is naturally and uniquely defined by this Riemann scheme for any values of parameters.

In [O1] we define a (generalized) Riemann scheme

$$\begin{aligned}\{\lambda_{\mathbf{m}}\} &= \big([\lambda_{j,\nu}]_{(m_{j,\nu})}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \\ &= \begin{Bmatrix} x=c_0=\infty & c_1 & \cdots & c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix}\end{aligned} \tag{1.4}$$

for a general Fuchsian ordinary differential equation $Pu=0$ of order n. Here $c_0, c_1, \dots, c_p$ are singular points of the equation and the sets of characteristic exponents of the equation at $x=c_j$ are

$$\{\lambda_{j,\nu}+i \mid i=0,1,\dots,m_{j,\nu}-1,\ \nu=1,\dots,n_j\}, \tag{1.5}$$

respectively, and moreover the local monodromies of the solutions of the equation at $x=x_j$ are semisimple if $\lambda_{j,\nu}-\lambda_{j,\nu'}\notin\mathbb{Z}$ for $1\le\nu<\nu'\le n_j$ and $j=0,\dots,p$. We say that $\{\lambda_{\mathbf{m}}\}$ is the Riemann scheme of P and the $(p+1)$ tuples of partitions

$$\mathbf{m}=(m_{j,1},\dots,m_{j,n_j})_{j=0,\dots,p} \tag{1.6}$$

of n is called the *spectral type* of the equation $Pu=0$ or the operator P and we put $\operatorname{ord}\mathbf{m}=n$ which equals $m_{j,1}+\cdots+m_{j,n_j}$. If there is no confusion, $\mathbf{m}$ is shortly expressed by $m_{0,1}\cdots m_{0,n_0},\cdots,m_{p,1}\cdots m_{p,n_p}$ and for example, the spectral type of Gauss hypergeometric equation is $11,11,11$.

The existence of such equation implies the Fuchs relation

$$|\{\lambda_{\mathbf{m}}\}| := \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu}-\operatorname{ord}\mathbf{m}+\tfrac12\operatorname{idx}\mathbf{m}=0. \tag{1.7}$$

Here the *index of rigidity* of $\mathbf{m}$ is defined by [Kz] as follows.

$$\operatorname{idx}\mathbf{m} := \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}^2-(p-1)(\operatorname{ord}\mathbf{m})^2. \tag{1.8}$$

On the other hand, we say that $\mathbf{m}$ is *irreducibly realizable* if there exists an irreducible Fuchsian equation $Pu = 0$ with the Riemann scheme (1.4) for generic parameters $\lambda_{j,\nu}$ under the Fuchs relation. A characterization of irreducibly realizable spectral types $\mathbf{m}$ is given in [O1]. Moreover if $\mathbf{m}$ is irreducibly realizable, then the equation $Pu = 0$ with the Riemann scheme (1.4) satisfying (1.7) has $(1 - \frac{1}{2}\operatorname{idx}\mathbf{m})$ accessory parameters and the differential operator P is polynomial functions of the parameters $\lambda_{j,\nu}$ and the accessory parameters, which is the *universal operator* in [O1, Theorem 6.14]. If an irreducibly realizable spectral type $\mathbf{m}$ satisfies

$$\operatorname{idx}\mathbf{m} = 2, \tag{1.9}$$

then $\mathbf{m}$ is called *rigid*. In this case the Fuchsian differential equation with the Riemann scheme (1.4) has no accessory parameters and hence the equation is uniquely determined by local structure, namely, by characteristic exponents and conjugacy classes of local monodromies at the singular points.

The middle convolution for a Fuchsian system

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - c_j} u \tag{1.10}$$

with constant square matrices $A_1, \dots, A_p$ of size n is introduced by [Kz] and [DR] to analyze the rigid local system (cf. § 3). Denoting $\partial = \frac{d}{dx}$ and $\vartheta = x\partial$, we define in [O1] the middle convolution $\operatorname{mc}_\mu(P)$ with a complex number μ by

$$\operatorname{mc}_\mu(P) = \partial^{-k} \sum_{i,j} c_{i,j} x^i (\vartheta - \mu)^j, \tag{1.11}$$

$$\partial^N P = \sum_{i,j} c_{i,j} \partial^i \vartheta^j \tag{1.12}$$

for an element P of the ring $W[x]$ of linear ordinary differential operators with polynomial coefficients. Here N is a sufficiently large integer so that $\partial^N P$ is of the form (1.12) with $c_{i,j} \in \mathbb{C}$ and then k is the maximal integer so that $\operatorname{mc}_\mu(P) \in W[x]$.

Theorem 1.1 ([O1]). *Suppose that the equation $Pu = 0$ with the Riemann scheme* (1.4) *is irreducible and the coefficients of P are polynomials without a common zero and moreover suppose*

$$\lambda_{j,1} - \lambda_{j,\nu} \notin \mathbb{Z} \text{ or } m_{j,1} \ge m_{j,\nu} \quad (\nu = 1, \dots, n_j,\ j = 0, \dots, p) \tag{1.13}$$

and

$$\lambda_{j,1} = 0 \quad (j = 1, \dots, p). \tag{1.14}$$

If $\lambda_{j,\nu}$ are generic (see [O1, Theorem 5.2] *for the precise condition) or the number*

$$d_1(\mathbf{m}) := m_{0,1} + \cdots + m_{p,1} - (p-1)\operatorname{ord}\mathbf{m} = 2\operatorname{ord}\mathbf{m} - \sum_{j=0}^{p}\sum_{\nu=2}^{n_j} m_{j,\nu} \tag{1.15}$$

is positive, the Riemann scheme $\{\lambda'_{\mathbf{m}'}\}$ *of* $P' = \mathrm{mc}_{\lambda_{0,1}-1}(P)$ *is determined by*

$$\begin{cases} \lambda'_{j,\nu} = \begin{cases} 2-\lambda_{0,1} & (j=0, \quad \nu=1),\\ 0 & (j=1,\dots,p,\ \nu=1),\\ \lambda_{j,\nu}-\lambda_{0,1}+1 & (j=0, \quad \nu=2,\dots,n_0),\\ \lambda_{j,\nu}+\lambda_{0,1}-1 & (j=1,\dots,p,\ \nu=2,\dots,n_j),\end{cases}\\ m'_{j,\nu} = \begin{cases} m_{j,1}-d_1(\mathbf{m}) & (j=0,\dots,p,\ \nu=1),\\ m_{j,\nu} & (j=0,\dots,p,\ \nu=2,\dots,n_j,)\end{cases}\end{cases} \tag{1.16}$$

and $P = \mathrm{mc}_{1-\lambda_{0,1}} \circ \mathrm{mc}_{\lambda_{0,1}-1}(P)$. *Moreover* $\mathrm{mc}_{\lambda_{0,1}-1}(P)$ *is irreducible if* $d_1(\mathbf{m}) \ge 0$. *Here we allow that some* $m_{j,1}$ *are 0.*

Remark 1.2. i) If $d_1(\mathbf{m}) > 0$, then $m_{j,1} > 0$ for $j = 0, \dots, p$ in the theorem.

ii) Suppose $P' = \mathrm{mc}_{\lambda_{0,1}-1}(P)$ is defined when the values of the parameters $\lambda_{j,\nu}$ are generic. Then we define P' for other values of the parameters by the analytic continuation of the parameters. In this case P and P' may be reducible for certain values of the parameters.

We define

$$\partial_1(\mathbf{m}) := \mathbf{m}' \tag{1.17}$$

by (1.15) and (1.16). Since

$$d_1(\mathbf{m}) \cdot \mathrm{ord}\,\mathbf{m} = \mathrm{idx}\,\mathbf{m} + \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}(m_{j,1}-m_{j,\nu})m_{j,\nu}, \tag{1.18}$$

we have $d_1(\mathbf{m}) > 0$ if $\mathrm{idx}\,\mathbf{m} = 2$ and moreover $\mathbf{m}$ is monotone, namely

$$m_{j,1} \ge m_{j,\nu} \quad (\nu = 1,\dots,n_j,\ j = 0,\dots,p). \tag{1.19}$$

The equation $Pu = 0$ is called rigid if it is irreducible and it has rigid spectral type. By the gauge transformation $u(x) \mapsto v(x) = \prod_{i=1}^{p}(x-c_i)^{\mu_i}u(x)$, the characteristic exponents of the Riemann scheme are changed from $\lambda_{j,\nu}$ to $\lambda_{0,\nu} - \sum_{i=1}^{p}\mu_i$ and $\lambda_{j,\nu} + \mu_j$ according to $j = 0$ and $j = 1, \dots, p$, respectively. The corresponding transformation of P is called addition and we denote the transformation of P by $\mathrm{RAd}_{\mu_1,\dots,\mu_p}(P)$. Here

$$\mathrm{RAd}_{\mu_1,\dots,\mu_p}(P) \in W[x] \cap W(x)\prod_{j=1}^{p}(x-c_j)^{\mu_j}\cdot P \cdot \prod_{j=1}^{p}(x-c_j)^{-\mu_j}$$

and the coefficients of the differential operator $\mathrm{RAd}_{\mu_1,\dots,\mu_p}(P)$ has no common zero. We put $\mathrm{R}(P) = \mathrm{RAd}_{0,\dots,0}(P)$. The addition keeps the order of $P \in W[x]$.

Hence if the equation $Pu = 0$ is rigid and $\mathrm{ord}\,P > 1$, then we have $\mathrm{ord}(\mathrm{mc}_\mu \circ \mathrm{RAd}_{\mu_1,\dots,\mu_p}(P)) < \mathrm{ord}\,P$ with suitable numbers $\mu, \mu_1, \dots, \mu_p$. By a successive application of suitable additions and middle convolutions, the rigid equation $Pu = 0$ is transformed into the equation $\frac{du}{dx} = 0$. Since additions and middle convolutions are invertible, we can construct any rigid Fuchsian equation $Pu = 0$ by a successive application of suitable additions and middle convolutions to the equation $\frac{du}{dx} = 0$.

The combinatorial aspect of additions and middle convolutions are well interpreted by a star-shaped Kac-Moody root system (W,Π) as follows, which was introduced by [CB] to analyze irreducible Fuchsian systems. The set of simple roots is

$$\Pi = \{\alpha_0, \alpha_{j,\nu} \mid j = 0,1,\ldots,\ \nu = 1,2,\ldots\} = \{\alpha_i \mid i \in I\} \tag{1.20}$$

with

$$I := \{0, (j,\nu) \mid j = 0,1,\ldots,\ \nu = 1,2,\ldots\} \tag{1.21}$$

and the inner product of the roots are given by

$$\begin{gathered}(\alpha_0|\alpha_0) = 2,\ (\alpha_0 \mid \alpha_{j,\nu}) = -\delta_{\nu,1},\\ (\alpha_{j,\nu}|\alpha_{j',\nu'}) = \begin{cases} 2 & (j=j',\ \nu=\nu'),\\ -\delta_{j,j'} & (|\nu-\nu'|=1),\\ 0 & (j\neq j' \text{ or } |\nu-\nu'|>1).\end{cases}\end{gathered} \tag{1.22}$$

The Weyl group W is generated by the simple reflection

$$s_\alpha(x) = s_i(x) = x - (\alpha|x)\alpha \qquad (\alpha = \alpha_i \in \Pi,\ i \in I,\ x \in \sum_{\alpha\in\Pi}\mathbb{R}\alpha). \tag{1.23}$$

The set of positive real roots Σ_+^{re} and the set of negative real roots Σ_-^{re} are

$$\Sigma_+^{re} = \Bigl\{\sum_{\alpha\in\Pi} k_\alpha\alpha \in W\alpha_0 \mid k_\alpha \geq 0\Bigr\},\quad \Sigma_-^{re} = \bigl\{-\alpha \mid \alpha \in \Sigma_+^{re}\bigr\} \tag{1.24}$$

and then $W\Pi = \Sigma_+^{re} \cup \Sigma_-^{re}$. For a tuple of partition $\mathbf{m}$, [CB] attached an element $\alpha_{\mathbf{m}}$ of the root lattice $\sum_{\alpha\in\Pi}\mathbb{Z}\alpha$ by

$$\alpha_{\mathbf{m}} = \operatorname{ord}\mathbf{m}\cdot\alpha_0 + \sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\sum_{s=\nu+1}^{n_j} m_{j,s}\alpha_{j,\nu}. \tag{1.25}$$

Then $\operatorname{idx}\mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}})$, $s_{\alpha_0}(\alpha_{\mathbf{m}})$ corresponds to the middle convolution because $(\alpha_0|\alpha_{\mathbf{m}}) = d_1(\mathbf{m})$ and $s_{\alpha_{j,\nu}}(\alpha_{\mathbf{m}})$ corresponds to the transposition between $m_{j,\nu}$ and $m_{j,\nu+1}$. Moreover we have

Theorem 1.3 ([CB, O1])**.** *The spectral type* $\mathbf{m}$ *is rigid if and only if* $\alpha_{\mathbf{m}} \in \Sigma_+^{re}$.

Remark 1.4. i) This theorem is given by [O1, Chapter 7] and the corresponding theorem for the first order system of Schlesinger canonical form is proved by [CB].

ii) There exist positive real roots which do not correspond to the rigid spectral type. In fact we have

$$\begin{aligned}\Sigma_+^{re} &= \{\alpha_{\mathbf{m}} \mid \mathbf{m} \text{ are rigid spectral types}\}\\ &\quad \cup \{\alpha_{j,\nu} + \alpha_{j,\nu+1} + \cdots + \alpha_{j,\nu'} \mid 1 \leq \nu \leq \nu',\ j = 0,1,\ldots\}.\end{aligned} \tag{1.26}$$

We will examine the condition of irreducibility of the rigid equation $Pu = 0$ with the Riemann scheme (1.4). For an element $w \in W$, the expression $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_\ell}$ with the reflections s_{β_i} with respect to simple roots $\beta_i \in \Pi$ is called minimal if ℓ is smallest among this product expression and in this case the number ℓ is called the length of w and denoted by $L(w)$. For a positive real root $\alpha \in \Sigma_+^{re}$, an element $w_\alpha \in W$ is uniquely determined by the conditions $w_\alpha \alpha = \alpha_0$ and $L(w_\alpha)$ is minimal. Moreover we put

$$\Delta(w) := \Sigma_+^{re} \cap w^{-1}\Sigma_-^{re} \quad \text{and} \quad \Delta(\mathbf{m}) := \Delta(w_{\alpha_{\mathbf{m}}}) \tag{1.27}$$

for a rigid spectral type $\mathbf{m}$. Note that the number $|\Delta(w)|$ of elements of $\Delta(w)$ equals $L(w)$ and if $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_L}$ is a minimal expression, we have

$$\Delta(w) = \{\beta_L, s_{\beta_L}\beta_{L-1}, \ldots, s_{\beta_L} \cdots s_{\beta_2}\beta_1\}, \tag{1.28}$$

$$\Delta(ws_\beta) = \begin{cases} s_\beta\big(\Delta(w) \setminus \{\beta\}\big) & (\beta \in \Delta(w) \cap \Pi), \\ s_\beta\Delta(w) \cup \{\beta\} & (\beta \notin \Delta(w) \cap \Pi). \end{cases} \tag{1.29}$$

For a rigid spectral type $\mathbf{m}$, the set of positive integers

$$[\Delta(\mathbf{m})] := \{(\alpha|\alpha_{\mathbf{m}}) \mid \alpha \in \Delta(\mathbf{m})\} \tag{1.30}$$

is a partition of the non-negative integer $h(\alpha_{\mathbf{m}}) - 1$ which is called the type of $\Delta(\mathbf{m})$, where

$$h(\alpha) := k_0 + \sum_{j\ge 0}\sum_{\nu\ge 1} k_{j,\nu} \quad \text{for} \quad \alpha = k_0\alpha_0 + \sum_{j\ge 0}\sum_{\nu\ge 1} k_{j,\nu}\alpha_{j,\nu}. \tag{1.31}$$

Suppose $\mathbf{m}$ is rigid and monotone and $\operatorname{ord}\mathbf{m} > 1$. Let ν_j be the maximal integers satisfying $m_{j,1} - d_1(\mathbf{m}) < m_{j,\nu_j+1}$ for $j = 0, 1, \ldots$. Then [O1, Proposition 7.9] shows

$$\begin{aligned} \Delta(\mathbf{m}) &= s_0\Big(\prod_{\substack{j\ge 0\\ \nu_j>0}} s_{j,1}\cdots s_{j,\nu_j}\Big)\Delta(s\partial_1\mathbf{m}) \cup \{\alpha_0\} \\ &\quad \cup \bigcup_{j=1}^{p}\{\alpha_0 + \alpha_{j,1} + \cdots + \alpha_{j,\nu} \mid 1 \le \nu \le \nu_j\}, \\ [\Delta(\mathbf{m})] &= [\Delta(s\partial_1\mathbf{m})] \cup \{d_1(\mathbf{m})\} \\ &\quad \cup \bigcup_{j=1}^{p}\{m_{j,\nu+1} - m_{j,1} + d_1(\mathbf{m}) \in \mathbb{Z}_{>0} \mid 1 \le \nu \le \nu_j\}. \end{aligned} \tag{1.32}$$

Here $\mathbb{Z}_{>0}$ is a set of positive integers and $s\mathbf{m}'$ is a monotone spectral type obtained by permutations of the sequences $(m'_{j,1}, \ldots, m'_{j,n'_j})$ of integers. The transformation s is realized by an element of a subgroup W' of W generated by $\{s_{j,\nu} \mid j = 0, 1, \ldots,\ \nu = 1, 2, \ldots\}$.

Theorem 1.5 ([O1, Theorem 10.14])**.** *A rigid Fuchsian differential equation $Pu = 0$ with the Riemann scheme* (1.4) *is reducible if and only if at least one of the numbers*

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\mathbf{m}(\beta)_{j,\nu}\lambda_{j,\nu} \quad (\beta \in \Delta(\mathbf{m})) \tag{1.33}$$

is an integer. Here

$$\mathbf{m}(\beta)_{j,\nu} = k_{j,\nu-1} - k_{j,\nu} \quad (\nu = 1, \dots, n_j,\ j = 0, \dots, p) \tag{1.34}$$

by putting

$$\begin{cases}\beta = k_0\alpha_0 + \sum_{j=0}^{p}\sum_{\nu=1}^{n_j-1} k_{j,\nu}\alpha_{j,\nu}, \\ k_{j,0} = k_0 \quad \textit{and} \quad k_{j,\nu} = 0 \quad \textit{if} \quad \nu \geq n_j.\end{cases} \tag{1.35}$$

For example, suppose $\mathbf{m} = 11, 11, 11$, namely, the spectral type of Gauss hypergeometric equation. Then $\alpha_{\mathbf{m}} = 2\alpha_0 + \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1}$ and we have

$$\begin{aligned}2\alpha_0 + \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} &\xrightarrow{s_0} \alpha_0 + \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} \xrightarrow{s_{0,1}} \alpha_0 + \alpha_{1,1} + \alpha_{2,1} \\ &\xrightarrow{s_{1,1}} \alpha_0 + \alpha_{2,1} \xrightarrow{s_{2,1}} \alpha_0, \\ w_{\alpha_{\mathbf{m}}} &= s_{2,1}s_{1,1}s_{0,1}s_0, \\ \Delta(\mathbf{m}) &:= \{\alpha_0,\ s_0\alpha_{0,1},\ s_0s_{0,1}\alpha_{1,1},\ s_0s_{0,1}s_{1,1}\alpha_{2,1}\} \\ &= \{\alpha_0,\ \alpha_0 + \alpha_{0,1},\ \alpha_0 + \alpha_{1,1},\ \alpha_0 + \alpha_{2,1}\}.\end{aligned}$$

We rewrite the above in terms of tuples of partitions as follows.

$$\begin{array}{ccccccccc} 11,11,11 & \xrightarrow{\partial_1} & 01,01,01 & \xrightarrow{s_{0,1}} & 10,01,01 & \xrightarrow{s_{1,1}} & 10,10,01 & \xrightarrow{s_{2,1}} & 10,10,10 \\ 10,10,10 & \leftarrow & * & & & & & & \\ 01,10,10 & \leftarrow & -11,00,00 & \leftarrow & * & & & & \\ 10,01,10 & \leftarrow & 00,-11,00 & \leftarrow & 00,-11,00 & \leftarrow & * & & \\ 10,10,01 & \leftarrow & 00,00,-11 & \leftarrow & 00,00,-11 & \leftarrow & 00,00,-11 & \leftarrow & * \end{array}$$

and

$$\{10,10,10,\ \ 01,10,10,\ \ 10,01,10,\ \ 10,10,01\} = \{\mathbf{m}(\beta) \mid \beta \in \Delta(11,11,11)\}.$$

Then for the Riemann scheme (1.3) we have

$$\bigl\{\textstyle\sum_{j=0}^{2}\sum_{\nu=1}^{2}\mathbf{m}(\beta)_{j,\nu}\lambda_{j,\nu} \mid \beta \in \Delta(\mathbf{m})\bigr\} = \{a, b, 1 + a - c, c - b\}$$

and the condition for the reducibility of (1.1) by Theorem 1.5 (cf. (1.3)).

For a rigid spectral type $\mathbf{m}$ let $w_{\alpha_{\mathbf{m}}} = s_{i_1}s_{i_2}\cdots s_{i_L}$ be a minimal expression with respect to simple reflections. Put $w(j) = s_{i_1}s_{i_2}\cdots s_{i_j}$ and $\alpha(j) = w(j)^{-1}\alpha_0$ for $j = 1, \dots, L$. Then we have the expressions of $\alpha(j)$ inductively as follows.

$$\begin{aligned}\alpha(j) &= k_\nu\beta(j,\nu) + \gamma(j,\nu) \qquad (\nu = 1, \dots, j), \\ \beta(j,\nu) &= s_{i_j}\beta(j-1,\nu),\ \gamma(j,\nu) = s_{i_j}\gamma(j-1,\nu), \quad (\nu = 1, 2, \dots, j-1), \\ \beta(j,j) &= \alpha_{i_j},\ \gamma(j,j) = \alpha(j-1), \\ k_j &= -(\alpha(j-1)|\alpha_{i_j}) = (\alpha(j)|\alpha_{i_j}) \in \mathbb{Z}_{>0}, \\ \Delta(w(j)) &= \{\beta(j,\nu)|\nu = 1, \dots, j\}.\end{aligned} \tag{1.36}$$

Then we have

$$\Delta(\mathbf{m}) = \{\beta_\nu := \beta(L,\nu) \mid \nu = 1,\dots,L\}. \tag{1.37}$$

A successive application of $s\partial_1$ to a rigid monotone spectral type $\mathbf{m}$, we have a sequence

$$\mathbf{m} \xrightarrow{s\partial_1} \mathbf{m}^{(1)} \xrightarrow{s\partial_1} \mathbf{m}^{(2)} \xrightarrow{s\partial_1} \cdots \xrightarrow{s\partial_1} \mathbf{m}^{(k)} = 10\cdots, 10\cdots, 10\cdots, \dots \tag{1.38}$$

with a non-negative integer k. Then rewriting each transformation s as a product of transpositions defined by $s_{j,\nu}$, we have a diagram as above to get $\Delta(\mathbf{m})$. In the diagram each arrow corresponds to an element of $\Delta(\mathbf{m})$, which is expressed by

$$\begin{array}{ccccccccccc} \mathbf{m} = \mathbf{m}^{(L)} & \to & \cdots & \to & \mathbf{m}^{(j+1)} & \xrightarrow{s_{i_{j+1}}} & \mathbf{m}^{(j)} & \xrightarrow{s_{i_j}} & \mathbf{m}^{(j-1)} & \to \\ k_\ell \mathbf{n}^{(L,j)} & \leftarrow & \cdots & \leftarrow & k_\ell \mathbf{n}^{(j+1,j)} & \leftarrow & k_\ell \mathbf{n}^{(j,j)} & \leftarrow & * & \end{array} \tag{1.39}$$

for $j = 1,\dots,|\Delta(\mathbf{m})|$ and we have

$$\begin{aligned} &k_j \mathbf{n}^{(j,j)} = \mathbf{m}^{(j)} - \mathbf{m}^{(j-1)}, \quad \alpha_{\mathbf{n}^{(j,j)}} \in \Pi, \quad k_j = (\alpha_{\mathbf{m}} | \alpha_{\mathbf{n}^{(L,j)}}),\\ &\alpha_{\mathbf{n}^{(\ell+1,j)}} = s_{i_{\ell+1}} \alpha_{\mathbf{n}^{(\ell,j)}},\\ &\Delta(\mathbf{m}) = \{\alpha_{\mathbf{n}^{(L,j)}} \mid j = 1,\dots,|\Delta(\mathbf{m})|\}, \end{aligned} \tag{1.40}$$

and k_j is a greatest common divisor of $\mathbf{m}(j)_{i,\nu} - \mathbf{m}(j-1)_{i,\nu}$ with $i \ge 0$ and $\nu \ge 1$.

2. Condition for reducibility

In this section we will further examine the condition of reducibility of Fuchsian ordinary differential equations with a rigid spectral type. First we show a general lemma assuring that if the rank of the equation does not depend on the parameter, the condition for the reducibility does not depend on the realization of the equation nor an integral shift of the parameter. Then we will classify the numbers (1.33) giving the condition for the reducibility into three types, which is related to the reducibility of a Pfaffian system studied in §3. In particular we show that we may omit some numbers among them. Lastly in Theorem 2.7 we give a condition that a reducible equation has a non-trivial quotient without an apparent singularity, which is a *ground state* for the reducibility in view of a shift operator.

Put $x = (x_1,\dots,x_n)$ and denote the ring of differential operators of x with polynomial coefficients by $W[x]$, which is called a Weyl algebra. The ring of differential operators with coefficients in the field $\mathbb{C}(x)$ of rational functions of x is denoted by $W(x)$. We identify a system of linear differential equation

$$\mathcal{M} : \sum_{j=1}^{N} P_{i,j} u_j = 0 \quad (i = 1,\dots,M) \tag{2.1}$$

with a left $W(x)$-module. Here $P_{i,j} \in W(x)$ and $u_1,\dots,u_N$ are generators of the left $W(x)$-module and (2.1) defines fundamental relations among the generators. The *rank* of $\mathcal{M}$ is the dimension of the vector space of $\mathcal{M}$ over the field $W(x)$ and we denote it by $\operatorname{rank}\mathcal{M}$. If $n = 1$, the rank of the equation $Pu = 0$ with $P \in W(x)$

equals $\operatorname{ord} P$, the order of P. Suppose $\operatorname{rank} \mathcal{M} < \infty$. Then $\mathcal{M}$ is said to be reducible if $\mathcal{M}$ has a quotient left $W(x)$-module $\mathcal{M}'$ satisfying $0 < \operatorname{rank} \mathcal{M}' < \operatorname{rank} \mathcal{M}$.

Lemma 2.1. *Let $\mathcal{M}_t$ and $\mathcal{N}_t$ are systems of linear differential equations with holomorphic parameter $t \in D := \{t \in \mathbb{C} \mid |t| < 1\}$. Suppose there exist a positive integer K such that* $\operatorname{rank} \mathcal{M}_t = \operatorname{rank} \mathcal{N}_t = K$ *for any $t \in D$ and a homomorphism ϕ_t of $\mathcal{M}_t$ to $\mathcal{N}_t$. Here it means a homomorphism between left $W(x)$-modules. Assume that ϕ_t is holomorphically depend on t and ϕ_t is an isomorphism if $t \neq 0$. Then $\mathcal{M}_0$ is reducible if and only if $\mathcal{N}_0$ is reducible.*

Proof. Replacing ϕ_t by $t^m \phi_t$ with a suitable integer m, we may assume $\phi_0 \neq 0$. Then ϕ_0 is also a non-zero homomorphism of $\mathcal{M}_0$ to $\mathcal{N}_0$ by analytic continuation. If ϕ_0 is an isomorphism, the claim of the theorem is clear. If ϕ_0 is not bijective, then the kernel and the image of ϕ_0 is non-trivial proper invariant $W(x)$-submodules of $\mathcal{M}_0$ and $\mathcal{N}_0$, respectively, and hence we have the lemma. □

Hereafter in this section we examine the reducible condition for a Fuchsian ordinary differential equations with a rigid spectral type. Let

$$P(\lambda)u = 0 \tag{2.2}$$

be a differential equation with the Riemann scheme (1.4) satisfying (1.7). We assume that the spectral type $\mathbf{m}$ of the equation is rigid. Then we have the following remark, which also follows from Theorem 1.5.

Remark 2.2. Let $\epsilon_{j,\nu}$ be integers satisfying $\sum_{j,\nu} m_{j,\nu}\epsilon_{j,\nu} = 0$. Then [O1, Theorem 11.2] shows that there exists a homomorphism ϕ_λ of the equation $P(\lambda)u = 0$ to the equation $P(\lambda+\epsilon)v = 0$. Since ϕ_λ is holomorphically depend on λ, the theorem implies that $P(\lambda)u = 0$ is reducible if and only if $P(\lambda+\epsilon)v = 0$ is reducible.

Lemma 2.3. *Let $P(\lambda)$ be the universal operator with a rigid Riemann scheme $\{\lambda_{\mathbf{m}}\}$. Suppose that the characteristic exponents $\lambda_{j,\nu}$ holomorphically depend on $t \in D$. Assume that $P(\lambda(t))u = 0$ is irreducible if $t \neq 0$ and*

$$d_1(\mathbf{m}) := m_{j,0} + m_{j,1} + \cdots + m_{j,p} - (p-1)\operatorname{ord}\mathbf{m} > 0.$$

Put

$$\begin{aligned}\mu(t) &= \lambda_{0,1}(t) + \lambda_{1,1}(t) + \cdots + \lambda_{p,1}(t),\\ \tilde{P}(\lambda(t)) &= \mathrm{RAd}_{\lambda_{1,1}(t),\ldots,\lambda_{p,1}(t)}\, P(\lambda(t)),\\ Q(\lambda(t)) &= \mathrm{mc}_{1-\mu(t)}\tilde{P}(\lambda(t))\end{aligned}$$

and let $\mathbf{m}'$ be the spectral type of $Q(\lambda(t))$.

i) *If $\mu(0) = 1$, $P(\lambda(0))u = 0$ has a quotient* $\mathrm{RAd}_{-\lambda_{1,1}(0),\ldots,-\lambda_{p,1}(0)}\, Q(\lambda(0))u = 0$.

ii) *Let d' be a positive integer satisfying $1 \leq d' \leq d_1(\mathbf{m})$. If $\mu(0) = 1 - d'$, $P(\lambda(0))u = 0$ has solutions $r(x)\prod_{j=1}^p (x-c_j)^{\lambda_{j,1}}$, where $r(x)$ are arbitrary polynomials of degree $< d'$ and in particular $\prod_{j=1}^p (x-c_j)^{\lambda_{j,1}}$ is a solution.*

Proof. We may assume $Q(\lambda(t))u=0$ is irreducible if $t\neq 0$. Put $n=\operatorname{ord}\mathbf{m}$.

Note that $\operatorname{ord}Q(\lambda(t))=n-d_1(\mathbf{m})$ and the Riemann scheme of $\tilde P(\lambda(t))$ is

$$\begin{Bmatrix} x=c_0=\infty & c_j\ (j=1,\dots,p)\\ [\sum_{i=0}^p\lambda_{i,1}]_{(m_{0,1})} & [0]_{(m_{j,\nu})}\\ [\lambda_{0,\nu}+\sum_{i=1}^p\lambda_{i,1}]_{(m_{0,\nu})}\ (\nu=1,\dots,n_0) & [\lambda_{j,\nu}-\lambda_{j,1}]_{(m_{j,\nu})}\ (\nu=1,\dots,n_j)\end{Bmatrix}.$$

If $\mu(0)=1$, the definition of $\mathrm{mc}_{1-\mu(t)}$ implies that $Q(\lambda(0))$ is a quotient of $\tilde P(\lambda(0))$, which proves i).

Put $f(x)=\prod_{j=1}^p(x-c_j)^{n-m_{j,1}}$. Then the Riemann scheme of $P^\vee u=0$ with $P^\vee=(f(x)^{-1}\tilde P)^*$ equals

$$\begin{Bmatrix} x=c_0=\infty & c_j\ (j=1,\dots,p)\\ [2-n-m_{0,1}-\sum_{i=0}^p\lambda_{i,1}]_{(m_{0,1})} & [n-m_{j,1}]_{(m_{j,\nu})}\\ [2-n-m_{0,\nu}-\lambda_{0,\nu}-\sum_{i=1}^p\lambda_{i,1}]_{(m_{0,\nu})} & [n-m_{j,\nu}-\lambda_{j,\nu}+\lambda_{j,1}]_{(m_{j,\nu})}\end{Bmatrix}$$

as was given in [O1, Theorem 4.19 ii)] and that of $\tilde P^*u=0$ equals

$$\begin{Bmatrix} x=c_0=\infty & c_j\ (j=1,\dots,p)\\ [2-d_1-\sum_{i=0}^p\lambda_{i,1}]_{(m_{0,1})} & [0]_{(m_{j,\nu})}\\ [2-d_1-m_{0,\nu}+m_{0,1}-\lambda_{0,\nu}-\sum_{i=1}^p\lambda_{i,1}]_{(m_{0,\nu})} & [m_{j,1}-m_{j,\nu}-\lambda_{j,\nu}+\lambda_{j,1}]_{(m_{j,\nu})}.\end{Bmatrix}.$$

Here $d_1=d_1(\mathbf{m})$. Note that $P(\lambda(t))^*$ is the universal operator with the above Riemann scheme and $2-d_1-\sum_{i=0}^p\lambda_{i,1}(0)=1+d'-d_1$. Then in the case $d'=d_1(\mathbf{m})$, the claim ii) follows from the definition of $P^\vee$ and $\mathrm{mc}_{1-\mu(t)}$.

We may split the characteristic exponent $[\lambda_{0,1}]_{(m_{0,1})}$ to

$$[\lambda_{0,1}+d_1-d']_{(m_{0,1}-d_1+d')}\quad\text{and}\quad[\lambda_{0,1}]_{(d_1-d')}.$$

Then $m_{0,1}$, $d_1(\mathbf{m})$ and n_0 changed into $m_{0,1}-d_1(\mathbf{m})+d'$, d', n_0+1, respectively, and the same argument as above proves ii). □

Definition 2.4. The elements $\beta\in\Delta(\mathbf{m})$ are classified into three types

$$\begin{aligned}&(\text{Type }1)\qquad \operatorname{ord}s_\beta\alpha_{\mathbf{m}}>0,\\ &(\text{Type }2)\qquad \operatorname{ord}s_\beta\alpha_{\mathbf{m}}=0,\\ &(\text{Type }3)\qquad \operatorname{ord}s_\beta\alpha_{\mathbf{m}}<0.\end{aligned}$$

Putting $\gamma:=s_\beta\alpha_{\mathbf{m}}$, we have $\gamma=\alpha_{\mathbf{m}}-k_\beta\beta$ with $k_\beta=(\alpha_{\mathbf{m}}|\beta)$ and therefore

$$\alpha_{\mathbf{m}}=k_\beta\beta+\gamma. \tag{2.3}$$

Here we put $\operatorname{ord}\gamma=n_0$ if $\gamma=\sum_{i\in I}n_i\alpha_i$.

Proposition 2.5. i) *We have* $\operatorname{ord}\beta>0$ *for any* $\beta\in\Delta(\mathbf{m})$.

ii) *If* $k_\beta=1$, *then* β *is of Type 1.*

iii) *The numbers* (1.33) *for* $\beta\in\Delta(\mathbf{m})$ *of Type 3 may be omitted for Theorem* 1.5.

Proof. Under the notation in the preceding section, we have $\beta = \beta(L, \nu_o)$ with a certain ν_o in (1.36).

i) Since $\mathbf{m}$ is monotone, if $\alpha \in \Sigma_+^{re}$ satisfies $(\alpha|\alpha_{\mathbf{m}}) > 0$, then $\operatorname{ord}\alpha > 0$.

ii) The claim is given in [O1, Proposition 7.9 iv)].

iii) Note that the condition $\sum_{j,\nu} \mathbf{m}(\beta)_{j,\nu}\lambda_{j,\nu} \in \mathbb{Z}$ implies $\sum_{j,\nu} \mathbf{m}(\gamma)_{j,\nu}\lambda_{j,\nu} \in \mathbb{Z}$ because of the Fuchs relation. Since $\gamma(\nu_o, \nu_o) \in \Sigma_+^{re}$, there exists $j > \nu_o$ satisfying $\beta(j, j) = \alpha_{i_j} = \gamma(j-1, \nu_o) = -\gamma(j, \nu_o) \in \Pi$. Then $\beta(L, j) = -\gamma(L, \nu_o) \in \Delta(\mathbf{m})$. Hence the condition for the number (1.33) with $\beta(L, \nu_o)$ can be omitted for Theorem 1.5.

Put $\beta_\nu = \beta(L, \nu)$, $\gamma_\nu = \gamma(L, \nu)$ and $k_\nu = k_{\beta_\nu}$ for simplicity. Then

$$\alpha_{\mathbf{m}} = k_{\nu_o}\beta_{\nu_o} + \gamma_{\nu_o} \text{ with } k_{\nu_o} = (\alpha_{\mathbf{m}}|\beta_{\nu_o})$$

and

$$k_j = (\alpha_{\mathbf{m}}|\beta_j) = (\alpha_{\mathbf{m}}|k_{\nu_o}\beta_{\nu_o} - \alpha_{\mathbf{m}}) = k_{\nu_o}^2 - 2 > 1.$$

If $\beta(L, j)$ is of Type 3, we repeat the same way and this procedure ends in finite steps because $\nu_o < j \le L$. □

Remark 2.6. i) There always exists $\beta \in \Delta(\mathbf{m})$ of Type 1 and $k_\beta = 1$ in (2.3) if $\operatorname{ord}\mathbf{m} > 1$ (cf. [O1, Proposition 10.7]). In this case we have

$$|\{\lambda_{\mathbf{m}(\beta)}\}| + |\{\lambda_{\mathbf{m}(\gamma)}\}| = 1$$

and the condition $\sum \mathbf{m}(\beta)_{j,\nu}\lambda_{j,\nu} \in \mathbb{Z}$ is equivalent to $\sum \mathbf{m}(\gamma)_{j,\nu}\lambda_{j,\nu} \in \mathbb{Z}$.

ii) Type 2 appears only in the case when there exist j_o and $1 \le \nu_o < \nu_o' \le n$ such that $m_{j_o,\nu_o} + m_{j_o,\nu_o'}$ and $m_{j,\nu}$ with $(j,\nu) \notin \{(j_o,\nu_o),(j_o,\nu_o')\}$ are divisible by a common integer k larger than 1. There is an example of Type 2 in § 3.4.

iii) There is an example of Type 3 in § 3.7.

Theorem 2.7. *Let $P(\lambda)$ be the universal operator with the Riemann scheme* (1.4) *with rigid spectral type $\mathbf{m}$. Let $\beta \in \Delta(\mathbf{m})$. Then if*

$$1 \le d' := \operatorname{ord}\beta - \sum_{j=0}^{p}\sum_{\nu=1}^{n_j} \mathbf{m}(\beta)_{j,\nu}\lambda_{j,\nu} \le (\alpha_{\mathbf{m}}|\beta), \tag{2.4}$$

$P(\lambda) \in W(x)Q(\lambda)$. Here $Q(\lambda)$ is the universal operator with the spectral type $\mathbf{m}(\beta)$ and if $d' = 1$, the Riemann scheme of $Q(\lambda)$ equals $\{\lambda_{\mathbf{m}(\beta)}\}$. Here $0 < \operatorname{ord} Q(\lambda) = \operatorname{ord}\beta < \operatorname{ord} P$.

If $(\alpha_{\mathbf{m}}|\beta) = 1$, then replacing β by $\gamma = \alpha_{\mathbf{m}} - \beta$, the above statement also holds.

Proof. First note that the condition $d' = 1$ for β is the Fuchs relation of $Q(\lambda)$ written by $|\{\lambda_{\mathbf{m}(\beta)}\}| = 0$ and hence this number d' is invariant under the action of W.

We will prove the theorem by the induction with respect to the number k in (1.38). Suppose $\beta = \alpha_0$ or $\alpha_0 + \alpha_{j,1} + \cdots + \alpha_{j,\nu}$ in (1.32). Applying Lemma 2.3 ii) with $d' = 1$ to P after the transposition of the indices $(j, \nu+1)$ and $(j, 1)$ if necessary, we have the claim of the theorem for β. We have also the claim of the theorem for γ by Lemma 2.3 i).

Put $w' = (\prod_{\substack{j\ge 0\\ \nu_j>0}} s_{j,1}\cdots s_{j,\nu_j})s_0$ in (1.32) and let $\beta \in {w'}^{-1}\Delta(s\partial_1\mathbf{m})$. Then β corresponds to an element $\beta' \in \Delta(s\partial_1\mathbf{m})$ with $\beta' = w'\beta$ and $w' \in W$ corresponds to the map $s\partial_1$, which corresponds to $\psi = \mathrm{RAd}_{\mu_1,\dots,\mu_p}\circ\mathrm{mc}_\mu$, the combination of a middle convolution and an addition.

Here we note that β, $\beta' \in \Sigma_+^{re}$, $\operatorname{ord}\beta > 0$ and $\operatorname{ord}\beta' > 0$.

Then the statement for β is obtained by applying ψ^{-1} to the operator $\psi(P)$ and the operator corresponding to $\beta' \in \Delta(s\partial_1\mathbf{m})$ whose existence is assured by the hypothesis of the induction.

If β is of Type 1, then $\gamma' := w'\gamma \in \Sigma_+^{re}$ and $\operatorname{ord}\gamma' > 0$, we have also the last claim. □

3. Hypergeometric equations with several variables

Any 4 points t_0, t_1, t_∞, t_x in the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ can be transformed to 0, 1, ∞, x by the fractional transformation defined by

$$x = \frac{(t_x - t_0)(t_\infty - t_1)}{(t_1 - t_0)(t_\infty - t_x)}.$$

Hence Gauss hypergeometric function $F(a,b,c;x)$ is naturally considered as a hypergeometric function on the configuration space of 4 points in $P^1_{\mathbb{C}}$ by

$$F(a,b,c;\tfrac{t_x-t_0}{t_1-t_0}\tfrac{t_\infty-t_1}{t_\infty-t_x}).$$

Then the Riemann scheme of this function is

$$\begin{Bmatrix} t_x = t_0 & t_x = t_1 & t_x = t_\infty & t_0 = t_1 & t_0 = t_\infty & t_1 = t_\infty \\ 0 & 0 & a & a & 0 & 0 \\ 1-c & 1-a-b & b & b & 1-a-b & 1-c \end{Bmatrix}.$$

The solutions of the universal rigid Fuchsian equations $Pu = 0$ with a rigid spectral type $\mathbf{m}$ have natural integral representations (cf. [O1]). When $\mathbf{m}$ is a $(p+1)$ tuples of partitions, P has $(p+1)$ singular points. We may specialize the points as 0, 1, ∞, y_1, …, y_{p-2} and then the solutions $u(x, y_1, \dots, y_{p-2})$ has $(p-1)$ variables by the integral representation. These functions are a kind of hypergeometric functions with several variables and also they can be considered hypergeometric functions on the configuration space of $(p+2)$ points in $\mathbb{P}^1_{\mathbb{C}}$.

For these hypergeometric functions with several variables, it will be convenient to use the differential equations of Pfaffian form satisfied by the functions. They have been studied by [DR] for the case of single variable and by [Ha] for several variables. We will shortly explain it in the case of two variables x and y as an example. Then the Pfaffian system is

$$\mathcal{M}_\lambda\ :\ du = \left(A_1\frac{dx}{x} + A_2\frac{d(x-y)}{x-y} + A_3\frac{d(x-1)}{x-1} + A_4\frac{dy}{y} + A_5\frac{d(y-1)}{y-1}\right)u. \quad (3.1)$$

Here A_j are square matrices of size n. They are called the residue matrices along the corresponding hypersurfaces. If the eigenvalues of A_j are $\lambda_{j,\nu}$ with multiplicity $m_{j,\nu}$ for $\nu = 1, \dots, n_j$ and A_j are semisimple, the set of characteristic exponents corresponding to A_j is defined by $\{[\lambda_{j,1}]_{m_{j,1}}, \dots, [\lambda_{j,n_j}]_{m_{j,n_j}}\}$, and we can define the generalized Riemann scheme for this Pfaffian system. For example, the set of characteristic exponents at $x = 0$, $x = y$ and $x = 1$ are these sets for $j = 1$, $j = 2$ and $j = 3$, respectively, and the set of characteristic exponents at $x = \infty$ is define by the matrix $A_0 = -(A_1 + A_2 + A_3)$.

The addition with parameters $\lambda_1, \lambda_2, \lambda_3$ (for x-variable) is defined by

$$\mathrm{Ad}_{\lambda_2,\lambda_2,\lambda_3}(A_1, A_2, A_3, A_4, A_5) \mapsto (A_1 + \lambda_1, A_2 + \lambda_2, A_3 + \lambda_3, A_4, A_5).$$

The middle convolution mc_μ to these matrices is defined by

$$\mathrm{mc}_\mu(A_j) := \tilde{A}_j \mod \mathcal{K}_\mu := \begin{pmatrix} \ker A_1 \\ \ker A_2 \\ \ker A_3 \end{pmatrix} \oplus \ker(\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3) \tag{3.2}$$

with

$$\tilde{A}_1 = \begin{pmatrix} A_1 + \mu & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ A_1 & A_2 + \mu & A_3 \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1 & A_2 & A_3 + \mu \end{pmatrix},$$

$$\tilde{A}_4 = \begin{pmatrix} A_4 + A_2 & -A_2 & 0 \\ -A_1 & A_4 + A_1 & 0 \\ 0 & 0 & A_4 \end{pmatrix} \text{ and } \tilde{A}_5 = \begin{pmatrix} A_5 & 0 & 0 \\ 0 & A_5 + A_3 & -A_3 \\ 0 & -A_2 & A_5 + A_2 \end{pmatrix}.$$

Here $\tilde{A}_j$ are considered to be linear maps on the vector space $\mathbb{C}^{3n}$ and the space $\mathcal{K}_\mu$ are invariant by these maps and then we represent A_j as square matrices of size $3n - \dim \mathcal{K}_\mu$.

Considering y as a parameter and forgetting A_4 and A_5, the above definition is due to [DR]. In this case, $x = 0, 1, y, \infty$ are regular singular points of the differential equation with the variable x. The above operation for A_4 and A_5 is defined and studied by [Ha], where x and y are equally considered as variables.

When we regard y as a parameter, the structure of operations of additions and middle convolutions are compatible to the corresponding operations of Fuchsian ordinary differential equations, which are briefly explained in the previous sections. Moreover any rigid monodromy group is equally realized by solutions of both type of equations. Assume that the spectral type $\mathbf{m}$ is rigid. Then there is a homomorphism ψ_λ of the universal Fuchsian differential equation $P(\lambda)u = 0$ to the rigid Pfaffian system $\mathcal{M}_\lambda$, where ψ_λ is meromorphically depend on λ and it defines an onto isomorphism between two equations for generic λ, which follows from the fact that they are constructed from the trivial equation by successive applications of additions and middle convolutions.

Table 1 is the number of rigid spectral types of order at most 15 whose numbers of singular points are smaller than 7.

It is well-known that the integral representation of the solution of Jordan-Pochhammer equation, which is characterized by rigid spectral type $21, 21, 21, 21,$

gives a solution of Appell's F_1 (in general, Lauricella's F_D with more variables). As is given in the table, the rigid Pfaffian system corresponding to the spectral type $31,31,22,211$ corresponds to Appell's F_2 and F_3 and that to $31,22,22,22$ corresponds to Appell's F_4 as was shown in [Ha]. Hence we have a plenty of generalizations of Appell's hypergeometric functions.

Remark 3.1. i) The number of parameters of the equation with a rigid spectral type after a suitable addition is given by

$$\#\text{parameters} = \sum\bigl(\#\text{ blocks at singular points}-1\bigr) = \sum_{j=0}^{p}(n_j-1) \tag{3.3}$$

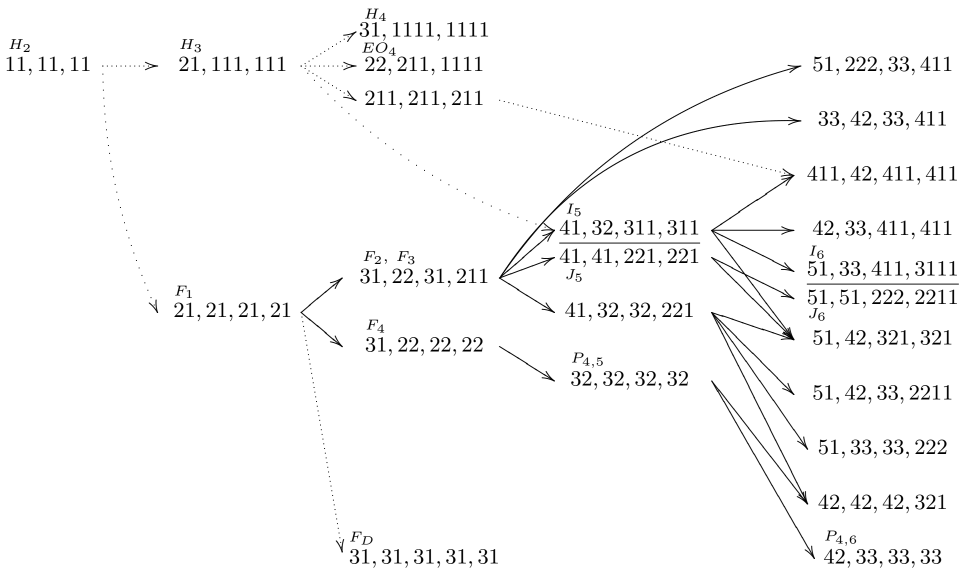
as in the case of the Gauss hypergeometric function.

ii) In Table 2, the arrow shows that two spectral types are connected by an addition and a middle convolution. Moreover in Table 2 we see that F_2 and F_3

TABLE 1. Hypergeometric equations with less than 6 variables

Order	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1 variable	1	1	3	5	13	20	45	74	142	212	421	588	1004	1481
2 variables		1	2	4	11	16	35	58	109	156	299	402	685	924
3 variables			1	1	3	5	12	17	43	52	104	135	263	327
4 variables				1	0	1	3	5	8	14	24	39	60	79
5 variables					1	0	0	2	3	4	6	6	14	20

TABLE 2. Hierarchy of rigid quartets (cf. [O1])



are obtained by restrictions of a Pfaffian system to different complex lines and the pair of I_5 and J_5 has the same property.

iii) It follows from [O1, Theorem 11.2] that there exists a non-zero homomorphism realizing any integral shift of characteristic exponents in the Pfaffian system corresponding to a rigid Fuchsian ordinary differential equation. Lemma 2.1 assures that the irreducibility of the system is invariant under the integral shift.

iv) There exists a universal Fuchsian equation with a rigid spectral type as is stated in §1. But in the case of a Pfaffian system the extension to a special value of λ which corresponds to a reducible monodromy group is not unique. Consider the Gauss hypergeometric system with the Riemann scheme $\begin{Bmatrix}\lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix}$ for generic values of parameters under the condition $\lambda_{0,j}+\lambda_{1,j}+\lambda_{2,j}=0$ for $j=1$ and 2. Then the Riemann scheme of its irreducible quotient is $\{\lambda_{0,1}\ \lambda_{1,1}\ \lambda_{2,1}\}$ or $\{\lambda_{0,2}\ \lambda_{1,2}\ \lambda_{2,2}\}$. But there is no natural way to determine it and it depends on its construction using additions and middle convolutions.

Theorem 3.2. *Let*

$$du=\Bigl(\sum_{i=1}^{n}\sum_{k=1}^{q}A_{i,k}\frac{dy_i}{y_i-c_k}+\sum_{1\le i<j\le n}B_{i,j}\frac{d(y_i-y_j)}{y_i-y_j}\Bigr)u \tag{3.4}$$

be a completely integrable Pfaffian system with variables $y_1,\dots,y_n$, where $A_{i,k}$ and $B_{i,j}$ are constant $N\times N$ matrices and $c_1,\dots,c_q$ are mutually different complex numbers. Suppose

$$\frac{du}{dx}=\Bigl(\sum_{k=1}^{q}\frac{A_{1,k}}{x-c_k}+\sum_{j=2}^{n}\frac{B_{1,j}}{x-y_j}\Bigr)u \tag{3.5}$$

is an irreducible rigid Fuchsian system with mutually different complex numbers $c_1,\dots,c_q,y_2,\dots,y_n$. Then (3.4) *is irreducible and it is obtained by a successive application of additions and middle convolutions extended by Haraoka for the variable $x=y_1$ to a Pfaffian differential equation*

$$dv=\Bigl(\sum_{i=2}^{n}\sum_{k=1}^{q}\alpha_{i,k}\frac{dy_i}{y_i-c_k}+\sum_{2\le i<j\le n}\beta_{i,j}\frac{d(y_i-y_j)}{y_i-y_j}\Bigr)v \tag{3.6}$$

of the first order. Here $\alpha_{i,k}$ and $\beta_{i,j}$ are complex numbers. Note that the solution of (3.6) *is a constant multiple of $\prod_{i=2}^{n}\prod_{k=1}^{q}(y_i-c_k)^{\alpha_{i,k}}\cdot\prod_{2\le i<j\le n}(y_i-y_j)^{\beta_{i,j}}$.*

Proof. This theorem is not clearly stated in [Ha] but it is essentially obtained in or easily obtained by [Ha].

Since (3.5) is rigid, it is reduced to the trivial equation $du=0$ by a successive application of additions and middle convolutions. Then we get an equation of the form (3.6) by the successive application of the corresponding operations for y_1 variable to (3.4). Since these operations are invertible, we obtain (3.4) from (3.6) by these operations.

Since (3.5) is irreducible, the monodromy group of the solutions of (3.4) is irreducible even if $y_2,\dots,y_2$ are fixed. Hence the equation (3.4) is irreducible. □

Theorem 3.3. *Let* $\mathbf{m}$ *be a rigid spectral type and let* (3.5) *be the Fuchsian system with the rigid spectral type* $\mathbf{m}$*. Let* $\big\{[\lambda_{j,\nu}]_{m_{j,\nu}}\big\}_{\substack{0\le j\le p\\ 1\le\nu\le n_j}}$ *be the Riemann scheme of the Fuchsian system* (3.5) *with* $p = q + n - 1$*. Let* (3.4) *be the corresponding completely integrable Pfaffian system of rank* $N = \operatorname{ord}\mathbf{m}$*.*

i) *Suppose any one of the numbers* (1.33) *is not an integer, the Pfaffian system* (3.4) *is irreducible.*

i) *Suppose there exists* $\beta \in \Delta(\mathbf{m})$ *such that*

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\mathbf{m}(\beta)_{j,\nu}\lambda_{j,\nu} \in \mathbb{Z}. \tag{3.7}$$

If β *is of Type 1 or Type 3 (cf. Definition* 2.4*), then* (3.4) *is reducible.*

Suppose (3.4) *is irreducible. Fix a suitable base of local solutions of* (3.4) *in a neighborhood of a generic point* $y^o = (y_1^0, \dots, y_n^o)$ *of* $\mathbb{C}^n$*. Then*

$$\tilde A_\nu = \tilde A'_\nu \otimes I_r \quad (1 \le \nu \le p) \tag{3.8}$$

are the corresponding monodromy matrices along the closed loops γ_ν *starting from* y^o *in the* y_1*-plane and* r *is an integer larger than* 1*. Here putting* $c_{q+j-1} = y_j$ *for* $2 \le j \le n$*, we denote by* γ_ν *the closed loops starting from* y^o *in the* y_1*-plane which satisfy* $\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_\nu}\frac{dy_1}{y_1 - c_{\nu'}} = \delta_{\nu,\nu'}$ *with* ν, $\nu' \in \{1, 2, \dots, p\}$*.*

Proof. The claim i) follows from Theorem 3.2.

Fix $\beta \in \Delta(\mathbf{m})$. We may fix generic $\lambda_{j,\nu}$ satisfying (3.7) and the Fuchs relation. Then (3.5) is reducible. Let $u(y)$ be a local solution of (3.4) in a neighborhood of y^o such that a component of $u(x, y_2, \dots, y_n)$ is a solution of an irreducible Fuchsian ordinary differential equation with the variable x so that the order R of the equation is smaller than N. Since the coefficients of the equation are rational functions of (x, y), the analytic continuations of the component satisfy the same equation.

Suppose (3.4) is irreducible. Then the dimension of the linear span V of the local solutions obtained by analytic continuations of $u(y)$ with respect to the variables $y_1, \dots, y_n$ equals N. Hence there exist $u_{\ell,1}(y), \dots, u_{\ell,R}(y)$ in V for $1 \le \ell \le r$ with $N = rR$ such that the spaces spanned by $u_{\ell,1}(y), \dots, u_{\ell,R}(y)$ are stable under the analytic continuation along the loops γ_ν and moreover the dimension of the space $\sum_{\ell=1}^{r}\sum_{i=1}^{R}\mathbb{C}u_{\ell,i}(y)$ equals N. We may moreover assume that the monodromies along γ_ν with respect to the base $\{u_{\ell,1}(y), \dots, u_{\ell,R}(y)\}$ do not depend on ℓ. Then the multiplicity of the eigenvalues of local monodromy matrices at each singular points of (3.5) is divisible by r. This never happens if β is of Type 1 nor Type 3 because of the genericity condition for the values $\lambda_{j,\nu}$. Hence β is of Type 2 and we have the theorem. □

Remark 3.4. Under the notation in Theorems 1.5, Theorem 3.2 and Theorem 3.3 the ordinary differential equation (3.5) is reducible if and only if at least one of the numbers (1.33) for $\beta \in \Delta(\mathbf{m})$ of Type 1 or Type 2 is an integer. But it seems that the system (3.4) is reducible if and only if at least one of the numbers (1.33) for

$\beta \in \Delta(\mathbf{m})$ of Type 1 is an integer (cf. §3.4). We note that for most rigid spectral types $\mathbf{m}$ there is no $\beta \in \Delta(\mathbf{m})$ of Type 2 (cf. Remark 2.6 ii)) and in this case the system (3.4) is reducible if and only if so is the equation (3.5).

We will examine all of our Pfaffian systems with two variables corresponding to a rigid spectral type $\mathbf{m}$ whose ranks are at most 5 (cf. [O3]). The Pfaffian systems satisfied by Appell's hypergeometric functions are described in [HK].

3.1. Computer algebra

A computer algebra `Risa/Asir` with the library [O4] calculates $\Delta(\mathbf{m})$ and mc_μ in (1.11) and (3.2) and in particular the examples given in this section.

3.2. Appell's F_1

$\mathbf{m} = 21,21,21,21$: rank $= 3$ with 4 parameters and $[\Delta(\mathbf{m})] = 1^4 \cdot 2^1$

$21,21,21,21 \ \to\ 01,01,01,01 \quad H_2 : 11,11,11,20$ (by a middle convolution)
$= 10,10,10,01 \oplus 11,11,11,20$ (redicibility: 4 cases by the symmetry)
$= 2(10,10,10,10) \oplus 01,01,01,01$ (1 case)

The generalized Riemann scheme for the two variables (x,y) is

$$\begin{Bmatrix} x=0 & x=1 & x=y & x=\infty & y=0 & y=1 & y=\infty \\ [0]_2 & [0]_2 & [0]_2 & [e]_2 & [0]_2 & [0]_2 & [-c-e]_2 \\ a & b & c & d & a+c+2e & b+c+2e & d \end{Bmatrix}$$

with the following Fuchs relation.

$a+b+c+d+2e = 2, 0$ (Fuchs rel.)
$10,10,10,01 : d \notin \mathbb{Z}$ (irred. cond.)
$10,10,01,10 : c+e \notin \mathbb{Z}$
$10,01,10,10 : b+e \notin \mathbb{Z}$
$01,10,10,10 : a+e \notin \mathbb{Z}$
$10,10,10,10 : e \notin \mathbb{Z}$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		21	21	21	21	2
t_0	21		21	21	21	2
t_y	21	21		21	21	2
t_1	21	21	21		21	2
t_x	21	21	21	21		2

In the Fuchs relation, "$= 2$" is valid for a single Fuchsian differential equation and "$= 0$" is valid for the corresponding Pfaffian system.

The set of 5 points $\{t_0, t_1, t_\infty, t_x, t_y\}$ in $\mathbb{P}^1_{\mathbb{C}}$ is transformed into $\{0, 1, \infty, x, y\}$. Hence a Pfaffian system with 5 variables $t_0, t_1, t_\infty, t_x, t_y$ is defined through this map and the spectral type at each singular hypersurface is given in the above.

The above 5 conditions for the irreducibility are obtained by the following procedure, which is explained in §1.

$$\underline{2}1,\underline{2}1,\underline{2}1,\underline{2}1 \ \underset{2}{\overset{-2}{\Rightarrow}}\ \underline{0}1,\underline{0}1,\underline{0}1,\underline{0}1 \underset{1}{\to} \underline{10},01,01,01 \underset{1}{\to} 10,\underline{10},01,01 \underset{1}{\to} 10,10,\underline{10},01 \underset{1}{\searrow}$$
$$2(10,10,10,10) \overset{\pm 2}{\Leftarrow}\ * \qquad\qquad 10,10,10,10$$
$$01,10,10,10 \ \overset{\pm 1}{\Leftarrow}\ \underline{-1}1,\underline{0}0,\underline{0}0,\underline{0}0 \leftarrow\ *$$
$$10,01,10,10 \ \overset{\pm 1}{\Leftarrow}\ \underline{0}0,\underline{-1}1,\underline{0}0,\underline{0}0 \leftarrow \underline{00},-11,00,00 \leftarrow\ *$$
$$10,10,01,10 \ \overset{\pm 1}{\Leftarrow}\ \underline{0}0,\underline{0}0,\underline{-1}1,\underline{0}0 \leftarrow \underline{00},00,-11,00 \leftarrow 00,\underline{00},-11,00 \leftarrow\ *$$
$$10,10,10,01 \ \overset{\pm 1}{\Leftarrow}\ \underline{0}0,\underline{0}0,\underline{0}0,\underline{-1}1 \leftarrow \underline{00},00,00,-11 \leftarrow 00,\underline{00},00,-11 \leftarrow 00,00,\underline{00},-11 \leftarrow *$$

The Pfaffian system corresponds to the hypergeometric function

$$F_1(\alpha;\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_m\,(\beta')_n}{(\gamma)_{m+n}\,m!\,n!}\,x^m y^n$$

with

$$a=\beta'-\gamma+1,\quad b=\gamma-\alpha-\beta-1,\quad c=-\beta-\beta',\quad d=\alpha,\quad e=\beta$$

and then the necessary and sufficient condition for the irreducibility is

$$\{\alpha,\ \beta,\ \beta',\ \alpha-\gamma,\ \beta+\beta'-\gamma\}\cap\mathbb{Z}=\emptyset. \tag{3.9}$$

3.3. Appell's F_2, F_3

$211,22,31,31$: rank$=4$, 5 parameters, $(1^6\cdot 2^2)$

$\to\ F_1 : 201,21,21,21\quad H_2 : 011,02,11,11$

$= 010,01,10,10\oplus 201,21,21,21\quad (4)$

$= 101,11,11,20\oplus 110,11,20,11\quad (2)$

$= 2(100,01,10,10)\oplus 011,20,11,11\quad (2)$

These are of Type 1.

$2a+b+c+2d+e+f=3,\ 0$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		211	211	211	211	-8
t_0	211		31	31	22	2
t_y	211	31		22	31	2
t_1	211	31	22		31	2
t_x	211	22	31	31		2

$$\begin{Bmatrix}x=0 & x=y & x=1 & x=\infty & y=0 & y=1 & y=\infty & x=y=\infty\\ [0]_2 & [0]_3 & [0]_3 & [d]_2 & [0]_3 & [0]_2 & [-a-b-e-d]_2 & [f]_3\\ [a]_2 & b & c & e & a+b+2d & [e-f]_2 & f & -a-b-e\\ & & & f & & & a+f & \end{Bmatrix}$$

$$\underline{2}11,\underline{2}2,\underline{3}1,\underline{3}1 \underset{2}{\overset{-2}{\Rightarrow}} \underline{01}1,02,11,11 \to 1\underline{01},02,11,11 \to 110,\underline{02},11,11 \to 110,20,11,11 \Rightarrow$$

$$2(100,10,10,10)\overset{+2}{\Leftarrow}\ *$$

$$010,10,10,10\ \overset{+1}{\Leftarrow}\ \underline{-1}10,\underline{0}0,\underline{0}0,\underline{0}0 \leftarrow\ *$$

$$001,10,10,10\ \overset{+1}{\Leftarrow}\ \underline{-1}01,\underline{0}0,\underline{0}0,\underline{0}0 \leftarrow\ \underline{0-1}1,\underline{0}0,\underline{0}0,\underline{0}0 \leftarrow\ *$$

$$2(100,01,10,10)\overset{+1}{\Leftarrow}2(\underline{0}00,\underline{-1}1,\underline{0}0,\underline{0}0)\leftarrow 2(\underline{00}0,-11,00,00)\leftarrow 2(0\underline{00},-11,00,00)$$

$$010,01,10,10\ \overset{+0}{\Leftarrow}\ \underline{0}10,\underline{0}1,\underline{1}0,\underline{1}0 \leftarrow \underline{10}0,01,10,10 \leftarrow 1\underline{00},01,10,10 \leftarrow 100,\underline{10},10,10$$

$$001,01,10,10\ \overset{+0}{\Leftarrow}\ \underline{0}01,\underline{0}1,\underline{1}0,\underline{1}0 \leftarrow \underline{00}1,01,10,10 \leftarrow 1\underline{10},01,10,10 \leftarrow 010,\underline{10},10,10$$

$$110,11,11,20\ \overset{+1}{\Leftarrow}\ \underline{0}10,\underline{0}1,\underline{0}1,\underline{1}0 \leftarrow \underline{10}0,01,01,10 \leftarrow 1\underline{00},01,01,10 \leftarrow 100,\underline{10},01,10$$

$$110,11,20,11\ \overset{+1}{\Leftarrow}\ \underline{0}10,\underline{0}1,\underline{1}0,\underline{0}1 \leftarrow \underline{10}0,01,10,01 \leftarrow 1\underline{00},01,10,01 \leftarrow 100,\underline{10},10,01$$

The condition for the irreducibility is

$$\{d,\ e,\ a+d,\ a+e,\ a+b+d+e,\ a+c+d+e,\ f,\ a+f\}\cap\mathbb{Z}=\emptyset. \tag{3.10}$$

This system corresponds to the equation satisfied by $F_2(\alpha;\beta,\beta';\gamma,\gamma';x,1-y)$ with

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_m\,(\beta')_n}{(\gamma)_m\,(\gamma')_n\,m!\,n!}\,x^m y^n,$$

$$a=1-\gamma,\quad b=\gamma+\gamma'-\alpha-\beta-\beta'-2,\quad c=\gamma-\alpha-\beta+\beta'-1,$$
$$d=\beta,\qquad e=\alpha-\gamma'+1,\qquad f=\alpha$$

and (3.10) equals

$$\{\alpha,\ \beta,\ \beta',\ \alpha-\gamma,\ \alpha-\gamma',\ \beta-\gamma,\ \beta'-\gamma',\ \alpha-\gamma-\gamma'\}\cap\mathbb{Z}=\emptyset. \tag{3.11}$$

3.4. Appell's F_4

$22,22,31,22$: rank $=4$, 4 parameters,

$(1^8\cdot 2^1)\quad\to\quad F_1:12,12,21,12$

$=01,01,10,01\oplus 21,21,21,21\quad(8)$

$=2(11,11,20,11)\oplus 00,00,(-1)1,00\quad(1)$

The last one is of Type 2.

$a+2b+2c+2d+2e=3,\ 0$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		211	22	211	22	-4
t_0	211		22	211	22	-4
t_y	22	22		22	31	2
t_1	211	211	22		22	-4
t_x	22	22	31	22		2

$$\begin{Bmatrix} x=0 & x=1 & x=y & y=0 & y=1 & x=\infty & y=\infty & x=y=\infty \\ [0]_2 & [0]_2 & [0]_3 & [0]_2 & [0]_2 & [d]_2 & [b+c+d]_2 & b+c+2d \\ [b]_2 & [c]_2 & a & [-b]_2 & [-c]_2 & [e]_2 & [b+c+e]_2 & b+c+2e \\ & & & & & & & [0]_2 \end{Bmatrix}$$

The addition corresponding to $u\mapsto y^b(1-y)^c u$ changes the above into

$$\begin{Bmatrix} x=0 & x=1 & x=y & y=0 & y=1 & x=\infty & y=\infty & x=y=\infty \\ [0]_2 & [0]_2 & [0]_3 & [0]_2 & [0]_2 & [d]_2 & [d]_2 & 2d \\ [b]_2 & [c]_2 & a & [b]_2 & [c]_2 & [e]_2 & [e]_2 & 2e \\ & & & & & & & [-b-c]_2 \end{Bmatrix}.$$

$$\begin{array}{l}
\underline{3}1,\underline{2}2,\underline{2}2,\underline{2}2 \underset{1}{\overset{-1}{\Rightarrow}} 21,\underline{12},12,12 \ \to 21,21,\underline{12},12 \ \to 21,21,21,\underline{12} \to 21,21,21,21 \Rightarrow \\
10,10,10,10 \ \overset{\pm1}{\Leftarrow} \ * \\
10,01,10,10 \ \overset{\pm1}{\Leftarrow} \ \underline{0}0,\underline{-1}1,\underline{0}0,\underline{0}0 \leftarrow \ * \\
10,10,01,10 \ \overset{\pm1}{\Leftarrow} \underline{00},00,-11,00 \leftarrow 00,\underline{00},-11,00 \leftarrow \ * \\
10,10,10,01 \ \overset{\pm1}{\Leftarrow} \underline{0}0,\underline{0}0,\underline{0}0,\underline{-1}1 \leftarrow 00,\underline{00},00,-11 \leftarrow 00,00,\underline{00},-11 \leftarrow \ * \\
2(20,11,11,11) \overset{\pm1}{\Leftarrow} 2(\underline{1}0,\underline{0}1,\underline{0}1,\underline{0}1) \leftarrow 2(10,\underline{10},01,01) \leftarrow 2(10,10,\underline{10},01) \leftarrow 2(10,10,10,\underline{10}) \\
21,21,21,21 \ \overset{\pm2}{\Leftarrow} \ \underline{0}1,\underline{0}1,\underline{0}1,\underline{0}1 \ \leftarrow \ 01,\underline{01},01,01 \ \leftarrow \ 01,01,\underline{10},01 \ \leftarrow \ 01,10,10,\underline{10} \\
10,10,01,01 \ \overset{\pm0}{\Leftarrow} \ \underline{1}0,\underline{1}0,\underline{0}1,\underline{0}1 \ \leftarrow \ 10,\underline{01},01,01 \ \leftarrow \ 10,01,\underline{10},01 \ \leftarrow \ 10,01,10,\underline{10} \\
10,01,10,01 \ \overset{\pm0}{\Leftarrow} \ \underline{1}0,\underline{0}1,\underline{1}0,\underline{0}1 \ \leftarrow \ 10,\underline{10},10,01 \ \leftarrow \ 10,10,\underline{01},01 \ \leftarrow \ 10,10,01,\underline{10} \\
10,01,01,10 \ \overset{\pm0}{\Leftarrow} \ \underline{1}0,\underline{0}1,\underline{0}1,\underline{1}0 \ \leftarrow \ 10,\underline{10},01,10 \ \leftarrow \ 10,01,\underline{10},10 \ \leftarrow \ 10,10,10,\underline{01}
\end{array}$$

Kato [K1, K2] gives the equation

$$\begin{cases} x(1-x)\dfrac{\partial^2 u}{\partial x^2}+\big(\gamma-(\alpha+\beta+1)x\big)\dfrac{\partial u}{\partial x}-\alpha\beta u+\epsilon\dfrac{y-1}{x-y}\Big(x\dfrac{\partial u}{\partial x}-y\dfrac{\partial u}{\partial y}\Big)=0,\\ y(1-y)\dfrac{\partial^2 u}{\partial y^2}+\big(\gamma-(\alpha+\beta+1)y\big)\dfrac{\partial u}{\partial y}-\alpha\beta u+\epsilon\dfrac{x-1}{y-x}\Big(y\dfrac{\partial u}{\partial y}-x\dfrac{\partial u}{\partial x}\Big)=0 \end{cases}$$

satisfied by $u(x,y)=F_4\big(\alpha,\beta;\gamma,\gamma';xy,(1-x)(1-y)\big)$ with

$$F_4(\alpha,\beta;\gamma,\gamma';x,y)=\sum_{m=n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_{m+n}}{(\gamma)_m\,(\gamma')_n\,m!\,n!}\,x^m y^n,$$
$$\epsilon=\gamma+\gamma'-\alpha-\beta-1. \tag{3.12}$$

We note that when $\epsilon=0$, the above equation has the solution

$$u(\alpha,\beta,\gamma;x,y)=F(\alpha,\beta,\gamma;x)\cdot F(\alpha,\beta,\gamma;y). \tag{3.13}$$

Hence the monodromy group defined by the space of solutions is irreducible when $\epsilon = 0$ (generally $2\epsilon \in \mathbb{Z}$ by Lemma 2.1 or by the Riemann scheme) and α, β and γ are generic. The corresponding Pfaffian system (3.1) is given by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 \\ 0 & \epsilon & 0 & 1 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 1 \\ 0 & 0 & 1-\gamma & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & -\epsilon & 0 \\ 0 & -\epsilon & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha\beta & -\gamma' & \epsilon & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha+\epsilon)(\beta+\epsilon) & -\gamma' \end{pmatrix},\ A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta & \epsilon & -\gamma' & 0 \\ 0 & -(\alpha+\epsilon)(\beta+\epsilon) & 0 & -\gamma' \end{pmatrix},$$

with

$$a = 2\epsilon, \quad b = 1-\gamma, \quad c = -\gamma', \quad d = \alpha, \quad e = \beta.$$

Then the condition for the irreducibility is

$$\{d,\, e,\, b+d,\, b+e,\, c+d,\, c+e,\, b+c+d,\, b+c+e\} \cap \mathbb{Z} = \emptyset \tag{3.14}$$

or equivalently

$$\{\alpha,\, \beta,\, \alpha-\gamma,\, \beta-\gamma,\, \alpha-\gamma',\, \beta-\gamma',\, \alpha-\gamma-\gamma',\, \beta-\gamma-\gamma'\} \cap \mathbb{Z} = \emptyset. \tag{3.15}$$

Note that Theorem 1.5 says that under the condition (3.14) the differential equation $\frac{du}{dx} = \bigl(\frac{A_1}{x} + \frac{A_2}{x-y} + \frac{A_3}{x-1}\bigr)u$ is irreducible if and only if $a \notin \mathbb{Z}$.

3.5. Rank 5 with 6 parameters

I_5 $41, 32, 311, 311,$ J_5 $41, 41, 221, 221$: $(1^6 \cdot 2^4)$

$41, 41, 221, 221 \ \to\ F_1 : 21, 21, 021, 021$
$41, 32, 311, 311 \ \to\ H_2 : 11, 02, 011, 011$
$41, 41, 221, 221$
$= 10, 10, 001, 010 \oplus 31, 31, 220, 211$ (4)
$= 20, 11, 110, 110 \oplus 21, 30, 111, 111$ (2)
$= 2(10, 10, 100, 100) \oplus 21, 21, 021, 021$ (4)

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		311	311	221	221	-10
t_0	311		32	311	41	2
t_y	311	32		311	41	2
t_1	221	311	311		221	-10
t_x	221	41	41	221		2

3.6. Rank 5 with 5 parameters

$41, 32, 32, 221 :$ $(1^7 \cdot 2^3)$
$\to$ $31, 22, 22, 220$ $F_2 : 31, 22, 31, 121$
$F_1 : 21, 12, 12, 021$
$= 10, 10, 10, 001 \oplus 31, 22, 22, 220$ (1)
$= 10, 01, 10, 010 \oplus 31, 31, 22, 211$ (4)
$= 20, 11, 11, 101 \oplus 21, 21, 21, 120$ (2)
$= 2(10, 10, 10, 100) \oplus 21, 12, 12, 021$ (2)
$= 2(20, 11, 11, 110) \oplus 01, 10, 10, 001$ (1)

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		311	32	2111	32	-6
t_0	311		32	2111	32	-6
t_y	32	32		221	41	2
t_1	2111	2111	221		221	-18
t_x	32	32	41	221		2

3.7. Rank 5 with 4 parameters

$P_{4,5}$ $32, 32, 32, 32 : (1^8 \cdot 2^2)$

$\to F_4 : 22, 22, 22, 31 \quad F_1 : 12, 12, 12, 12$

$= 10, 10, 10, 01 \oplus 22, 22, 22, 31 \quad (4)$

$= 21, 21, 21, 12 \oplus 11, 11, 11, 20 \quad (4)$

$= 2(10, 10, 10, 10) \oplus 12, 12, 12, 12 \quad (1)$

$= 2(21, 21, 21, 21) \oplus -(10, 10, 10, 10) \ (1)$

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		221	221	211	32	−10
t_0	221		221	221	32	−10
t_y	221	221		221	33	−10
t_1	221	221	221		32	−10
t_x	32	32	32	32		2

The last one is of Type 3 and then $10, 10, 10, 10$ appears in the preceding decomposition (cf. Proposition 2.5 iii)).

Acknowledgements. The author thanks the referee for several improvements of the paper, which makes this paper more readable.

References

[CB] W. Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspaces and sum zero*, Duke Math. J. **118** (2003), 339–352.

[DR] M. Dettweiler, S. Reiter, *An algorithm of Katz and its applications to the inverse Galois problems*, J. Symbolic Comput. **30** (2000), 761–798.

[Ha] Y. Haraoka, *Middle convolution for completely integrable systems with logarithmic singularities along hyperplane arrangements*, Adv. Studies in Pure Math. **62**(2012), 109–136.

[HK] Y. Haraoka and T. Kikukawa, *Rigidity of monodromies for Appell's hypergeometric functions*, Opuscula Math. **5**(2015), 567–594.

[K1] M. Kato, *A Pfaffian system of Appell's F4*, Bull. College Educ. Univ. Ryukyus **33** (1988), 331–334.

[K2] M. Kato, *Connection formulas for Appell's system F4 and some applications*, Funkcial, Ekvac. **38** (1995), 243–266.

[Kz] H. M. Katz, *Rigid Local Systems*, Annals of Mathematics Studies **139**, Princeton University Press, 1995.

[O1] T. Oshima, *Fractional calculus of Weyl algebra and Fuchsian differential equations*, MSJ Memoirs **28**, Mathematical Society of Japan, Tokyo, 2012.

[O2] T. Oshima, *An elementary approach to the Gauss hypergeometric function*, Josai Mathematical Monographs **6** (2013), 2–23.

[O3] T. Oshima, *Transformations of KZ type equations*, to appear in RIMS Kôkyûroku Bessatsu, Kyoto University.

[O4] T. Oshima, `os_muldif.rr`, `ftp://akagi.ms.u-tokyo.ac.jp/pub/math/muldif`, 2008-2016.

Toshio Oshima
Josai University
Faculty of Science
2-3-20, Hirakawacho, Chiyodaku, Tokyo 102-0093, Japan
e-mail: `t-oshima@josai.ac.jp`