Mathematische Zeitschrift manuscript No.
(will be inserted by the editor)

On Oka's Extra-Zero Problem and Examples

Makoto Abe · **Sachiko Hamano** · **Junjiro Noguchi**

the date of receipt and acceptance should be inserted later

Abstract After the solution of Cousin II problem by K. Oka III in 1939, he thought an extra-zero problem in 1945 (his posthumous paper) asking if it is possible to solve an arbitrarily given Cousin II problem adding some extra-zeros whose support is disjoint from the given one. By the secondly named author, some special case was affirmatively confirmed in dimension two and a counter-example in dimension three or more was given. The purpose of the present paper is to give a complete solution of this problem with examples and some new questions.

Keywords Oka principle · Cousin II problem · Stein space · extra-zero problem

Mathematics Subject Classification (2000) 32E10 · 32C18 · 32L10

1 Introduction

After the solution of Cousin II problem by K. Oka [10, III] he thought the following *extra-zero problem* in 1945 (his posthumous paper [11], no. 2, p. 31, Problem 2; see §2).

Research supported in part by Grants-in-Aid for Scientific Research (C) 23540217, for Young Scientists (B) 23740098 and for Scientific Research (B) 23340029 of Japan Society for the Promotion of Science.

Makoto Abe
Division of Mathematical and Information Sciences, Faculty of Integrated Arts and Sciences, Hiroshima University, 1-7-1 Kagamiyama, Higashi-Hiroshima, 739-8521, Japan
E-mail: abem@hiroshima-u.ac.jp

Sachiko Hamano
Department of Mathematics, Faculty of Human Development and Culture, Fukushima University, Kanayagawa, Fukushima, 960-1296, Japan
E-mail: hamano@educ.fukushima-u.ac.jp

Junjiro Noguchi
Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo, 153-8914, Japan
E-mail: noguchi@ms.u-tokyo.ac.jp

Oka's Extra-Zero Problem *Let X be a domain of holomorphy and let D be an effective divisor on X. Find an effective divisor E on X such that their supports have no intersection,*

$$(\mathrm{Supp}\, D) \cap (\mathrm{Supp}\, E) = \emptyset,$$

and Cousin II *problem for $D+E$ is solvable on X.*

The divisor E in the above problem is called an *extra-zero* of D. Let $L(D)$ denote the line bundle determined by D, let $N(D) = L(D)|_{(\mathrm{Supp}\, D)} \to \mathrm{Supp}\, D$ be the normal bundle of D over the support $\mathrm{Supp}\, D$ of D, and let $\mathbf{1}_X$ denote the trivial line bundle over X. Then Cousin II problem is equivalent to ask if $L(D) \cong \mathbf{1}_X$. The *Oka Principle* [10, III] says that $L(D) \cong \mathbf{1}_X$ if and only if the the first Chern class $c_1(L(D)) = 0$ in the cohomology group $H^2(X, \mathbf{Z})$. Since the problem is trivial for D such that $L(D) \cong \mathbf{1}_X$, Oka's extra-zero problem makes sense for D with $c_1(L(D)) \neq 0$. For a general reference of the Oka Principle, cf. Forstnerič [4].

In [6] a counter-example was constructed in $\dim X \geq 3$, and if $\dim X = 2$, some partial affirmative answer was shown.

The purpose of this paper is to give a complete answer to Oka's extra-zero problem with examples and some new questions based on this problem, on which we would like to put equal emphasis as well (see §§4 and 5). It is also a point of this paper to have the analytic expressions of some topological invariants of Stein manifolds (cf. Stein [12]). In the general case we have

Theorem 1.1 *Let D be an effective Cartier divisor on a (reduced) Stein space X. Then Oka's extra-zero problem is solvable if and only if $c_1(N(D)) = 0$ in $H^2(\mathrm{Supp}\, D, \mathbf{Z})$. In particular, if $\dim X = 2$, Oka's extra-zero problem is always solvable.*

The last statement is due to $H^2(\mathrm{Supp}\, D, \mathbf{Z}) = 0$, since $\dim \mathrm{Supp}\, D \leq 1$ (cf. [7]).

N.B.

1. K. Oka [11] almost proved Theorem 1.1 (see Theorem 2.1). Referring to Oka's Theorem 2.1, one may say that Theorem 1.1 is an infinitesimalization of the topological condition from a neighborhood of D to D itself. This is not difficult now by many well-established results.
2. By the proof in §3 it is in fact not necessary to assume D to be effective; even in this case, the extra-zero E is kept to be effective.

Let E be an extra-zero of D in Oka's extra-zero problem. By definition $L(E) = L(-D)$. Thus the problem is equivalent to find a holomorphic section $\sigma \in \Gamma(X, L(-D))$ such that

$$\mathrm{Supp}(\sigma) \cap \mathrm{Supp}\, D = \emptyset. \tag{1}$$

Here we consider only σ whose zero set is nowhere dense in X and hence defines a divisor (σ) on X. From this viewpoint it is interesting to see

Proposition 1.2 *Let the notation be as in Theorem* 1.1. *Then Oka's extra-zero problem is solvable if and only if there exists a section $\tau \in \Gamma(X, L(D))$ with nowhere dense zero set and*

$$\mathrm{Supp}(\tau) \cap \mathrm{Supp}\, D = \emptyset. \tag{2}$$

N.B. For τ in (2) (resp. σ in (1)) we required that the zero set of τ (resp. σ) is nowhere dense in X. This is, however, not a restriction. For if τ vanishes constantly on an irreducible component of X, we let X'_ν ($\nu = 1, 2, \ldots$) be all such components. Then we take a section $\tau'_\nu \in \Gamma(X, L(D))$ such that $\tau'_\nu|_{X'_\nu} \not\equiv 0$ and $\tau'_\nu \equiv 0$ on every irreducible component of X other than X'_ν. We set $\hat{\tau} = \tau + \sum_\nu \tau'_\nu$. Then $\{\hat{\tau} = 0\} \subset \{\tau = 0\}$ as sets and the analytic subset $\{\hat{\tau} = 0\}$ is nowhere dense in X. This is the same for σ in (1).

Acknowledgements After the counter-example constructed by [6] which was a reducible divisor, Professor T. Ueda asked if there is an irreducible counter-example; his question forms a part of the motivation of the present paper. Professor S. Takayama gave an interesting example of §4. The authors are very grateful to all of them.

2 Oka's notes

Here we summarize in short the contents of the posthumous paper [11]. We should first notice that it is dated 28 February 1945 before Oka's Coherence Theorem [10, VII]. Roughly speaking, he developed the following study.

1. He wished to reformulate Cousin II problem by relaxing the conclusion so that it is solvable on every domain of holomorphy.
2. He recalled the Oka Principle for Cousin II problem on a domain of holomorphy, and reduced the essential key-part of the problem to the following:

   Let $\bar{\Omega} \Subset \mathbf{C}^n$ be a bounded closed domain with a fundamental system of holomorphically convex neighborhoods. Let D be a divisor on a neighborhood of $\bar{\Omega}$. Then the Cousin II problem for D is solvable in a neighborhood of $\bar{\Omega}$ if and only if $c_1(L(D)) = 0$ in a neighborhood of $\bar{\Omega}$.[1]

3. He then proposed the *Extra-Zero Problem* as Problem 2. Let Ω and D (effective) be as in the above item. Then he asks to find an effective divisor E in a neighborhood of $\bar{\Omega}$ such that $\operatorname{Supp} D \cap \operatorname{Supp} E = \emptyset$ and Cousin II problem for $D + E$ is solvable in a neighborhood of $\bar{\Omega}$.
4. He proved a result as Theorem 8 which is stated as follows:

   The extra-zero problem is solvable for D in a neighborhood of $\bar{\Omega}$ if and only if there is a neighborhood V of $D \cap \bar{\Omega}$ with $c_1(L(D)|_V) = 0$.

5. After confirming the above topological obstruction for the extra-zero problem, he proved that there always exists an effective divisor F in a neighborhood of $\bar{\Omega}$ such that Cousin II problem for $D + F$ is solvable. Furthermore he proved that there are at most $n + 1$ holomorphic functions f_j, $1 \leq j \leq n + 1$, in a neighborhood of $\bar{\Omega}$ such that in a neighborhood W of every point of $D \cap \bar{\Omega}$ one of zeros of f_j is exactly $D \cap W$.

Taking into account of the items 2 and 4 above, we may assume that he obtained or at least recognized the following statement.

Theorem 2.1 **(**Oka [11]) *Let $\Omega \subset \mathbf{C}^n$ be a domain of holomorphy, and let D be an effective divisor on Ω. Then the extra-zero problem for D is solvable if and only if there is a neighborhood V of D satisfying $c_1(L(D)|_V) = 0$ in V.*

[1] Here his term is "*balayable*" used in Oka [10, III]; the meaning is that the given Cousin II distribution is continuously deformable to a zero-free continuous Cousin II distribution. The Cousin II problem on a domain X of holomorphy is solvable if and only if D is *balayable* on X.

K. Oka wrote that it strongly attracts his interest from a number of viewpoints to decide if this Extra-Zero Problem is always solvable or there is a counter-example, and the problem would have a wide influence in future.[2]

It is now necessary to know what is the most general form of his statement (Theorem 2.1), and it is Theorem 1.1.

3 Proofs

(a) Proof of Theorem 1.1

Suppose first that Oka's extra-zero problem is solvable. Let E be an extra-zero of D, and let $\sigma \in \Gamma(X, L(E))$ with $(\sigma) = E$. Set $Y = \operatorname{Supp} D$. Then the restriction $\sigma|_U$ to $U = X \setminus \operatorname{Supp} E$ has no zero over the neighborhood U of Y. Therefore $L(-D)|_U = L(E)|_U \cong \mathbf{1}_U$, and then $N(D) \cong \mathbf{1}_Y$, so that $c_1(N(D)) = 0$.

Conversely, assume that $c_1(N(D)) = 0$. Then, of course, $c_1(N(-D)) = 0$. Since Y is Stein, $N(-D) \cong \mathbf{1}_Y$, so that there exists a holomorphic section $\varphi \in \Gamma(Y, N(-D))$ which has no zero. By the Fundamental Theorem of Oka-Cartan (Oka [10] I–II, VII–VIII; Grauert-Remmert [5]) φ extends to an element $\tilde{\varphi} \in \Gamma(X, L(-D))$ with nowhere dense zero set. Since $\tilde{\varphi}$ has no zero on Y, the divisor $(\tilde{\varphi})$ gives rise to an extra-zero of D.

(b) Proof of Proposition 1.2

We keep the notation used in (a). Suppose that Oka's extra-zero problem is solvable. Then the above $\sigma \in \Gamma(X, L(E))$ has no zero on Y. Therefore, $N(D) = L(D)|_Y = L(-E)|_Y \cong \mathbf{1}_Y$. By the Fundamental Theorem of Oka-Cartan, $\sigma^{-1}|_Y$ extends holomorphically to a section $\tau \in \Gamma(X, L(D))$ with nowhere dense zero set. By definition, $\operatorname{Supp}(\tau) \cap Y = \emptyset$.

Suppose the existence of $\tau \in \Gamma(X, L(D))$ with nowhere dense zero set such that $\operatorname{Supp}(\tau) \cap Y = \emptyset$. Then the same argument implies the existence of $\sigma \in \Gamma(X, L(-D))$ with nowhere dense zero set such that $\operatorname{Supp}(\sigma) \cap Y = \emptyset$, and hence (σ) is an extra-zero of D.

4 Examples

(a) A generalized example due to K. Stein

The case of $X = (\mathbf{C}^*)^2$ with $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ may be the most fundamental for non-trivial $H^2(X, \mathbf{Z}) \neq 0$. In fact, K. Stein [12] studied this case. The torus $T = S^1 \times S^1 \subset X$ gives the generator of $H_2(X, \mathbf{Z}) \cong H^2(X, \mathbf{Z})$. Let $(z, w) \in X$ be the natural coordinates, and let $\tau \in \mathbf{C}$ with $\Im \tau > 0$. Then we take an analytic hypersurface given by

$$D_\tau^+ : \quad w = z^\tau = e^{\tau \log z}. \tag{3}$$

We are going to show that D_τ^+ has the first Chern class T. The case of $\tau = i$ is the example of Stein [12]. We set

$$F_\tau^+(z, w) = \exp\left(\frac{\tau}{4\pi i}(\log z)^2 + \frac{\tau + 1}{2} \log z\right) \tag{4}$$

[2] He did not give an explicit problem here.

$$\times \prod_{k=0}^{\infty}\left(1-\frac{w}{e^{\tau \log z - 2k\pi i\tau}}\right)\times \prod_{k=1}^{\infty}\left(1-\frac{1}{we^{-\tau \log z - 2k\pi i\tau}}\right),$$

where we take a branch $\log 1 = 0$. Let $\mathscr{L}_z$ (resp. $\mathscr{L}_w$) denote the analytic continuation as the variable z (resp. w) runs over the unit circle in the anti-clockwise direction. Then $\mathscr{L}_z \log z = \log z + 2\pi i$, and

$$\mathscr{L}_z F^+_\tau(z,w) = wF^+_\tau(z,w), \quad \mathscr{L}_w F^+_\tau(z,w) = F^+_\tau(z,w). \tag{5}$$

Thus we have that

$$D^+_\tau = \{F^+_\tau(z,w) = 0\}.$$

It follows from (5) that

$$\frac{|F^+_\tau(z,w)|}{|w|^{\frac{1}{2\pi}\arg z}} \tag{6}$$

is one-valued on X. Let $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ be the polar coordinates. Then

$$d = \sum_{j=1}^{2}\left(\frac{\partial}{\partial r_j} dr_j + \frac{\partial}{\partial \theta_j} d\theta_j\right),$$
$$d^c = \frac{i}{4\pi}(\bar{\partial} - \partial) = \frac{1}{4\pi}\sum_{j=1}^{2}\left(r_j \frac{\partial}{\partial r_j} d\theta_j - \frac{1}{r_j}\frac{\partial}{\partial \theta_j} dr_j\right).$$

By (6) we see that $|w|^{\frac{1}{\pi}\arg z} = r_2^{\theta_1/\pi}$ gives a hermitian metric in $L(D^+_\tau)$, and we compute the Chern form:

$$dd^c \log |w|^{\frac{1}{\pi}\arg z} = d\left\{\frac{1}{4\pi}\sum_{j=1}^{2}\left(r_j \frac{\partial}{\partial r_j} d\theta_j - \frac{1}{r_j}\frac{\partial}{\partial \theta_j} dr_j\right)\frac{\theta_1}{\pi}\log r_2\right\}$$
$$= \frac{1}{4\pi^2}\left(d\log r_1 \wedge d\log r_2 + d\theta_1 \wedge d\theta_2\right).$$

The above first term is d-exact, since

$$d\log r_1 \wedge d\log r_2 = d((\log r_1)\cdot d\log r_2).$$

Hence the Chern class $c_1(L(D^+_\tau)) \in H^2(X,\mathbf{Z})$ is represented by $\dfrac{1}{4\pi^2} d\theta_1 \wedge d\theta_2$. By pairing we have

$$\langle c_1(L(D^+_\tau)), T\rangle = \int_{S^1\times S^1} \frac{1}{4\pi^2} d\theta_1 \wedge d\theta_2 = 1.$$

In the same way, taking $\tau' \in \mathbf{C}$ with $\Im\,\tau' > 0$ we set

$$D^-_{\tau'}: \quad w = z^{-\tau'} = e^{-\tau' \log z},$$
$$F^-_{\tau'}(z,w) = F^+_{\tau'}\left(\frac{1}{z}, w\right).$$

Thus

$$\mathscr{L}_z F^-_{\tau'}(z,w) = \frac{1}{w}F^-_{\tau'}(z,w), \quad \mathscr{L}_w F^-_{\tau'}(z,w) = F^-_{\tau'}(z,w). \tag{7}$$

From (4) one obtains

$$F^-_{\tau'}(z,w) = \exp\left(\frac{\tau'}{4\pi i}(\log z)^2 - \frac{\tau'+1}{2}\log z\right) \tag{8}$$

$$\times \prod_{k=0}^{\infty}\left(1-\frac{w}{e^{-\tau'\log z-2k\pi i\tau'}}\right)\times\prod_{k=1}^{\infty}\left(1-\frac{1}{we^{\tau'\log z-2k\pi i\tau'}}\right).$$

Therefore, $L(D^+_\tau + D^-_{\tau'}) \cong \mathbf{1}_X$, however in this example, $D^+_\tau \cap D^-_{\tau'} \neq \emptyset$; in fact, $D^+_\tau \cap D^-_{\tau'}$ is a countably infinite set.

By Theorem 1.1 there is an extra-zero E of D^+_τ, but it is unknown what is E. Therefore it is very interesting to ask

Question 4.1 Find an analytic expression of E.

On the other hand we may give an example for Proposition 1.2. We let $\lambda \in \mathbf{C}$ and further set

$$D^+_{\lambda,\tau}: \quad w = e^{\lambda} z^{\tau}.$$

Then $D^+_{\lambda,\tau}$ is the image of an embedding

$$\zeta \in \mathbf{C} \to (e^{\zeta}, e^{\lambda}\cdot e^{\tau\zeta}) \in X,$$

and is also given as a zero set of $F^+_\tau\left(z, e^{-\lambda}w\right)$. Thus, $D^+_\tau = D^+_{0,\tau}$ and $L(D^+_\tau) = L(D^+_{\lambda,\tau})$ for all λ. There is a small neighborhood Ω of $0 \in \mathbf{C}$ such that

$$\Phi : (\zeta,\lambda) \in \mathbf{C}\times\Omega \to (e^{\zeta}, e^{\lambda}\cdot e^{\tau\zeta}) \in (\mathbf{C}^*)^2 = X \tag{9}$$

is an into-biholomorphism; in particular,

$$D^+_\tau \cap D^+_{\lambda,\tau} = \emptyset, \quad \lambda \in \Omega\setminus\{0\}.$$

This describes precisely why D^+_τ is *balayable* in a neighborhood of D^+_τ (see §2 and its footnote).

N.B. We do not know a method how to produce the analytic expression $F^+_\tau(z,w)$ of (4) from the Chern class $c_1(L(D^+_\tau)) = T$, and it is an interesting problem to find it.

(b) Examples for Theorem 1.1 with $c_1(N(D)) \neq 0$

(1) (Reducible divisor) A counter example in $\dim X \geq 3$ was first given by [6] in a domain of $\mathbf{C}^n$ ($n \geq 3$). Using a similar idea, we give another counter example of a divisor on $(\mathbf{C}^*)^3$ for which Oka's extra-zero problem has no solution.

Now we let $X = (\mathbf{C}^*)^2 \times \mathbf{C}^* = (\mathbf{C}^*)^3$ with projection $p : X \to (\mathbf{C}^*)^2$. Let $D^+_\tau \subset (\mathbf{C}^*)^2$ be as in the above (a), and set

$$\begin{aligned} &D_1 = D^+_\tau \times \mathbf{C}^*, \quad D_2 = (\mathbf{C}^*)^2 \times \{1\}, \\ &D_3 = D_1 + D_2. \end{aligned} \tag{10}$$

Since $L(D_2) \cong \mathbf{1}_X$, $L(D_3) \cong L(D_1) \cong p^*L(D^+_\tau)$. Therefore $N(D_3)|_{D_2} \cong L(D^+_\tau) \not\cong \mathbf{1}_{D_2}$ with $D_2 \cong (\mathbf{C}^*)^2$, so that $N(D_3) \not\cong \mathbf{1}_{D_3}$. One sees that D_3 has no extra-zero on X.

(2) (Irreducible divisor) The above example of D_3 is reducible, and we like to have an irreducible analytic hypersurface that has no extra-zero. We are going to modify the example (1).

Let $\tau = i$, let $\mathbf{Z}[i] = \mathbf{Z} + i\mathbf{Z}$ be the lattice of Gaussian integers and put

$$\mathbf{C} \to \mathbf{C}/\mathbf{Z}[i] \cong \mathbf{C}^* \xrightarrow{\lambda_0} \mathbf{C}^*/\mathbf{Z} = E \cong \mathbf{C}/\mathbf{Z}[i].$$

Then E is an elliptic curve with complex multiplication $a \in E \to ia \in E$. Set

$$\iota : (a,b) \in E^2 \to (ia,b) \in E^2$$

and let $\Delta \subset E^2$ be the diagonal divisor. Renewing the index numbering, we set

$$\begin{aligned} D_1 &= \iota^*\Delta \subset E^2, \\ \lambda_1 &= \lambda_0 \times \lambda_0 : (\mathbf{C}^*)^2 \to E^2, \\ \hat{D}_1 &= \lambda^* D_1 \subset (\mathbf{C}^*)^2. \end{aligned}$$

Note that the example D_i^+ with $\tau = i$ in the above (a) is a connected component $\hat{D}'_1$ of $\hat{D}_1$. It follows that the Chern class

$$c_1(L(\hat{D}_1)) \neq 0 \quad \text{in } H^2((\mathbf{C}^*)^2, \mathbf{Z}).$$

(This is equivalent to the non-solvability of Cousin II for $\hat{D}_1$, or to the non-triviality of the line bundle $L(\hat{D}_1)$ over $(\mathbf{C}^*)^2$.)

In fact, it follows from (a) of this section that

$$\langle c_1(L(\hat{D}_1)), T \rangle = \langle c_1(L(\hat{D}'_1)), T \rangle = 1. \tag{11}$$

Now we set

$$\begin{aligned} \lambda_2 &: X = (\mathbf{C}^*)^2 \times \mathbf{C}^* \to E^2 \times E \quad (\text{the quotient map}), \\ D_2 &= D_1 \times E + E^2 \times \{0\}, \\ \hat{D}_2 &= \lambda_2^* D_2. \end{aligned}$$

Then $L(\lambda_2^*(E^2 \times \{0\}))$ is trivial on X and so $L(\hat{D}_2) = L(\lambda_2^*(D_1 \times E))$, which is the pull-back of $L(\hat{D}_1)$ over $(\mathbf{C}^*)^2$ by the projection $X \to (\mathbf{C}^*)^2$. Therefore, $L(\hat{D}_2) \not\cong \mathbf{1}_X$. Furthermore, we see that the normal bundle $N(\hat{D}_2) = L(\hat{D}_2)|_{\hat{D}_2} \to \hat{D}_2$ is non-trivial. For $N(\hat{D}_2)|_{(\mathbf{C}^*)^2 \times \{1\}} \cong L(\hat{D}_1)$. Therefore we obtain

Lemma 4.2 *Let the notation be as above. Then $L(\hat{D}_2) \not\cong \mathbf{1}_X$ and $N(\hat{D}_2) \not\cong \mathbf{1}_{\hat{D}_2}$.*

N.B. This means that Cousin II problem for $\hat{D}_2$ on X is not solvable and there is no extra zero for $\hat{D}_2$.

We would like to deform $\hat{D}_2$ to a smooth irreducible divisor, but this is not trivial. Thus we are going to deform D_2 on E^3, but D_2 is not ample. To make it ample, we add the divisor $\{0\} \times E^2$ to D_2 with setting

$$D_3 = D_2 + \{0\} \times E^2,$$

which is then ample, and we put $\hat{D}_3 = \lambda_2^* D_3$ on X. Since $\lambda_2^* L(\{1\} \times E^2) = L(\lambda_2^{-1}\{1\} \times (\mathbf{C}^*)^2) \cong \mathbf{1}_X$, we have that $L(\hat{D}_3) \cong L(\hat{D}_2)$. Thus Lemma 4.2 holds for $\hat{D}_3$, too:

Lemma 4.3 *Let the notation be as above. Then we have that $L(\hat{D}_3) \not\cong \mathbf{1}_X$ and $N(\hat{D}_3) \not\cong \mathbf{1}_{\hat{D}_3}$.*

It is well known that $L(3D_3)$ is very ample. We take a smooth irreducible hyperplane section D_4 by a holomorphic section of $L(3D_3)$, and set

$$\hat{D}_4 = \lambda_2^* D_4.$$

Proposition 4.4 (Example) *Let the notation be as above. Then $\hat{D}_4$ is a smooth irreducible divisor on X such that $L(\hat{D}_4) \not\cong \mathbf{1}_X$ and $N(\hat{D}_4) \not\cong \mathbf{1}_{\hat{D}_4}$; equivalently,*

$$c_1(L(\hat{D}_4)) \neq 0 \quad \text{in } H^2(X, \mathbf{Z}),$$
$$c_1(N(\hat{D}_4)) \neq 0 \quad \text{in } H^2(\hat{D}_4, \mathbf{Z}).$$

Proof It is clear due to the construction that $\hat{D}_4$ is smooth and irreducible (or connected). Now we look at the 2-cycle T in (11). We regard $T = S^1 \times S^1 \times \{1\} \in H_2(X, \mathbf{Z})$. Then this cycle T comes from a 2-cycle of E^3, which is again denoted by the same $T \in H_2(E^3, \mathbf{Z})$. Then it follows that

$$\langle c_1(L(D_4), T \rangle = 3, \tag{12}$$

so that $c_1(L(\hat{D}_4)) \neq 0$.

It remains to show that $c_1(N(\hat{D}_4)) \neq 0$. By Lefschetz' hyperplane-section theorem the natural morphism

$$H_2(D_4, \mathbf{Z}) \to H_2(E^3, \mathbf{Z}) \to 0$$

is surjective, and then there is a 2-cycle $T' \in H_2(D_4, \mathbf{Z})$ which is mapped to T. Then T' can be lifted to a 2-cycle in $H_2(\hat{D}_4, \mathbf{Z})$, denoted by the same T'. We see by (12) that

$$\langle c_1(N(\hat{D}_4)), T' \rangle = 3.$$

Thus $c_1(N(\hat{D}_4)) \neq 0$; this finishes the proof. □

(3) (Takayama's irreducible example) Let $z_j = x_j + iy_j$, $1 \le j \le n$, be the natural complex coordinates of $\mathbf{C}^n$ with the standard basis e_j, $1 \le j \le n$. Then e_j, ie_j, $1 \le j \le n$, form real basis of $\mathbf{C}^n$ and we define a lattice $\Gamma \subset \mathbf{C}^n$ by

$$\Gamma = \langle e_1, \dots, e_n, ie_1, \dots, ie_n \rangle.$$

We set $A = \mathbf{C}^n / \Gamma$ and a sequence of covering maps

$$\mathbf{C}^n \xrightarrow{\rho} (\mathbf{C}^*)^n \xrightarrow{\pi} A, \tag{13}$$

where ρ is the quotient map by $\langle ie_1, e_2, \dots, e_n \rangle$ and π is that by $\langle e_1, ie_2, \dots, ie_n \rangle$. We set $X = (\mathbf{C}^*)^n$.

Let L be the line bundle over A whose Chern class is represented by

$$\omega = \mu i \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j + i \sum_{j \neq k} dz_j \wedge d\overline{z}_k, \quad \mu \in \mathbf{Z}.$$

Then L is ample for $\mu \ge 2$, and very ample if $\mu \ge 4$.

Claim 4.5 *$\pi^* \omega \neq 0$ in $H^2(X, \mathbf{Z})$; in particular, the pairing, $\omega \cdot (ie_1 \wedge e_j) \neq 0$, $j \ge 2$, where $ie_1 \wedge e_j \in H_2(A, \mathbf{Z})$.*

Proof We consider the two pull-back morphisms

$$\pi^* : H^q(A,\mathbf{Z}) \to H^q(X,\mathbf{Z}), \quad q = 1,2.$$

Then $\pi^* dx_1 = 0$ and $\pi^* dy_k = 0$, $k \geq 2$; on the other hand, $\pi^* dy_1 \neq 0$ and $\pi^* dx_k \neq 0$, $k \geq 2$. It follows that

$$i dz_j \wedge d\bar{z}_j = 2dx_j \wedge dy_j = 0, \quad 1 \leq j \leq n \ (\mathrm{mod}\ dx_1, dy_k, k \geq 2).$$

Therefore we have

$$\begin{aligned}\pi^* i(dz_1 \wedge d\bar{z}_j + dz_j \wedge d\bar{z}_1) &= i\pi^*(idy_1 \wedge dx_j + dx_j \wedge (-idy_1)) \\ &= -2\pi^*(dy_1 \wedge dx_j)\end{aligned}$$

for $j \geq 2$. □

Now we assume $n \geq 3$ and $\mu \geq 4$. Then L is very ample.

Proposition 4.6 (Example) *We take a smooth irreducible divisor $D \in |L|$ and set $\tilde{D} = \pi^{-1}D \subset X$. Then the divisor $\tilde{D}$ is smooth irreducible and has no extra-zero on X.*

Proof Since $H_1(D,\mathbf{Z}) \cong H_1(A,\mathbf{Z})$ (Lefschetz' Theorem), $\tilde{D}$ is connected. Again by Lefschetz' Theorem the natural morphism is surjective:

$$H_2(D,\mathbf{Z}) \to H_2(A,\mathbf{Z}) \to 0.$$

There is an element $\xi \in H_2(D,\mathbf{Z})$ which is mapped to $ie_1 \wedge e_j$ ($j \geq 2$). Let $\iota : D \hookrightarrow A$ be the inclusion map and let $\tilde{\iota} : \tilde{D} \to X$ be the lifting. It follows from (13) that there is an element $\tilde{\xi} \in H_2(\tilde{D},\mathbf{Z})$ with $\tilde{\iota}_* \tilde{\xi} = \xi$. Note that $c_1(L(\tilde{D})) = \pi^*\omega$. We have that

$$c_1(L(\tilde{D})) \cdot \tilde{\xi} = \omega \cdot (ie_1 \wedge e_j) \neq 0.$$

Therefore we see that $c_1(L(\tilde{D})) \neq 0$ and that $c_1(N(\tilde{D})) \neq 0$; equivalently, the smooth irreducible divisor $\tilde{D}$ has no extra-zero on X. □

The problem that we are here dealing with may be considered as a special case of the intersection theory of analytic cycles. Let X be an affine algebraic manifold. Then there are theories of cycles in algebraic and analytic categories; there is a difference even in a simplest case as follows. Let $X \subset \mathbf{C}^2$ be an affine elliptic curve with a point at infinity, and let $a \in X$ be a point, which is an algebraic divisor. There is no regular rational function on X with exact zero a, but there exists such a holomorphic function on X.

In general, let $\bar{X}$ be a compact complex space, let $Z \subset \bar{X}$ be a reduced complex subspace, and set $X = \bar{X} \setminus Z$. Let $\mathscr{I}\langle Z\rangle$ be the geometric ideal sheaf of Z. Let $f \in \mathscr{O}(X)$ be a holomorphic function on X. We consider a positive number ρ' for which the following condition holds: There are a neighborhood $U \subset \bar{X}$ of every point $a \in Z$ with generators σ_j, $1 \leq j \leq \ell$, of $\mathscr{I}\langle Z\rangle$ over U, and positive constants C_1, C_2 such that

$$\log|f(x)| \leq \frac{C_1}{(\max_j |\sigma_j(x)|)^{\rho'}} + C_2, \quad {}^\forall x \in U \cap X.$$

We call the infinimum $\rho = \inf\{\rho'\}$ ($\leq \infty$) of all those ρ' the *order of f at infinity*; if no ρ' exists, $\rho = \infty$ by definition. Because of the coherence of $\mathscr{I}\langle Z\rangle$ (Oka [10, VII–VIII] and Cartan [1]), this definition makes sense.

Problem 4.7 Let $X = \bar{X} \setminus Z$ be Stein and algebraic. Let D be an effective algebraic divisor on X with $c_1(L(D)) = 0$ (resp. $c_1(N(D)) = 0$). Do there exist an algebraic compactification $\bar{X}$ of X and a holomorphic function $f \in \mathscr{O}(X)$ with zero divisor D (resp. locally in a neighborhood of $\mathrm{Supp}\,D$) such that f has order at most one at infinity?

(c) Stein's example from the viewpoint of the value distribution theory

Let $f : \zeta \in \mathbf{C} \to (e^{\zeta}, e^{i\zeta}) \in (\mathbf{C}^*)^2 = X$ be the example (3) due to Stein in (a). Then f is algebraically non-degenerate; that is, there is no proper algebraic subset $Y \subset X$ with $f(\mathbf{C}) \subset Y$. In fact, let $P(z,w)$ $(\neq 0)$ be any non-zero polynomial in $(z,w) \in X$. We write

$$P(z,w) = \sum_{j,k} c_{jk} z^j w^k.$$

Suppose that $f(\mathbf{C}) \subset \{P = 0\}$. Then

$$\sum_{j,k} c_{jk} e^{(j+ik)\zeta} \equiv 0.$$

This is absurd, since $e^{(j+ik)\zeta}$ are linearly independent over $\mathbf{C}$.

According to the main result of Noguchi-Winkelmann-Yamanoi [8], [9], and Corvaja-Noguchi [2], the intersection set $f(\mathbf{C}) \cap D$ is infinite for an arbitrary algebraic divisor D on X, but we have that for an extra-zero E of $D^+ = f(\mathbf{C})$,

$$f(\mathbf{C}) \cap E = \emptyset.$$

Problem 4.8 Let $g : \mathbf{C} \to X$ be an analytically non-degenerate entire curve. Then, is $g(\mathbf{C}) \cap A \neq \emptyset$ for an arbitrary analytic divisor A of X? Moreover, is $g(\mathbf{C}) \cap A$ an infinite set? Here it is natural to generalize X to a semi-abelian variety.

References

1. Cartan, H.: Idéaux et modules de fonctions analytiques de variables complexes. Bull. Soc. Math. France **78**, 29–64 (1950)
2. Corvaja, P., Noguchi, J.: A new unicity theorem and Erdös' problem for polarized semi-abelian varieties. Math. Ann. **353**, 439–464 (2012)
3. Cornalba, M., Shiffman, B.: A counterexample to the "Transcendental Bezout problem", Ann. Math. (2) **96**, 402–406 (1972)
4. Forstnerič, F.: Stein Manifolds and Holomorphic Mappings: The Homotopy Principle in Complex Analysis. *Ergeb. der Math. und ihrer Grenzgebiete 3*, vol. 56, Springer, Berlin (2011)
5. Grauert, H., Remmert, R.: Theory of Stein Spaces. *Grundl. Math. Wiss.*, vol 236. Springer, Berlin (1979). Translated by Huckleberry, A.
6. Hamano, S.: Oka's extra zero problem and application. Kyushu J. Math. **61**, 155–167 (2007)
7. Hamm, H.A., Mihalache, N.: Deformation retracts of Stein spaces. Math. Ann. **308**, 333–345 (1997)
8. Noguchi, J., Winkelmann, J., Yamanoi, K.: The second main theorem for holomorphic curves into semi-abelian varieties. Acta Math. **188**, 129-161 (2002)
9. Noguchi, J., Winkelmann, J., Yamanoi, K.: The second main theorem for holomorphic curves into semi-abelian varieties II. Forum Math. **20**, 469–503 (2008)
10. Oka, K.[3]:
    - Sur les fonctions analytiques de plusieurs variables.
        - I – Domaines convexes par rapport aux fonctions rationnelles. J. Sci. Hiroshima Univ. Ser. A **6**, 245–255 (1936) [Rec. 1 mai 1936]
        - II – Domaines d'holomorphie. J. Sci. Hirosima Univ. Ser. A **7**, 115–130 (1937) [Rec. 10 déc 1936]

[3] It is now very difficult to find a complete correct record of Oka's publications without errors. The most commonly referred reference of K. Oka's works is "*Kiyoshi Oka Collected Papers*", translated by Narasimhan, R., with commentaries by Cartan, H., edited by Remmert, R., Springer, Berlin (1984), where there are some mistakes in the records and moreover all records of the received dates were deleted. Therefore the authors think that it is meaningful and useful to provide a complete list of his publications with the received dates in one place.

III – Deuxieme problème de Cousin. J. Sci. Hirosima Univ. Ser. A **9**, 7–19 (1939) [Rec. 20 jan 1938]
IV – Domaines d'holomorphie et domaines rationnellement convexes. Japan. J. Math. **17**, 517–521 (1941) [Rec. 27 mar 1940]
V – L'intégrale de Cauchy. Japan. J. Math. **17**, 523–531 (1941) [Rec. 27 mar 1940]
VI – Domaines pseudoconvexes. Tôhoku Math. J. **49**, 15–52 (1942(+43)) [Rec. 25 oct 1941]
VII – Sur quelques notions arithmétiques. Bull. Soc. Math. France **78**, 1–27 (1950) [Rec. 15 oct 1948]
VIII – Lemme fondamental. J. Math. Soc. Japan **3**, no. 1, 204–214; no. 2, 259–278 (1951) [Rec. 15 mar 1951]
IX – Domaines finis sans point critique intérieur. Japan. J. Math. **23**, 97–155 (1953) [Rec. 20 oct 1953]
X – Une mode nouvelle engendrant les domaines pseudoconvexes. Japan. J. Math. **32**, 1–12 (1962) [Rec. 20 sep 1962]

- Note sur les familles de fonctions multiformes etc. J. Sci. Hiroshima Univ. Ser. A **4**, 93–98 (1934) [Rec. 20 jan 1934]
- Sur les domaines pseudoconvexes. Proc. Imp. Acad. Tokyo **17**, 7–10 (1941) [Comm. 13 jan 1941]
- Note sur les fonctions analytiques de plusieurs variables. Kōdai Math. Sem. Rep., no. 5–6, 15–18 (1949) [Rec. 19 déc 1949]

11. Oka, K.: On analytic functions in several variables XII: Generalization of Cousin second problem (in Japanese). In: Nishino, T., Takeuchi, A. (eds.) Posthumous Papers of Kiyoshi Oka, no. 2(/7), pp. 30–50, Kyoto (1981) URL http://www.lib.nara-wu.ac.jp/oka/fram/ikos.html
12. Stein, K.: Topologische Bedingungen für die Existenz analytischer Funktionen komplexer Veränderichen zu vorgegebenen Nullstellenflächen. Math. Ann. **117**, 727–757 (1941)