Inverse of Abelian Integrals and Ramified Riemann Domains*

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Abstract

We deal with the Levi problem (Hartogs' inverse problem) for ramified Riemann domains by introducing a positive scalar function $\rho(a, X)$ for a complex manifold X with a global frame of the holomorphic cotangent bundle by closed Abelian differentials, which is an analogue of Hartogs' radius. We obtain some geometric conditions in terms of $\rho(a, X)$ which imply the validity of the Levi problem for finitely sheeted ramified Riemann domains over $\mathbf{C}^n$. On the course, we give a new proof of the Behnke–Stein Theorem.

1 Introduction and main results

1.1 Introduction

In 1943 K. Oka wrote a manuscript in Japanese, solving affirmatively the Levi problem (Hartogs' inverse problem) for unramified Riemann domains over complex number space $\mathbf{C}^n$ of arbitrary dimension $n \geq 2$,[1)] and in 1953 he published Oka IX [26] to solve it by making use of his First Coherence Theorem proved in Oka VII [24][2)]; there, he put a special emphasis on the difficulties of the ramified case (see [26], Introduction 2 and §23, [25], Introduction). H. Grauert also emphasized the problem to generalize Oka's Theorem (IX) to the case of ramified Riemann domains in his lecture at OKA 100 Conference Kyoto/Nara 2001. Oka's Theorem (IX) was generalized for unramified Riemann domains

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1) This fact was written twice in the introductions of his two papers, [25] and [26]: The manuscript was written as a research report *dated 12 Dec. 1943*, sent to Teiji Takagi, then Professor at the Imperial University of Tokyo, and now one can find it in [29].

2) It is noted that Oka VII [24] is different to his original, Oka VII in [27]; therefore, there are two versions of Oka VII. The English translation of Oka VII in [28] was taken from the latter, but unfortunately in [28] all the records of the received dates of the papers were deleted.

over complex projective n-space $\mathbf{P}^n(\mathbf{C})$ by R. Fujita [10] and A. Takeuchi [32]. On the other hand, H. Grauert [18] gave a counter-example to the problem for ramified Riemann domains over $\mathbf{P}^n(\mathbf{C})$, and J.E. Fornæss [7] gave a counter-example to it over $\mathbf{C}^n$. Therefore, it is natural to look for geometric conditions which imply the validity of the Levi problem for ramified Riemann domains.

Under a geometric condition (Cond A, 1.1) on a complex manifold X, we introduce a new scalar function $\rho(a, \Omega)(> 0)$ for a subdomain $\Omega \subset X$, which is an analogue of the boundary distance function in the unramified case (cf. Remark 2.1 (i)). We prove an *estimate of Cartan-Thullen type* ([4]) for the holomorphically convex hull $\hat{K}_\Omega$ of a compact subset $K \Subset \Omega$ with $\rho(a, \Omega)$ (see Theorem 1.3).

In the one-dimensional case, by making use of $\rho(a, \Omega)$ we give a *new proof of Behnke–Stein's Theorem*: Every open Riemann surface is Stein. In the known methods one uses a generalization of the Cauchy kernel or some functional analytic method (cf. Behnke–Stein [2], Kusunoki [16], Forster [8], etc.). Here we use *Oka's Jôku-Ikô* combined with Grauert's Finiteness Theorem, which is now a rather easy result by a simplification of the proof, particularly in the one-dimensional case (see §1.2.2): *Oka's Jôku-Ikô* (transform to a higher space) is a principal method of K. Oka to reduce a difficult problem over a certain general space to the one over a simpler space such as a polydisk, but of higher dimension, and to solve it (cf. K. Oka [27], e.g., [20]). We see here how the scalar $\rho(a, \Omega)$ works well in this case.

Now, let $\pi : X \to \mathbf{C}^n$ be a Riemann domain, possibly ramified, such that X satisfies Cond A. Then, we prove that a domain $\Omega \Subset X$ is a *domain of holomorphy*[3)] *if and only if Ω is holomorphically convex* (see Theorem 1.12). Moreover, if X is exhausted by a continuous family of relatively compact domains of holomorphy, then X is Stein (see Theorem 1.17; see §3 (a) for a counter-example which does not satisfy Cond A).

We next consider a boundary condition (Cond B, 1.18) with $\rho(a, X)$. We assume that X satisfies Cond A and that $X \xrightarrow{\pi} \mathbf{C}^n$ satisfies Cond B and is finitely sheeted. We prove that *if X is locally Stein over $\mathbf{C}^n$, then X is Stein* (see Theorem 1.19; see §3 (a) for a counter-example, not satisfying the conditions).

We give the proofs in §2. In §3 we will discuss some examples and properties of $\rho(a, X)$.

Acknowledgment. The author is very grateful to Professor J.E. Fornæss for the clarification that his example ([7]) does not satisfy Cond A (§3 (a)), and to Professor Makoto Abe for interesting discussions on the present theme.

[3)] Cf. Definition 1.2

1.2 Main results

1.2.1 Scalar $\rho(a, \Omega)$

Let X be a connected complex manifold of dimension n with holomorphic cotangent bundle $\mathbf{T}(X)^*$. We assume:

Condition 1.1 (Cond A). There exists a global frame $\omega = (\omega^1, \ldots, \omega^n)$ of $\mathbf{T}(X)^*$ over X such that $d\omega^j = 0$, $1 \le j \le n$.

Let $\Omega \subset X$ be a subdomain. With Cond A we consider an Abelian integral (a path integral) of ω in Ω from $a \in \Omega$:

$$\alpha : x \in \Omega \longrightarrow \zeta = (\zeta^j) = \left(\int_a^x \omega^1, \ldots, \int_a^x \omega^n \right) \in \mathbf{C}^n. \tag{1.1}$$

We denote by $\mathrm{P}\Delta = \prod_{j=1}^n \{|\zeta^j| < 1\}$ the unit polydisk of $\mathbf{C}^n$ with center at 0 and and set

$$\rho \mathrm{P}\Delta = \prod_{j=1}^n \{|\zeta^j| < \rho\}$$

for $\rho > 0$. Then, $\alpha(x) = \zeta$ has the inverse $\phi_{a,\rho_0}(\zeta) = x$ on a small polydisk $\rho_0 \mathrm{P}\Delta$:

$$\phi_{a,\rho_0} : \rho_0 \mathrm{P}\Delta \longrightarrow U_0 = \phi_{a,\rho_0}(\rho_0 \mathrm{P}\Delta) \subset \Omega. \tag{1.2}$$

Then we extend analytically ϕ_{a,ρ_0} to $\phi_{a,\rho} : \rho\mathrm{P}\Delta \to X$, $\rho \ge \rho_0$, as much as possible, and set

$$\rho(a, \Omega) = \sup\{\rho > 0 : \exists \phi_{a,\rho} : \rho\mathrm{P}\Delta \to X,\ \phi_{a,\rho}(\rho\mathrm{P}\Delta) \subset \Omega\} \le \infty. \tag{1.3}$$

Then we have the inverse of the Abelian integral α on the polydisk of the maximal radius

$$\phi_a : \rho(a, \Omega)\mathrm{P}\Delta \longrightarrow \Omega. \tag{1.4}$$

To be precise, we should write

$$\rho(a, \Omega) = \rho(a, \omega, \Omega) = \rho(a, \mathrm{P}\Delta, \omega, \Omega), \tag{1.5}$$

but unless confusion occurs, we use $\rho(a, \Omega)$ for notational simplicity.

We immediately see that (cf. §2.1)

(i) $\rho(a, \Omega)$ is finitely valued and continuous, unless $\rho(a, \Omega) \equiv \infty$;

(ii) $\rho(a, \Omega) \le \inf\{|v|_\omega : v \in \mathbf{T}(X)_a,\ F_\Omega(v) = 1\}$, where F_Ω denotes the Kobayashi hyperbolic infinitesimal form of Ω, and $|v|_\omega = \max_j |\omega^j(v)|$, the maximum norm of v with respect to $\omega = (\omega^j)$.

For a subset $A \subset \Omega$ we write

$$\rho(A, \Omega) = \inf\{\rho(a, \Omega) : a \in A\}.$$

For a compact subset $K \Subset \Omega$ we denote by $\hat{K}_\Omega$ the holomorphically convex hull of K defined by

$$\hat{K}_\Omega = \left\{ x \in \Omega : |f(x)| \le \max_K |f|, \ \forall f \in \mathcal{O}(\Omega) \right\},$$

where $\mathcal{O}(\Omega)$ is the set of all holomorphic functions on Ω. If $\hat{K}_\Omega \Subset \Omega$ for every $K \Subset \Omega$, Ω is called a holomorphically convex domain.

Definition 1.2. For a relatively compact subdomain $\Omega \Subset X$ of a complex manifold X we may naturally define the notion of *domain of holomorphy*: i.e., there is no point $b \in \partial\Omega$ such that there are a connected neighborhood U of b in X and a non-empty open subset $V \subset U \cap \Omega$ satisfying that for every $f \in \mathcal{O}(\Omega)$ there exists $g \in \mathcal{O}(U)$ with $f|_V = g|_V$.

The following theorem of the Cartan–Thullen type (cf. [4]) is our first main result.

Theorem 1.3. *Let X be a complex manifold satisfying Cond A. Let $\Omega \Subset X$ be a relatively compact domain of holomorphy, let $K \Subset \Omega$ be a compact subset, and let $f \in \mathcal{O}(\Omega)$. Assume that*

$$|f(a)| \le \rho(a, \Omega), \quad \forall a \in K.$$

Then we have

$$|f(a)| \le \rho(a, \Omega), \quad \forall a \in \hat{K}_\Omega. \tag{1.6}$$

In particular, we have

$$\rho(K, \Omega) = \rho(\hat{K}_\Omega, \Omega). \tag{1.7}$$

Corollary 1.4. *Let $\Omega \Subset X$ be a domain of a complex manifold X, satisfying Cond A. Then, Ω is a domain of holomorphy if and only if Ω is holomorphically convex.*

1.2.2 The Behnke–Stein Theorem for open Riemann surfaces

We apply the scalar $\rho(a, \Omega)$ introduced above to give a new proof of the Behnke–Stein Theorem for the Steinness of open Riemann surfaces, which is one of the most basic facts in the theory of Riemann surfaces: Here, we do not use the Cauchy kernel generalized on a Riemann surface (cf. [2], [16]), nor a functional analytic method (cf., e.g., [8]), but use Oka's Jôku-Ikô together with Grauert's Finiteness Theorem. This is the very difference of our new proof to the known ones.

To be precise, we recall the definition of a Stein manifold:

Definition 1.5. A complex manifold M of pure dimension n is called a Stein manifold if the following Stein conditions are satisfied:

(i) M satisfies the second countability axiom.

(ii) For distinct points $p, q \in M$ there is an $f \in \mathcal{O}(M)$ with $f(p) \neq f(q)$.

(iii) For every $p \in M$ there are $f_j \in \mathcal{O}(M)$, $1 \leq j \leq n$, such that $df_1(p) \wedge \cdots \wedge df_n(p) \neq 0$.

(iv) M is holomorphically convex.

We will rely on the following H. Grauert's Finiteness Theorem in the one-dimensional case, which is now a rather easy consequence of the Oka–Cartan Fundamental Theorem, thanks to a very simplified proof of L. Schwartz's Finiteness Theorem based on the idea of Demailly's Lecture Notes [5], Chap. IX (cf. [20], §7.3 for the present form):

L. Schwartz's Finiteness Theorem. *Let E be a Fréchet space and let F be a Baire vector space. Let $A : E \to F$ be a continuous linear surjection, and let $B : E \to F$ be a completely continuous linear map. Then, $(A + B)(E)$ is closed and the cokernel* $\mathrm{Coker}(A + B)$ *is finite dimensional.*

Here, a Baire space is a topological space such that Baire's category theorem holds. The statement above is slightly generalized than the original one, in which F is also assumed to be Fréchet (cf. L. Schwartz [30], Serre [31], Bers [3], Grauert-Remmert [13], Demailly [5]).

Grauert's Theorem in dimension 1. *Let X be a Riemann surface, and let $\Omega \Subset X$ be a relatively compact subdomain. Then,*

$$\dim H^1(\Omega, \mathcal{O}_\Omega) < \infty. \tag{1.8}$$

Here, $\mathcal{O}_\Omega$ denotes the sheaf of germs of holomorphic functions over Ω. In case $\Omega(= X)$ itself is compact, this theorem reduces to the Cartan–Serre Theorem in dimension 1.

N.B. It is the very idea of Grauert to claim only the finite dimensionality, weaker than a posteriori statement, $H^1(\Omega, \mathcal{O}_\Omega) = 0$: It makes the proof considerably easy.

By making use of this theorem we prove an intermediate result:

Lemma 1.6. *Every relatively compact domain Ω of X is Stein.*

Let $\Omega \Subset \tilde{\Omega} \Subset X$ be subdomains of an open Riemann surface X. Since $\tilde{\Omega}$ is Stein by Lemma 1.6 and $H^2(\tilde{\Omega}, \mathbf{Z}) = 0$, we see by the Oka Principle that the line bundle of holomorphic 1-forms over $\tilde{\Omega}$ is trivial, and so we have:

Corollary 1.7. *There exists a holomorphic* 1*-form ω on $\tilde{\Omega}$ without zeros.*

By making use of ω above we define $\rho(a, \Omega)$ as in (1.3) with $X = \tilde{\Omega}$.

Applying Oka's Jôku-Ikô combined with $\rho(a, \Omega)$, we give the proofs of the following approximations of the Runge type:

Lemma 1.8. *Let Ω' be a domain such that $\Omega \Subset \Omega' \Subset \tilde{\Omega}$, and let $K \Subset \Omega$ be a compact subset. Assume that****

$$\max_{b \in \partial\Omega} \rho(b, \Omega') < \rho(K, \Omega). \tag{1.9}$$

Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on K by elements of $\mathcal{O}(\Omega')$.

Theorem 1.9. *Assume that no component of $\tilde{\Omega} \setminus \bar{\Omega}$ is relatively compact in $\tilde{\Omega}$. Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compact subsets of Ω by elements of $\mathcal{O}(\tilde{\Omega})$.*

Finally we give another proof of

Theorem 1.10 (Behnke–Stein [2])**.** *Every open Riemann surface X is Stein.*

1.2.3 Riemann domains

Let X be a complex manifold, and let $\pi : X \to \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) be a holomorphic map.

Definition 1.11. We call $\pi : X \to \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) a *Riemann domain* (over $\mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$)) if every fiber $\pi^{-1}z$ with $z \in \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) is discrete; if $d\pi$ has the maximal rank everywhere, it is called an *unramified* Riemann domain (over $\mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$)). A Riemann domain which is not unramified, is called a *ramified Riemann domain.* If the cardinality of $\pi^{-1}z$ is bounded in $z \in \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$), then we say that $\pi : X \to \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) is *finitely sheeted* or *k-sheeted* with the maximum k of the cardinalities of $\pi^{-1}z$ ($z \in \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$)).

If $\pi : X \to \mathbf{C}^n$ (resp. $\mathbf{P}^n(\mathbf{C})$) is a Riemann domain, then the pull-back of the Euclidean metric (resp. the Fubini–Study metric) by π is a degenerate (pseudo-)hermitian metric on X, which leads a distance function on X; hence, X satisfies the second countability axiom.

Note that unramified Riemann domains over $\mathbf{C}^n$ naturally satisfy Cond A.

We have:

Theorem 1.12. *Let $\pi : X \to \mathbf{C}^n$ be a Riemann domain possibly ramified such that X satisfies Cond A.*

(i) *Let $\Omega \Subset X$ be a subdomain. Then, Ω is a domain of holomorphy if and only if Ω is Stein.*

(ii) *If X is Stein, then $-\log \rho(a, X)$ is either identically $-\infty$, or continuous plurisubharmonic.*

Definition 1.13 (Locally Stein). (i) Let X be a complex manifold. We say that a subdomain $\Omega \Subset X$ is *locally Stein* if for every $a \in \bar{\Omega}$ (the topological closure) there is a neighborhood U of a in X such that $\Omega \cap U$ is Stein.

(ii) Let $\pi : X \to \mathbf{C}^n$ be a Riemann domain, possibly ramified. If for every point $z \in \mathbf{C}^n$ there is a neighborhood V of z such that $\pi^{-1}V$ is Stein or empty, X is said to be *locally Stein over* $\mathbf{C}^n$ (cf. [7]).

In general, the Levi problem is the one to asks if a locally Stein domain (over $\mathbf{C}^n$) is Stein.

Remark 1.14. The following statement is a direct consequence of Elencwajg [6], Théorème II combined with Andreotti–Narasimhan [1], Lemma 5:

Theorem 1.15. *Let $\pi : X \to \mathbf{C}^n$ be a ramified Riemann domain, and let $\Omega \Subset X$ be a subdomain. If Ω is locally Stein, then Ω is a Stein manifold.*

Therefore the Levi problem for a ramified Riemann domain $X \xrightarrow{\pi} \mathbf{C}^n$ is essentially at the "*infinity*" of X.

Definition 1.16. Let X be a complex manifold in general. A family $\{\Omega_t\}_{0 \le t \le 1}$ of subdomains Ω_t of X is called a *continuous exhaustion family of subdomains of* X if the following conditions are satisfied:

(i) $\Omega_t \Subset \Omega_s \Subset \Omega_1 = X$ for $0 \le t < s < 1$,

(ii) $\bigcup_{t<s} \Omega_t = \Omega_s$ for $0 < s \le 1$,

(iii) $\partial\Omega_t = \bigcap_{s>t} \overline{\Omega_s \setminus \overline{\Omega}_t}$ for $0 \le t < 1$.

Theorem 1.17. *Let $\pi : X \to \mathbf{C}^n$ be a Riemann domain, possibly ramified. Assume that there is a continuous exhaustion family $\{\Omega_t\}_{0 \le t \le 1}$ of subdomains of X such that for $0 \le t < 1$,*

(i) *Ω_t satisfies Cond A,*

(ii) *Ω_t is a domain of holomorphy (or equivalently, Stein).*

Then, X is Stein, and for any fixed $0 \le t < 1$ a holomorphic function $f \in \mathcal{O}(\Omega_t)$ can be approximated uniformly on compact subsets by elements of $\mathcal{O}(X)$.

Let $\pi : X \to \mathbf{C}^n$ be a Riemann domain such that X satisfies Cond A and let ∂X denote the ideal boundary of X over $\mathbf{C}^n$ (called the accessible boundary in Fritzsche–Grauert [9], Chap. II §9). We set

$$\Gamma = \overline{\pi(\partial X)} \quad \text{(the topological closure)}.$$

To deal with the total space X we consider the following condition which is a sort of localization principle:

Condition 1.18 (Cond B). (i) For any sequence $\{a_\nu\}_{\nu=1}^{\infty}$ of points of X such that it has no accumulation point in X and $\{\pi(a_\nu)\}_{\nu=1}^{\infty}$ is convergent, $\lim_{\nu\to\infty} \rho(a_\nu, X) = 0$.

(ii) For every point $z \in \Gamma$ there are arbitrarily small neighborhoods $V \Subset W$ of z in $\mathbf{C}^n$ such that

$$\rho(a, X) = \rho(a, \widetilde{W}), \quad \forall a \in \widetilde{V}, \tag{1.10}$$

where $\widetilde{V}$ (resp. $\widetilde{W}$) is an arbitrary connected component of $\pi^{-1}V$ (resp. $\pi^{-1}W$) with $\widetilde{V} \subset \widetilde{W}$.

For the Levi problem we prove:

Theorem 1.19. *Let $\pi : X \to \mathbf{C}^n$ be a finitely sheeted ramified Riemann domain. Assume that Cond A and Cond B are satisfied. If X is locally Stein over $\mathbf{C}^n$, X is a Stein manifold.*

Remark 1.20. Fornæss' counter-example ([7]) for the Levi problem in the ramified case is a 2-sheeted Riemann domain over $\mathbf{C}^n$, but it does not satisfy Cond A (see §3 (a)).

2 Proofs

2.1 Scalar $\rho(a, \Omega)$

Let X be a complex manifold satisfying Cond A. We here deal with some elementary properties of $\rho(a, \Omega)$ defined by (1.3) for a subdomain $\Omega \subset X$. We use the same notion as in §1.2.1.

First, we suppose that $\rho(a_0, \Omega) = \infty$ at a point $a_0 \in \Omega$. Then, $\phi_{a_0} : \mathbf{C}^n \to \Omega$ is surjective, and $\rho(a, \Omega) \equiv \infty$ for $a \in \Omega$. In fact, for any $a \in \Omega$ we take a path C_a from a_0 to a in Ω and set $\zeta = \alpha(a)$. By the definition, $\phi_{a_0}(\zeta) = a$, and it follows that $\rho(a, \Omega) = \infty$. Thus, we have:

$$\text{either} \quad \rho(a, \Omega) \equiv \infty, \text{ or } \ \rho(a, \Omega) < \infty, \ \forall a \in \Omega. \tag{2.1}$$

Suppose that the latter case above holds. We identify $\rho_0 \mathrm{P}\Delta_0$ and U_0 in (1.2). For $b, c \in \rho_0 \mathrm{P}\Delta$ we have

$$\rho(b, \Omega) \geq \rho(c, \Omega) - |b - c|,$$

where $|b - c|$ denotes the maximum norm with respect to the coordinate system $(\zeta^j) \in \rho_0 \mathrm{P}\Delta$. Thus,

$$\rho(c, \Omega) - \rho(b, \Omega) \leq |b - c|.$$

Changing b and c, we have the converse inequality, so that

$$|\rho(b, \Omega) - \rho(c, \Omega)| \leq |b - c|, \quad b, c \in \rho_0 \mathrm{P}\Delta \cong U_0. \tag{2.2}$$

Therefore, $\rho(a, \Omega)$ is a continuous function in $a \in \Omega$.

Let $v = \sum_{j=1}^{n} v^j \left(\frac{\partial}{\partial \zeta^j}\right)_a \in \mathbf{T}(\Omega)_a$ be a holomorphic tangent vector at $a \in \Omega$. Then, we set

$$|v|_\omega = \max_{1 \le j \le n} |v^j|.$$

With $|v|_\omega = 1$ we have by the definition of the Kobayashi hyperbolic infinitesimal metric F_Ω (cf. [15], [21])

$$F_\Omega(v) \le \frac{1}{\rho(a, \Omega)}.$$

Therefore we have

$$\rho(a, \Omega) \le \inf_{v : F_\Omega(v) = 1} |v|_\omega. \tag{2.3}$$

Provided that $\partial\Omega \neq \emptyset$, it immediately follows that

$$\lim_{a \to \partial\Omega} \rho(a, \Omega) = 0. \tag{2.4}$$

Remark 2.1. (i) We consider an *unramified* Riemann domain $\pi : X \to \mathbf{C}^n$. Let $(z^1, \ldots, z^n)$ be the natural coordinate system of $\mathbf{C}^n$ and put $\omega = (\pi^* dz^j)$ (Cond A). Then the boundary distance function $\delta_{\mathrm{P}\Delta}(a, \partial X)$ to the ideal boundary ∂X with respect to the unit polydisk $\mathrm{P}\Delta$ is defined as the supremum of such $r > 0$ that X is univalent onto $\pi(a) + r\mathrm{P}\Delta$ in a neighborhood of a (cf., e.g., [14], [20]). Therefore, in this case we have that

$$\rho(a, X) = \delta_{\mathrm{P}\Delta}(a, \partial X), \tag{2.5}$$

and Cond B is naturally satisfied. As for the difficulty to deal with the Levi problem for ramified Riemann domains, K. Oka wrote in IX [26], §23:

> “ Pour le deuxième cas les rayons de *Hartogs* cessent de jouir du rôle; ceci présente une difficulté qui m'apparait vraiment grande.”

The above “le deuxième cas” is the ramified case.

(ii) For X satisfying Cond A one can define *Hartogs' radius* $\rho_n(a, X)$ as follows. Consider $\phi_{a,(r_j)} : \mathrm{P}\Delta(r_j) \to X$ for a polydisk $\mathrm{P}\Delta(r_j)$ about 0 with a poly-radius $(r_1, \ldots, r_n)$ $(r_j > 0)$, which is an inverse of α given by (1.1). Then, one defines $\rho_n(a, X)$ as the supremum of such $r_n > 0$; for other j, it is similarly defined. Hartogs' radius $\rho_n(a, \Omega)$ is not necessarily continuous, but lower semi-continuous. In the present paper, the scalar $\rho(a, X)$ defined under Cond A plays the role of “Hartogs' radius”.

Remark 2.2. Even if $\rho(a, \omega, X) = \infty$ (cf. (1.5)), “$\rho(a, \omega', X) < \infty$” may happen for another choice of ω' (cf. §3).

2.2 Proof of Theorem 1.3

For $a \in \Omega$ we let

$$\phi_a : \rho(a, \Omega)\mathrm{P}\Delta \longrightarrow \Omega$$

be as in (1.4). We take an arbitrary element $u \in \mathcal{O}(\Omega)$. With a fixed positive number $s < 1$ we set

$$L = \bigcup_{a \in K} \phi_a \left(s|f(a)|\, \overline{\mathrm{P}\Delta} \right).$$

Then it follows from the assumption that L is a compact subset of Ω. Therefore there is an $M > 0$ such that

$$|u| < M \text{ on } L.$$

Let ∂_j be the dual vector fields of ω^j, $1 \leq j \leq n$, on X. For a multi-index $\nu = (\nu_1, \ldots, \nu_n)$ with non-negative integers $\nu_j \in \mathbf{Z}^+$ we put

$$\begin{aligned} \partial^\nu &= \partial_1^{\nu_1} \cdots \partial_n^{\nu_n}, \\ |\nu| &= \nu_1 + \cdots + \nu_n, \\ \nu! &= \nu_1! \cdot \cdots \cdot \nu_n!\,. \end{aligned}$$

By Cauchy's inequalities for $u \circ \phi_a$ on $s|f(a)|\,\overline{\mathrm{P}\Delta}$ with $a \in K$ we have

$$\frac{1}{\nu!}|\partial^\nu u(a)| \cdot |sf(a)|^{|\nu|} \leq M, \quad \forall a \in K,\ \forall \nu \in (\mathbf{Z}^+)^n.$$

Note that $(\partial^\nu u) \cdot f^{|\nu|} \in \mathcal{O}(\Omega)$. By the definition of $\hat{K}_\Omega$,

$$\frac{1}{\nu!}|\partial^\nu u(a)| \cdot |sf(a)|^{|\nu|} \leq M, \quad \forall a \in \hat{K}_\Omega,\ \forall \nu \in (\mathbf{Z}^+)^n. \tag{2.6}$$

For $a \in \hat{K}_\Omega$ we consider the Taylor expansion of $u \circ \phi_a(\zeta)$ at a:

$$u \circ \phi_a(\zeta) = \sum_{\nu \in (\mathbf{Z}^+)^n} \frac{1}{\nu!} \partial^\nu u(a) \zeta^\nu. \tag{2.7}$$

We infer from (2.6) that (2.7) converges at least on $s|f(a)|\,\mathrm{P}\Delta$. Since Ω is a domain of holomorphy, we have that $\rho(a, \Omega) \geq s|f(a)|$. Letting $s \nearrow 1$, we deduce (1.6).

By definition, $\rho(K, \Omega) \geq \rho(\hat{K}_\Omega, \Omega)$. The converse is deduced by applying the result obtained above for a constant function $f \equiv \rho(K, \Omega)$; thus (1.7) follows. □

Proof of Corollary 1.4: Assume that $\Omega \Subset X$ is a domain of holomorphy. Let $K \Subset \Omega$. It follows from (1.7) that $\hat{K}_\Omega \Subset \Omega$, and hence Ω is holomorphically convex. The converse is clear. □

Remark 2.3. (i) Replacing $\mathrm{P}\Delta$ by the unit ball B with center at 0, one may define similarly $\rho(a, \Omega)$. Then Theorem 1.3 remains to hold. Note that the union of all unitary rotations of $\frac{1}{\sqrt{n}}\mathrm{P}\Delta$ is B.

(ii) Note that $\mathrm{P}\Delta$ may be an arbitrary polydisk with center at 0; still, Theorem 1.3 remains valid. We use the unit polydisk just for simplicity.

2.3 Proof of the Behnke–Stein Theorem

2.3.1 Proof of Lemma 1.6

(a) We take a subdomain $\tilde{\Omega}$ of X such that $\Omega \Subset \tilde{\Omega} \Subset X$. Let $c \in \partial\Omega$ be any point, and take a local coordinate neighborhood system (W_0, w) in $\tilde{\Omega}$ with holomorphic coordinate w such that $w = 0$ at c. We consider Cousin I distributions for $k = 1, 2, \ldots$:

$$\begin{array}{ccl} \dfrac{1}{w^k} & \text{on} & W_0, \\ 0 & \text{on} & W_1 = \tilde{\Omega} \setminus \{c\}. \end{array}$$

These induce cohomology classes

$$\left[\frac{1}{w^k}\right] \in H^1(\{W_0, W_1\}, \mathcal{O}_{\tilde{\Omega}}) \hookrightarrow H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}), \quad k = 1, 2, \ldots.$$

Since $\dim H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}) < \infty$ by (1.8) (Grauert's Theorem), there is a non-trivial linear relation over $\mathbf{C}$

$$\sum_{k=1}^{\nu} \gamma_k \left[\frac{1}{w^k}\right] = 0 \in H^1(\tilde{\Omega}, \mathcal{O}_{\tilde{\Omega}}), \quad \gamma_k \in \mathbf{C},\ \gamma_\nu \neq 0.$$

Hence there is a meromorphic function F on $\tilde{\Omega}$ with a pole only at c such that about c

$$F(w) = \frac{\gamma_\nu}{w^\nu} + \cdots + \frac{\gamma_1}{w} + \text{holomorphic term}. \tag{2.8}$$

Therefore the restriction $F|_\Omega$ of F to Ω is holomorphic and $\lim_{x \to c} |F(x)| = \infty$. Thus we see that Ω is holomorphically convex.

(b) We show the holomorphic separation property of Ω (Definition 1.5 (ii)). Let $a, b \in \Omega$ be any distinct points. Let F be the one obtained in (a) above. If $F(a) \neq F(b)$, then it is done. Suppose that $F(a) = F(b)$. We may assume that $F(a) = F(b) = 0$. Let (U_0, z) be a local holomorphic coordinate system about a with $z(a) = 0$. Then we have

$$F(z) = a_{k_0} z^{k_0} + \text{higher order terms}, \quad a_{k_0} \neq 0,\ k_0 \in \mathbf{N}, \tag{2.9}$$

where $\mathbf{N}$ denotes the set of positive integers. We define Cousin I distributions by

$$\begin{array}{ccl} \dfrac{1}{z^{kk_0}} & \text{on} & U_0,\ k \in \mathbf{N}, \\ 0 & \text{on} & U_1 = \Omega \setminus \{a\}, \end{array}$$

which lead cohomology classes

$$\left[\frac{1}{z^{kk_0}}\right] \in H^1(\{U_0, U_1\}, \mathcal{O}_\Omega) \hookrightarrow H^1(\Omega, \mathcal{O}_\Omega), \quad k = 1, 2, \ldots. \tag{2.10}$$

It follows from (1.8) that there is a non-trivial linear relation

$$\sum_{k=1}^{\mu} \alpha_k \left[\frac{1}{z^{kk_0}}\right] = 0, \quad \alpha_k \in \mathbf{C},\ \alpha_\mu \neq 0.$$

It follows that there is a meromorphic function G on Ω with a pole only at a, where G is written as

$$G(z) = \frac{\alpha_\mu}{z^{\mu k_0}} + \frac{\alpha_{\mu-1}}{z^{(\mu-1)k_0}} + \cdots + \frac{\alpha_1}{z^{k_0}} + \text{holomorphic term.} \tag{2.11}$$

With $g = G \cdot F^\mu$ we have $g \in \mathcal{O}(\Omega)$ and by (2.9) and (2.11) we see that

$$g(a) = \alpha_\mu a_{k_0}^\mu \neq 0, \quad g(b) = 0.$$

(c) Let $a \in \Omega$ be any point. We show the existence of an element $h \in \mathcal{O}(\Omega)$ with non vanishing differential $dh(a) \neq 0$ (Definition 1.5 (iii)). Let (U_0, z) be a holomorphic local coordinate system about a with $z(a) = 0$. As in (2.10) we consider

$$\left[\frac{1}{z^{kk_0-1}}\right] \in H^1(\{U_0, U_1\}, \mathcal{O}_\Omega) \hookrightarrow H^1(\Omega, \mathcal{O}_\Omega), \quad k = 1, 2, \ldots. \tag{2.12}$$

In the same as above we deduce that there is a meromorphic function H on Ω with a pole only at a, where H is written as

$$H(z) = \frac{\beta_\lambda}{z^{\lambda k_0 - 1}} + \cdots + \frac{\beta_1}{z^{k_0-1}} + \text{holomorphic term}, \quad \beta_k \in \mathbf{C},\ \beta_\lambda \neq 0,\ \lambda \in \mathbf{N}. \tag{2.13}$$

With $h = H \cdot F^\lambda$ we have $h \in \mathcal{O}(\Omega)$ and by (2.9) and (2.13) we get

$$\frac{dh}{dz}(a) = \beta_\lambda a_{k_0}^\lambda \neq 0.$$

Thus, Ω is Stein. $\square$

2.3.2 Proof of Lemma 1.8

We take a domain $\tilde{\Omega} \Subset X$ with $\tilde{\Omega} \Supset \Omega$. By Lemma 1.6, $\tilde{\Omega}$ is Stein, and hence there is a holomorphic 1-form on $\tilde{\Omega}$ without zeros. Then we define $\rho(a, \Omega)$ as in (1.3) with $X = \tilde{\Omega}$. With this $\rho(a, \Omega)$ we have by (1.7):

Lemma 2.4. *For a compact subset $K \Subset \Omega$ we get*

$$\rho(K, \Omega) = \rho(\hat{K}_\Omega, \Omega).$$

Lemma 2.5. *Let Ω' be a domain such that $\Omega \Subset \Omega' \Subset \tilde{\Omega}$. Assume that*

$$\max_{b \in \partial\Omega} \rho(b, \Omega') < \rho(K, \Omega). \tag{2.14}$$

Then,

$$\hat{K}_{\Omega'} \cap \Omega \Subset \Omega.$$

Proof. Since $\hat{K}_{\Omega'}$ is compact in Ω' by Lemma 1.6, it suffices to show that

$$\hat{K}_{\Omega'} \cap \partial\Omega = \emptyset.$$

Suppose that there is a point $b \in \hat{K}_{\Omega'} \cap \partial\Omega$. It follows from Lemma 2.4 that

$$\rho(b, \Omega') \geq \rho(\hat{K}_{\Omega'}, \Omega') = \rho(K, \Omega') \geq \rho(K, \Omega).$$

By the assumption, $\rho(b, \Omega') < \rho(K, \Omega)$; this is absurd. □

Proof of Lemma 1.8: Here we use Oka's Jôku-Ikô. By Lemma 1.8 there are holomorphic functions $\psi_j \in \mathcal{O}(\Omega')$ such that a finite union P, called an analytic polyhedron, of relatively compact connected components of

$$\{x \in \Omega' : |\psi_j(x)| < 1\}$$

satisfies "$\hat{K}_{\Omega'} \cap \Omega \Subset P \Subset \Omega$" and the Oka map

$$\Psi : x \in P \longrightarrow (\psi_1(x), \ldots, \psi_N(x)) \in \mathrm{P}\Delta_N$$

is a closed embedding into the N-dimensional unit polydisk $\mathrm{P}\Delta_N$.

Let $f \in \mathcal{O}(\Omega)$. We identify P with the image $\Psi(P) \subset \mathrm{P}\Delta_N$ and regard $f|_P$ as a holomorphic function on $\Psi(P)$. Let $\mathscr{I}$ denote the geometric ideal sheaf of the analytic subset $\Psi(P) \subset \mathrm{P}\Delta_N$. Then we have a short exact sequence of coherent sheaves:

$$0 \to \mathscr{I} \to \mathcal{O}_{\mathrm{P}\Delta_N} \to \mathcal{O}_{\mathrm{P}\Delta_N}/\mathscr{I} \to 0.$$

By Oka's Fundamental Lemma, $H^1(\mathrm{P}\Delta_N, \mathscr{I}) = 0$ (cf., e.g., [20], §4.3), which implies the surjection

$$H^0(\mathrm{P}\Delta_N, \mathcal{O}_{\mathrm{P}\Delta_N}) \to H^0(\mathrm{P}\Delta_N, \mathcal{O}_{\mathrm{P}\Delta_N}/\mathscr{I}) \cong \mathcal{O}(P) \to 0. \tag{2.15}$$

Since $f|_P \in \mathcal{O}(P)$, there is an element $F \in \mathcal{O}(\mathrm{P}\Delta_N)$ with $F|_P = f|_P$. We then expand F to a power series

$$F(w_1, \ldots, w_N) = \sum_{\nu} c_\nu w^\nu, \quad w \in \mathrm{P}\Delta_N,$$

where ν denote multi-indices in $\{1, \ldots, N\}$. For every $\epsilon > 0$ there is a number $l \in \mathbf{N}$ such that

$$\left| F(w) - \sum_{|\nu| \leq l} c_\nu w^\nu \right| < \epsilon, \quad w \in \Psi(K).$$

Substituting $w_j = \psi_j$, we have that

$$g(x) = \sum_{|\nu| \leq l} c_\nu \Psi^\nu(x) \in \mathcal{O}(\Omega'),$$

$$|f(x) - g(x)| < \epsilon, \quad \forall x \in K.$$

□

2.3.3 Proof of Theorem 1.9

We take a continuous exhaustion family $\{\Omega_t\}_{0\le t\le 1}$ of subdomains of $\tilde{\Omega}$ (cf. Definition 1.16) with $\Omega_0 = \Omega$. Let $K \Subset \Omega$ be a compact subset and let $f \in \mathcal{O}(\Omega)$. We set

$$T = \{t : 0 < t \le 1,\ \mathcal{O}(\Omega_t)|K \text{ is dense in } \mathcal{O}(\Omega)|K\},$$

where "dense" is taken in the sense of the maximum norm on K. Note that

(i) $\rho(a, \Omega_t)$ is continuous in t;

(ii) $\rho(K, \Omega) \le \rho(K, \Omega_s) < \rho(K, \Omega_t)$ for $s < t$;

(iii) $\lim_{t\searrow s} \max_{b\in\partial\Omega_s} \rho(b, \Omega_t) = 0$.

It follows from Lemma 1.8 that T is non-empty, open and closed. Therefore, $T \ni 1$, so that $\mathcal{O}(\tilde{\Omega})|K$ is dense in $\mathcal{O}(\Omega)|K$. □

2.3.4 Proof of Theorem 1.10

We owe the second countability axiom for the Riemann surface X to T. Radó. We take an increasing sequence of relatively compact domains $\Omega_j \Subset \Omega_{j+1} \Subset X$, $j \in \mathbf{N}$, such that $X = \bigcup_{j=1}^{\infty} \Omega_j$ and no connected component of $\Omega_{j+1} \setminus \bar{\Omega}_j$ is relatively compact in Ω_{j+1}. Then, (Ω_j, Ω_{j+1}) forms a so-called Rung pair (Theorem 1.9). Since every Ω_j is Stein (Lemma 1.6), the Steinness of X is deduced. □

2.4 Proofs for Riemann domains

2.4.1 Proof of Theorem 1.12

(i) Suppose that $\Omega(\Subset X)$ is a domain of holomorphy. It follows from the assumption and Corollary 1.4 that Ω is K-complete in the sense of Grauert and holomorphically convex. Thus, by Grauert's Theorem ([11]), Ω is Stein.

(ii) Let $Z = \{\det d\pi = 0\}$. Then, Z is a thin analytic subset of X. We first take a Stein subdomain $\Omega \Subset X$ and show the plurisubharmonicity of $-\log \rho(a, \Omega)$. By Grauert-Remmert [12] it suffices to show that $-\log \rho(a, \Omega)$ is plurisubharmonic in $\Omega \setminus Z$. Take an arbitrary point $a \in \Omega \setminus Z$, and a complex affine line $\Lambda \subset \mathbf{C}^n$ passing through $\pi(a)$. Let $\tilde{\Lambda}$ be the irreducible component of $\pi^{-1}\Lambda \cap \Omega$ containing a. Let Δ be a small disk about $\pi(a)$ such that $\tilde{\Delta} := \pi^{-1}\Delta \cap \tilde{\Lambda} \Subset \tilde{\Lambda} \setminus Z$.

Claim. The restriction $-\log \rho(x, \Omega)|_{\tilde{\Lambda}\setminus Z}$ is subharmonic.

By a standard argument (cf., e.g., [14], Proof of Theorem 2.6.7) it suffices to prove that if a holomorphic function $g \in \mathcal{O}(\tilde{\Lambda})$ satisfies

$$-\log \rho(x, \Omega) \le \Re g(x), \quad x \in \partial\tilde{\Delta},$$

then

$$-\log \rho(x, \Omega) \le \Re g(x), \quad x \in \tilde{\Delta}, \tag{2.16}$$

where $\Re$ denotes the real part. Now, we have that

$$\rho(x, \Omega) \geq |e^{g(x)}|, \quad x \in \partial \tilde{\Delta}.$$

Since Ω is Stein, there is a holomorphic function $f \in \mathcal{O}(\Omega)$ with $f|_{\tilde{\Lambda}} = g$ (cf. the arguments for (2.15)). Then,

$$\rho(x, \Omega) \geq |e^{f(x)}|, \quad x \in \partial \tilde{\Delta}.$$

Since $\widehat{\bar{\tilde{\Delta}}}_{\Omega} = \bar{\tilde{\Delta}}$, it follows from (1.6) that

$$\rho(x, \Omega) \geq |e^{f(x)}| = |e^{g(x)}|, \quad x \in \tilde{\Delta},$$

so that (2.16) follows.

Let $\{\Omega_\nu\}_{n=1}^{\infty}$ be a sequence of Stein domains of X such that $\Omega_\nu \Subset \Omega_{\nu+1}$ for all ν and $X = \bigcup_\nu \Omega_\nu$. Then, $-\log \rho(a, \Omega_\nu)$, $\nu = 1, 2, \ldots$, are plurisubharmonic and monotone decreasingly converges to $-\log \rho(a, X)$. Therefore, $-\log \rho(a, X)$ is either identically $-\infty$, or plurisubharmonic ($\not\equiv -\infty$). If $-\log \rho(a, X) \not\equiv -\infty$, it is everywhere finitely valued and continuous by (2.1). □

Corollary 2.6. *Let X be a Stein manifold satisfying Cond A. Then, $-\log \rho(a, X)$ is either identically $-\infty$ or continuous plurisubharmonic.*

Proof. Since X is Stein, there is a holomorphic map $\pi : X \to \mathbf{C}^n$ which forms a Riemann domain. The assertion is immediate from (ii) above. □

Remark 2.7. As a consequence, one sees with the notation in Corollary 2.6 that *if $\Omega \subset X$ is a domain of holomorphy, then Hartogs' radius $\rho_n(a, \Omega)$ (cf. Remark 2.1 (ii)) is plurisubharmonic.* This is, however, opposite to the history: The plurisubharmonicity or the pseudoconvexity of Hartogs' radius $\rho_n(a, \Omega)$ was found first through the study of the maximal convergence domain of a power series (Hartogs' series) in several complex variables (cf. Oka [22], VI [23], IX [26], Nishino [19], Chap. I, Fritzsche–Grauert [9], Chap. II).

Remark 2.8. We here give a proof of Theorem 1.15 under Cond A by making use of $\rho(a, \Omega)$. Since ω is defined in a neighborhood of $\bar{\Omega}$, Cond B is satisfied at every point of the boundary $\partial\Omega$; that is, for every $b \in \partial\Omega$ there are neighborhoods $U' \Subset U \Subset X$ of b such that

$$\rho(a, \Omega) = \rho(a, U \cap \Omega), \quad a \in U'.$$

If $U \cap \Omega$ is Stein, then $-\log \rho(a, \Omega)$ is plurisubharmonic in $a \in U'$ by Theorem 1.12 (iii). Therefore there is a neighborhood V of $\partial\Omega$ in X such that $-\log \rho(a, \Omega)$ is plurisubharmonic in $a \in V \cap \Omega$. Take a real constant C such that

$$-\log \rho(a, \Omega) < C, \quad a \in \Omega \setminus V.$$

Set

$$\psi(a) = \max\{-\log \rho(a, \Omega), C\}, \quad a \in \Omega.$$

Then, ψ is a continuous plurisubharmonic exhaustion function on Ω. By Theorem 2.10 of Andreotti–Narasimhan below, Ω is Stein. $\square$

2.4.2 Proof of Theorem 1.17

In the same way as Lemma 1.8 and its proof we have

Lemma 2.9. *Let $\pi : \tilde{\Omega} \to \mathbf{C}^n$ be a Riemann domain such that $\tilde{\Omega}$ satisfies Cond A. Let $\Omega \Subset \Omega'$ be relatively compact subdomains of $\tilde{\Omega}$ satisfying* (1.9)*: Then, every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on K by elements of $\mathcal{O}(\Omega')$.*

For the proof of the theorem it suffices to show that (Ω_t, Ω_s) is a Runge pair for $0 \le t < s < 1$. Since any fixed $\Omega_{s'}$ $(s < s' < 1)$ satisfies Cond A, we have the scalar $\rho(a, \Omega_s)$. Take a compact subset $K \Subset \Omega_t$. Then, for $s > t$ sufficiently close to t we have

$$\max_{b \in \partial\Omega_t} \rho(b, \Omega_s) < \rho(K, \Omega_t).$$

It follows from Lemma 2.9 that $\mathcal{O}(\Omega_s)|_K$ is dense if $\mathcal{O}(\Omega_t)|_K$. Then, the rest of the proof is the same as in §2.3.3. $\square$

2.4.3 Proof of Theorem 1.19

Here we will use the following result:

Theorem 2.10 (Andreotti–Narasimhan [1])**.** *Let $\pi : X \to \mathbf{C}^n$ be a Riemann domain. If X admits a continuous plurisubharmonic exhaustion function, then X is Stein.*

Let $z \in \Gamma$, $(z \in)V \Subset W$ and $\tilde{V} \subset \widetilde{W}$ be as in Cond B. Then,

$$\rho(a, X) = \rho(a, \widetilde{W}), \quad a \in \widetilde{V}. \tag{2.17}$$

By the assumption, $\widetilde{W}$ can be chosen to be Stein. By Theorem 1.12 (ii), $-\log \rho(a, \widetilde{W})$ is plurisubharmonic in $a \in \widetilde{V}$, and hence so is $-\log \rho(a, X)$ in $\widetilde{V}$. By covering Γ by those $V \Subset W$ and making use of Cond B (i), there is a closed subset $F \subset X$ such that

(i) $F \cap \{x \in X : \|\pi(x)\| \le R\}$ is compact for every $R > 0$,

(ii) $-\log \rho(a, X)$ is plurisubharmonic in $a \in X \setminus F$,

(iii) $\lim_{\nu \to \infty} -\log \rho(a_\nu, X) = \infty$ for every sequence $\{a_\nu\}$ of points of X with no accumulation point in X such that $\{\pi(a_\nu)\}$ is convergent in $\mathbf{C}^n$.

From this we may construct a continuous plurisubharmonic exhaustion function on X as follows:

We fix a point $a_0 \in F$, and may assume that $\pi(a_0) = 0$. Let X_ν be a connected component of $\{\|\pi\| < \nu\}$ containing a_0. Then, $\bigcup_\nu X_\nu = X$. Put

$$\Omega_\nu = X_\nu \setminus F \Subset X.$$

Take a real constant C_1 such that

$$-\log \rho(a, X) < C_1, \quad a \in \bar{\Omega}_1.$$

Then we set

$$\psi_1(a) = \max\{-\log \rho(a, X), C_1\}, \quad a \in X.$$

Then, ψ_1 is plurisubharmonic in X_1. We take a positive constant C_2 such that

$$-\log \rho(a, X) < C_1 + C_2(\|\pi(a)\|^2 - 1)^+, \quad a \in \bar{\Omega}_2,$$

where $(\cdot)^+ = \max\{\cdot, 0\}$. Put

$$p_2(a) = C_1 + C_2(\|\pi(a)\|^2 - 1)^+,$$
$$\psi_2(a) = \max\{-\log \rho(a, X), p_2(a)\}, \quad a \in X.$$

Then, we have:

(i) $p_2(a) \geq C_1 + 2C_2$ in $\{\|\pi\| \geq 2\}$;

(ii) $\psi_1(a) = \psi_2(a)$ in $a \in X_1$;

(iii) $\psi_2(a)$ is plurisubharmonic in X_2.

Similarly, we take $C_3 > C_2$ so that

$$-\log \rho(a, X) < p_2(a) + C_3(\|\pi(a)\|^2 - 2^2)^+, \quad a \in \bar{\Omega}_3.$$

Put

$$p_3(a) = p_2(a) + C_3(\|\pi(a)\|^2 - 2^2)^+,$$
$$\psi_3(a) = \max\{-\log \rho(a, X), p_3(a)\}, \quad a \in X.$$

We then obtain:

(i) $p_3(a) \geq C_1 + 3C_2 + 5C_3$ in $\{\|\pi\| \geq 3\}$;

(ii) $\psi_3(a) = \psi_2(a)$ in $a \in X_2$;

(iii) $\psi_3(a)$ is plurisubharmonic in X_3.

Inductively, we may take a continuous function $\psi_\nu(a)$, $\nu = 1, 2, \ldots$, such that ψ_ν is plurisubharmonic in X_ν and $\psi_{\nu+1}|_{X_\nu} = \psi_\nu|_{X_\nu}$. it is clear from the construction that

$$\psi(a) = \lim_{\nu \to \infty} \psi_\nu(a), \quad a \in X,$$

is a continuous plurisubharmonic exhaustion function of X.

Finally, by Theorem 2.10 of Andreotti–Narasimhan we see that X is Stein.

3 Examples and some more on $\rho(a, X)$

(a) (Fornæss' example). Fornæss [7] constructed a 2-sheeted ramified Riemann domain $\pi : M \to \mathbf{C}^2$ such that it is locally Stein, M is exhausted by an increasing sequence of relatively compact Stein subdomains, but M is *not* Stein. We here show that the holomorphic cotangent bundle $\mathbf{T}(M)^*$ does not carry a global frame, so that M *does not satisfy Cond A*.

For convenience, we use the same notation as in [7]. Assume that there exists a global frame $\{\lambda_1, \lambda_2\}$. With the coordinates (z, w) we write in a neighborhood $U = \{(z, w) : |z| < \delta, 1 - \delta < |w| < 1 + \delta\}$ ($\delta > 0$, sufficiently small) of $z = 0, |w| = 1$:

$$\lambda_1 = f(z, w)dz + g(z, w)dw,$$
$$\lambda_2 = h(z, w)dz + k(z, w)dw.$$

Then, we have

$$\lambda_1 \wedge \lambda_2 = (fk - gh)dz \wedge dw.$$

By the assumption, $fk - gh$ has no zero. Put

$$\nu_0 = \frac{1}{2\pi i}\int_{|w|=1} d\log\big(f(z, w)k(z, w) - g(z, w)h(z, w)\big) \in \mathbf{Z}, \quad |z| < \delta.$$

Then we may write

$$f(z, w)k(z, w) - g(z, w)h(z, w) = e^{A(z,w)}w^{\nu_0},$$

where $A(z, w)$ is a holomorphic function of z and w. We consider the analytic continuations of the above holomorphic functions as far as possible. In a neighborhood of $(1/n, w)$ with every sufficiently large natural number n and sufficiently small $|w|$ we have another chart,

$$z = \frac{1}{n} + C_n\eta w^{m_n} + \epsilon_n\eta^2, \quad w = w.$$

Then it follows that

$$\begin{aligned} \lambda_1 \wedge \lambda_2 &= (fk - gh)dz \wedge dw \\ &= e^{A(z,w)}w^{\nu_0} d\left(\frac{1}{n} + C_n\eta w^{m_n} + \epsilon_n\eta^2\right) \wedge dw \\ &= e^{A(z,w)}w^{\nu_0}(C_n w^{m_n} + 2\epsilon_n\eta)d\eta \wedge dw. \end{aligned} \tag{3.1}$$

Since at $z = 1/n$ and $\eta = 0$ the coefficient function of (3.1) should be holomorphic and should have no zero, we deduce that $m_n = -\nu_0$. But, $m_n \to \infty$ as $n \to \infty$: This is a contradiction.

(b) (Grauert's example). Grauert [18] gave a counter-example to the Levi problem for ramified Riemann domains over $\mathbf{P}^n(\mathbf{C})$: There is a locally Stein domain Ω in a complex torus M such that $\mathcal{O}(\Omega) = \mathbf{C}$. Then, M satisfies Cond A. One may assume that M is projective algebraic, so that there is a holomorphic finite map $\tilde{\pi} : M \to \mathbf{P}^n(\mathbf{C})$, which is a Riemann domain over $\mathbf{P}^n(\mathbf{C})$. Then, the restriction $\pi = \tilde{\pi}|_\Omega : \Omega \to \mathbf{P}^n(\mathbf{C})$ is a Riemann domain over $\mathbf{P}^n(\mathbf{C})$, which satisfies Cond A and Cond B. Therefore, Theorem 1.19 cannot be extended to a Riemann domain over $\mathbf{P}^n(\mathbf{C})$.

Remark 3.1. Let $\pi : \Omega \to \mathbf{P}^n(\mathbf{C})$ be Grauert's example as above. Let $\mathbf{C}^n$ be an affine open subset of $\mathbf{P}^n(\mathbf{C})$, and let $\pi' : \Omega' \to \mathbf{C}^n$ be the restriction of $\pi : \Omega \to \mathbf{P}^n(\mathbf{C})$ to $\mathbf{C}^n$. Then, Ω' *is Stein by Theorem 1.19.*

The Steinness of Ω' may be not inferred by a formal combination of the known results on pseudoconvexity, since it is an unbounded domain (cf., e.g., [18], [17]).

(c) Domains in the products of open Riemann surfaces and complex semi-tori (cf. [21], Chap. 5) serve for examples satisfying Cond A.

(d) An open Riemann surface X is *not* Kobayashi hyperbolic if and only if X is biholomorphic to $\mathbf{C}$ or $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ (For the Kobayashi hyperbolicity in general, cf. [15], [21]).

(d1) Let $X = \mathbf{C}$. If $\omega = dz$, then $\rho(a, dz, \mathbf{C}) \equiv \infty$ for every $a \in \mathbf{C}$. If $\omega = e^z dz$, then a simple calculation implies that

$$\rho(a, e^z dz, \mathbf{C}) = |e^a|.$$

(d2) Let $X = \mathbf{C}^*$. If $\omega = z^k dz$ with $k \in \mathbf{Z} \setminus \{-1\}$, then

$$\rho(a, z^k dz, \mathbf{C}^*) = \left| \frac{1}{k+1} a^{k+1} \right|.$$

Therefore, $\lim_{a \to 0} \rho(a, z^k dz, \mathbf{C}^*) = 0$ for $k \geq 0$, and $\lim_{a \to \infty} \rho(a, z^k dz, \mathbf{C}^*) = 0$ for $k \leq -2$. If $\omega = \frac{dz}{z}$, then $\rho(a, \frac{dz}{z}, \mathbf{C}^*) \equiv \infty$. It follows that

$$\psi(a) := \max\{-\log \rho(a, dz, \mathbf{C}^*), -\log \rho(a, z^{-2} dz, \mathbf{C}^*)\}$$

is continuous subharmonic in $\mathbf{C}^*$, and $\lim_{a \to 0, \infty} \psi(a) = \infty$.

Thus, the finiteness or the infiniteness of $\rho(a, \omega, X)$ depends on the choice of ω.

(e) For a Kobayashi hyperbolic open Riemann surface X we take a holomorphic 1-form ω without zeros, and write

$$\|\omega(a)\|_X = |\omega(v)|, \quad v \in \mathbf{T}(X)_a, \ F_X(v) = 1.$$

Then it follows from (2.3) that $\rho(a, \omega, X) \leq \|\omega(a)\|_X$. We set

$$\rho^+(a, X) = \sup\{\rho(a, \omega, X) : \omega \text{ hol. 1-form without zeros, } \|\omega(a)\|_X = 1\},$$
$$\rho^-(a, X) = \inf\{\rho(a, \omega, X) : \omega \text{ hol. 1-form without zeros, } \|\omega(a)\|_X = 1\}.$$

Clearly, $\rho^\pm(a, X) (\leq 1)$ are biholomorphic invariants of X, but we do not know the behavior of them.

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