# Kobayashi Hyperbolicity and Lang's Conjecture

Junjiro Noguchi[1]

The University of Tokyo/Tokyo Institute of Technology, Emeritus

## 1 Introduction

This paper is based on the talk given at the memorial conference for Shoshichi Kobayashi (1932–2012). Serge Lang has lived for similar years (1927–2005). The present article is dedicated to an homage to the both from the view point of Kobayashi hyperbolic geometry. They were interested not only in solving problems, but even more interested in understanding Mathematics.[2] The main aim of this article is to survey the development of the theory of Mordell's Conjecture and Lang's Conjecture in relation with the Kobayashi hyperbolicity.

The notion of Kobayashi hyperbolic manifolds was introduced in [Ko67a], [Ko67b] in 1967, and was treated by his monograph [Ko70]. S. Lang took it to discuss and formulate higher dimensional Diophantine problems in [La74]. This approach was further developed by Osgood and Vojta in relation with Nevanlinna theory in several complex analysis (cf. [Os81], [Vo87], [La91]). This leads to a problem of "*abc*-Conjecture" in higher dimension, of which counter-part in Nevanlinna theory is a second main theorem with properly truncated counting functions (cf., e.g., [No03b], [NW14]).

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## 2 Kobayashi hyperbolicity and Lang's conjecture

The present topics lie in the three fields, complex analysis, complex geometry and Diophantine geometry. We recall first the Kobayashi pseudo-distance $d_X$ of a (reduced) complex space $X$, which he called an *intrinsic pseudo-distance*: that is, $d_X$ is characterized to be the maximum one satisfying the following.

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[2]This is a personal impression of the present author.

(i) For each complex space a pseudo-distance $d_X$ is assigned.

(ii) For every holomorphic mapping $f : X \to Y$ between two complex spaces, $f^* d_Y \le d_X$ holds.

(iii) If $X$ is the unit disc $\Delta(1)$ of the Gaussian plane $\mathbf{C}$, $d_X = d_{\Delta(1)}$ is the Poincaré distance on $\Delta(1)$.

If $d_X$ is a "distance", then $X$ is called a *Kobayashi hyperbolic space.* It is immediate to see $d_{\mathbf{C}} \equiv 0$, and the following criterion of Brody is well-known:

**Theorem 2.1** ([Br78])**.** *Let $X$ be a compact complex space. Then $X$ is Kobayashi hyperbolic if and only if there is no non-constant entire curve $f : \mathbf{C} \to X$.*

Another interesting instance is the following.

**Theorem 2.2** (Royden [Ro70], [Ro71], [Ro75])**.** *Let $\mathbf{T}_g$ be the Teichmüller space of compact Riemann surfaces of genus $g \geqq 2$. Then the Teichmüller metric on $\mathbf{T}_g$ coincides with the Kobayashi hyperbolic metric on it.*

The importance of this theorem is that the Teichmüller metric was incorporated into the theory of holomorphic mappings between complex manifolds in general; the Teichmüller metric was important and useful by itself, but it had been isolated in the theory of holomorphic mappings before.

In 1974 S. Lang proposed the following conjectures.

**Conjecture 2.3** (Lang [La74])**.** *Let $k$ be a number field, and let $V_k$ be an algebraic variety defined over $k$. Suppose that $V_{\mathbf{C}}$ is Kobayashi hyperbolic for an embedding $k \hookrightarrow \mathbf{C}$. Then the set $V_k(k)$ of $k$-rational points of $V_k$ is at most finite.*

On the same time he formulated the analogue over function fields:

**Conjecture 2.4** ([La74])**.** *Let $X \to R$ be a proper fiber space of algebraic varieties over $\mathbf{C}$, and let $X(\mathbf{C}(R)) = \Gamma(R, X)$ denote the set of rational points (rational sections) over $R$. Assume that every $X_t$ $(t \in R)$ is Kobayashi hyperbolic.*

(i) *If the set $X(\mathbf{C}(R))$ is infinite, there exists a splitting subspace $Y \to R$ of $X \to R$; i.e., $Y \cong Y_0 \times R$.*

(ii) *(Splitting case) Let $V$ be a complex projective hyperbolic variety, and let $W$ be a complex algebraic variety. Then there is only a finite number of dominant rational maps of $W$ onto $V$.*

The statement (ii) above was motivated by de Franchis' Theorem: If an algebraic curve $C$ has genus $\geq 2$, then there are at most finitely many non-constant rational mappings from an algebraic variety $W$ to $C$. For generalizations of Mordell's Conjecture (now, Faltings' Theorem) he took the following two typical cases:

(a) Let $A$ be an abelian variety defined over a number field $k$. Let $V \subset A$ be a subvariety such that $V_{\mathbf{C}}$ does not contain a translate of a 1-parameter subgroup. Then $V(k)$ is finite.

(b) Let $V = \Omega/\Gamma$ be a fixed-point free quotient of a bounded domain $\Omega \subset \mathbf{C}^n$ by a discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$. Assume that $V$ is defined over a number field $k$. Then $V(k)$ is finite.

The common nature of $V$ of (a) and (b) above is the Kobayashi hyperbolicity.

The following conjecture due to Igor Shafarevich [Sh63] played an essential role in the solution of Mordell's Conjecture and is related to Kobayashi hyperbolicity, too (see *1988a*, *1988b* and *1990c* in §3).

**Conjecture 2.5** (Shafarevich)**.** *Let $R$ be a projective algebraic curve and let $S \subset R$ be a finite set. Then, for any given integer $g \geq 2$ there exists at most finitely many proper families $\pi : X \to R$ over $R$ such that the fiber $X_t := \pi^{-1}(t)$ is a smooth projective algebraic curve of genus $g$ for $t \in R \setminus S$.*

*Remark.* It is interesting to learn that I. Shafarevich was trying to find good problems in algebraic geometry modeled after number theory.

# 3 Chronicle of Mordell's and Lang's Conjectures

*1960/62*: S. Lang proposed Mordell's Conjecture over function fields, which is the 1-dimensional case of Conjecture 2.4 (i) ([La60], [La62]).

*1963*: Y. Manin [Ma63] gave a proof of Mordell's Conjecture over function fields proposed by S. Lang, but later a non-trivial gap was found in the proof (cf. below *1990a*). The method was analytic.

*1965*: H. Grauert [Gr65] proved Mordell's Conjecture over function fields by a method of algebraic geometry. Here the notion of Grauert's (weak) negativity was effectively used.

*1966*: P. Samuel published a comprehensive lecture notes on the above Grauert's proof ([Sa66]).

*1968*: A.N. Parshin proved the Shafarevich Conjecture 2.5 under the condition $S = \emptyset$, and observed that Shafarevich's Conjecture implies Mordell's ([Pa68]).

*1971*: S.Ju. Arakelov solved the general case of the Shafarevich Conjecture 2.5 (Parshin-Arakelov Theorem: [Ar71]).

*1975a*: Motivated by the Lang Conjecture 2.4 (ii), S. Kobayashi and T. Ochiai [KO75] proved a finiteness theorem:

*Let $V$ be a compact complex space of general type, and let $W$ be a complex algebraic variety. Then there are only finitely many dominant meromorphic mappings from $W$ to $V$.*

*1975b*: D. Mumford published a comprehensive concise lecture notes on the works of Parshin and Arakelov ([Mu75]).

*1979*: F.A. Bogomolov gave a proof of Mordell's Conjecture over functions fields by making use of algebraic surface theory (cf. [De79]).

*1981*: J. Noguchi generalized Grauert's result to the higher dimensional case under the assumption that the relative tangent bundle is negative ([No81]).

*1983a*: G. Faltings took up the question of the Shafarevich Conjecture (the Parshin-Arakelov Theorem) for abelian varieties and showed that the moduli of morphisms from an algebraic curve into Siegel space is of finite type ([Fa83a]). This led to the following break-through (*1983b*).

*1983b*: G. Faltings proved Mordell's Conjecture over number fields (now, Faltings' Theorem: [Fa83b]).

*1985*: J. Noguchi proved Lang's Conjecture 2.4 (i) for hyperbolic spaces with some geometric assumption for degenerate fibers that is automatically fulfilled in $\dim X_t = 1$. Therefore, in 1-dimensional case, this gives another proof for Faltings' Theorem (Mordell's Conjecture) over function fields by means of the *Kobayashi hyperbolicity* ([No85]).

*1987*: P. Vojta published a lecture notes on his approach to Diophantine geometry from the viewpoint of Nevanlinna theory ([Vo87]).

*1988a*: Noguchi studied the moduli of holomorphic maps from a Zariski open subset of a compact Kähler manifold into an arithmetic quotient $M$ of a bounded symmetric domain, and proved that the moduli is properly immersed onto a totally geodesic submanifold of $M$. In the proof the *Kobayashi* *hyperbolicity* played an important role. The result generalizes Faltings' Parshin-Arakelov Theorem for abelian varieties in *1983a* above ([No88]).

*1988b*: Y. Imayoshi and H. Shiga gave another proof of Parshin-Arakelov Theorem (the Shafarevich Conjecture 2.5) by means of Teichmüller theory, and hence Faltings' Theorem (Mordell's Conjecture) over function fields. Here, *Theorem 2.2 of Royden* played an essential role ([IS88]).

*1990a*: R.F. Coleman noticed that there was a gap in the above proof of Manin in 1963, and completed Manin's proof ([Co90], cf. also [Ch91]).

*1990b*: C. Horst proved Lang's Conjecture 2.4 (ii) for compact hyperbolic Kähler manifold $V$ ([Ho90]).

*1990c*: Using a method involving *Kobayashi hyperbolicity*, A.N. Parshin proved a finiteness theorem that contains Mordell's Conjecture over function fields as a special case ([Pa90]).

*1990d*: E. Bombieri gave a proof of Faltings' Theorem by Diophantine approximation theory ([Bo90]).

*1991*: P. Vojta gave a proof of Faltings' Theorem based on his function field version ([Vo91]), which was in fact earlier than Bombieri's proof above.

*1991/94*: Faltings proved Lang's Conjecture 2.3 for subvarieties of abelian varieties over number fields ([Fa91], [Fa94]).

*1992*: Noguchi proved Lang's Conjecture 2.4 (ii) for any compact Kobayashi hyperbolic complex space with no assumption on singularity, algebraicity nor being Kähler ([No92]). Cf. Makoto Suzuki [SuMk94] for the non-compact generalization.

*1996/99*: Vojta generalized Faltings' results of *1991/94* above for semi-abelian varieties ([Vo96], [Vo99]). Cf. M. McQuillan [Mc95] and A. Buium [Bu92] for related results.

*2002/08*: J. Noguchi, J. Winkelmann and K. Yamanoi proved the second main theorem with counting functions truncated to level one, which is an analytic analogue of "*abc*-Conjecture in semi-abelian varieties" ([No96]) for entire curves in semi-abelian varieties ([NWY02], [NWY08]).

*2003*: Noguchi showed an example of Kobayashi hyperbolic projective hypersurface (defined by one equation, the least constraint for algebraic varieties) defined over $\mathbf{Q}$, which carries only finitely many rational points over any fixed number field ([No03a]).

*Remark.* Cf. [No07] for an arithmetic Kobayashi pseudo-distance, and [Ru01], [BG06], [NW14] for more detailed treatment on Nevanlinna theory and Diophantine approximation.

# 4 Open Problems

Here we recall some open problems on Kobayashi hyperbolic spaces.

**Problem 4.1** ([No93])**.** *Let $X$ be a fixed compact complex space. Are there only finitely many pairs $(Y, f)$ up to isomorphism such that $Y$ is a compact Kobayashi hyperbolic space and $f : X \to Y$ is a surjective holomorphic map?*

**Problem 4.2.** *Does there exist a non-algebraic or a non-Kähler compact hyperbolic manifold?*

**Problem 4.3.** *Is a compact Kobayashi hyperbolic manifold of general type?*

A holomorphic map from the punctured disk $\Delta^*(\subset \mathbf{C})$ into a compact Kobayashi hyperbolic space extends holomorphically over the whole disk $\Delta$ to $X$. Therefore, it is interesting to ask

**Problem 4.4** ([No93])**.** *Let $E \subset \Delta(\subset \mathbf{C})$ be a subset of logarithmic capacity zero of a disk $\Delta$. Let $f : \Delta \setminus E \to X$ be a holomorphic map into a compact Kobayashi hyperbolic space (or manifold). Then, does $f$ extend holomorphically over $\Delta$ to $X$?*

*Remark.* T. Nishino [Ni79] proved the case of $\dim X = 1$ (cf. [SuMs87] for another proof), and Masakazu Suzuki [SuMs88] proved the case when $X$ has a universal covering biholomorphic to a bounded polynomially convex domain of $\mathbf{C}^n$ with $n = \dim X$.

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