Junjiro Noguchi

Analytic Function Theory of Several Variables

Elements of Oka's Coherence

Springer

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Preface

The title of this book was taken from the series of papers to which Dr. Kiyoshi Oka devoted his life:

"Sur les fonctions analytiques de plusieurs variables."

A term such as "complex function theory in several variables," "function theory in several complex variables," or "complex analysis in several variables" is used almost in the same sense as the present one. "Several variables" means not only the case where the independent variables are plural, but also where the dependent variables are plural, and the fundamental theory provided in this book is indispensable. The most fundamental part of the theory is the *Coherence Theorems* found and proved by K. Oka ([62], VII, VIII). These theorems together with the notion of coherence itself are indispensable, not only in the study of complex analysis, complex geometry or the theory of complex manifolds of general dimension, but also in a large area of modern Mathematics to which analytic function theory of several variables provides a foundation. For example, the theory of M. Sato's hyperfunctions is based on coherent sheaves and the Oka–Cartan Fundamental Theorem. The situation for complex function theory of one variable or the theory of Riemann surfaces is similar, when a little advanced content is involved.

The purpose of this book is to develop the theory of Oka's Coherence Theorems as a standard subject in a reasonable volume size for those students at the level of the first year of a graduate course in Mathematics, who have learned the elements of Mathematics such as the general theory of sets and topology, some algebra (groups, rings, modules, etc.), and complex function theory of one variable. It is an interesting question as to what kind of approach is the best to provide such contents in a course of Mathematics. It turns out that the best and the easiest is to begin with Oka's Coherence Theorem (Oka's First Coherence Theorem), opposite to the order in existing books, and then to deal with the Oka–Cartan Fundamental Theorem on holomorphically convex domains.

In view of the theory of Sato's hyperfunctions mentioned above, due to an introductory book by A. Kaneko ([34], p. 307) the Oka–Cartan Theorem on polynomially convex domains suffices for that purpose, and in the case of the present book it is included in the content up to Chap. 4, where the theorem is proved on holomorphically convex domains. Up to the proof of the Oka–Cartan Fundamental Theorem on holomorphically convex domains in Chap. 4, the notion of abstract manifolds will not appear. When the content at the end of Chap. 4 is presented, the definition of manifolds will have been taught in some other lectures. Then it is easy to introduce the notion of Stein manifolds, and the Oka–Cartan Fundamental Theorem on Stein manifolds.

We took account of the above considerations in organizing the materials of this book. It is intended to present the contents as comprehensively as possible for the readers who are starting to learn Mathematics. Citations from other books or sources are avoided or minimized, so that the readers just after finishing a standard textbook on complex function theory of one variable together with general topology and elementary algebra may be able to read the contents by themselves. In Chap. 2 very basic contents of algebra are cited from other books, but they may be already learned in class lectures or easily understood by referring to some textbooks. Although in Chap. 6 the existence of primitive elements in the finite field extension is cited, the facts from ring theory are proved.

The subjects taught in Mathematics major courses, such as general theory of sets and topology, complex analysis in one variable and algebra on groups, rings and modules are marvelously merged, so that such a far reaching result as the Oka–Cartan Fundamental Theorem is proved; therefore the contents of the present textbook may be suitable to be presented at the beginning of graduate courses in Mathematics. This book provides the complete self-contained proofs of the following:

- Oka's three Coherence Theorems ($\mathscr{O}_{\mathbf{C}^n}$, geometric ideal sheaves, and the normalization sheaves of complex spaces).
- The Oka–Cartan Fundamental Theorem.
- Oka's Theorem on Levi's Problem (Hartogs' Inverse Problem) for Riemann domains.

As seen in the list of references, there are already a number of excellent books on analytic function theory of several variables, each of which is specialized in its specific theme. But it is rather difficult to find a book dealing with all the above three themes in a self-contained manner at elementary level. The present textbook, for instance, should be read before reading Hörmander's book [33] on the theory of $\bar{\partial}$-equation based on the theory of Hilbert spaces, or Grauert and Remmert [27] or [28]. The present text shares a common part with those of S. Hitotsumatsu [31], R.C. Gunning and H. Rossi [29], or T. Nishino [49], but the overall structure is different, and may be easier than those for readers.

The theory presented in this textbook was established by the 1960s, and one can say without exaggeration that almost all the essential parts are due to K. Oka's ideas and his works; the central core is his coherence theorems. The standing viewpoint of this book is the one described in the introductions of Oka [62], VII and VIII. Being based on the coherence of analytic functions, one can see at a glance the path achieved in Oka [62], I–VI and can reach in a step to the forthcoming Levi's Problem of pseudoconvexity (Hartogs' Inverse Problem). When K. Oka was writing VII (*Oka's First Coherence Theorem*), he had in hand the proofs of the coherence of geometric ideal sheaves (*Oka's Second Coherence Theorem*, Oka VII, VIII) and the normalization sheaves of structure sheaves of complex spaces (*Oka's Third Coherence Theorem*, Oka VIII). In many references the coherence of geometric ideal sheaves is attributed to H. Cartan [10], but as H. Cartan pointed out in [10], K. Oka had already obtained its proof when he wrote VII. In fact, a key preparation of the proof of the coherence of geometric ideal sheaves was already discussed and proved in Oka VII (1948) (cf. Problème (K) in it), which was used by Cartan [10] (1950) and by Oka VIII (1951). In this way the works of Oka VII and VIII form one set of works: It might be the most plausible version of history that H. Cartan gave an independent proof of geometric ideal sheaves referring to Oka VII between Oka VII and VIII for his own aim of completing the program proposed in [9]. Therefore, we refer in this text to those three coherence theorems as:

- Oka's First Coherence Theorem (the sheaf of germs of holomorphic functions);
- Oka's Second Coherence Theorem (geometric ideal sheaves (ideal sheaves of analytic subsets);
- Oka's Third Coherence Theorem (normalization sheaves).

This will be discussed in more detail at Chap. 9.

In this textbook, we prove Oka's First Coherence Theorem first (Chap. 2) just after some necessary definitions and a preparation from one variable theory (Chap. 1). This is new, and different from the other existing standard monographs. In Chap. 3 we prepare the cohomology theory of sheaves. We prove the Oka–Cartan Fundamental Theorem on holomorphically convex domains in Chap. 4, at the end of which the definition of Stein manifolds is given and the Oka–Cartan Fundamental Theorem on them is proved.

In Chap. 5 we show the equivalence of domains of holomorphy and holomorphically convex domains. Then the solutions of Cousin Problems I and II and the Oka Principle are described.

In Chap. 6 we deal with the theory of analytic sets. We investigate the structure of analytic sets and prove Oka's Second Coherence Theorem claiming the coherence of geometric ideal sheaves. As a result, we see that the set of singular points of an analytic set is again analytic. Then we introduce the concept of complex spaces. After the definition of normality of structure sheaves, we prove Oka's Third Coherence Theorem on the normalization of the structure sheaf of a complex space.

In Chap. 7 we give a solution of Levi's Problem (Hartogs' Inverse Problem). K. Oka solved this in the two-dimensional case in Oka VI (1942), and then for Riemann domains (unramified covering domains) of general dimension in Oka IX

(1953). On the course we describe *plurisubharmonic functions* introduced by K. Oka VI (1942) in order to solve Levi's Problem (Hartogs' Inverse Problem).

As for Levi's Problem (Hartogs' Inverse Problem), there is an interesting comment on the reason why he dealt only with the two-dimensional case in VI, in footnote (3) of Oka VIII, Introduction: "Précisément dit, … pour le problème des convexités, nous l'avons expliqué pour les deux variables complexes, pour diminuer la répétition ultérieure inévitable". In the introduction of Oka VII (1948/1950) he had mentioned a possibility to apply his coherence theorems to this problem (but, that part was deleted by a modification by H. Cartan). Furthermore, in the first sentence of Oka VIII (1951), Oka was writing that the solution of Levi's Problem (Hartogs' Inverse Problem) for unramified covering domains over $\mathbf{C}^n$ had been written and sent in 1943 as a research report to Teiji Takagi, then professor at the University of Tokyo, which was written in Japanese. The manuscript was complete just before the translation into French. But, it was time for him to begin thinking of coherent sheaves, *idéal de domaines indéterminés* in his own terms; even the notion was not at all clear then. He probably preferred to use his time not to translate the manuscript but to concentrate on thinking of *idéal de domaines indéterminés*. Fortunately, his handwritten report to T. Takagi remains and can be seen on the website "K. Oka Library" [68] (Posthumous Papers, Vol. 1 §7, dated 12 December 1943). Up to Oka VIII, he had believed that it would be possible to solve Levi's Problem (Hartogs' Inverse Problem) even for ramified covering domains, and proved the series of the coherence theorems for that purpose. Oka, however, preferred to write his IX limited to the case of unramified covering, solving Levi's Problem (Hartogs' Inverse Problem). (Later, a counterexample for the ramified case was found, and his choice turned out to be correct.)

In Chap. 7 we deal first with domains in $\mathbf{C}^n$, and then with Riemann domains over $\mathbf{C}^n$. The method is due to Grauert's Theorem of the finite dimensionality of higher cohomologies of coherent sheaves on strongly pseudoconvex domains.

Finally, in Chap. 8 we describe the topology in the space of sections of coherent sheaves, and the convergence of holomorphic functions on a complex space in general. Then we prove the Cartan–Serre Theorem on the finite dimensionality of cohomologies of coherent sheaves over compact complex spaces, and establish the above-mentioned Grauert's Theorem on domains with strongly pseudoconvex boundary in a complex manifold. In the final section, we apply Grauert's Theorem to prove Kodaira's Embedding Theorem. It is very nice to see such a fundamental theorem, which gives a bridge of Kodaira–Hodge theory and of complex projective algebraic geometry, to be proved as an application of Grauert's Theorem, which shows a supple possibility of Oka's Coherence Theorems.

In Chap. 7, there are not many references to Chap. 6. Therefore it is possible to skip Chap. 6 to read it. On the other hand, for those readers who like to learn the basics of analytic sets and complex spaces, they may proceed with Chaps. 1–2, and then may go to Chap. 6.

This book is based on the lectures which the author has delivered intermittently for about ten years at the Department of Mathematics, the University of Tokyo. In the course of reading the notes and writing proofs from them, Professors Hideaki

Kazama and Shigeharu Takayama gave valuable suggestions. Professor Hiroshi Yamaguchi provided a great deal of advice and suggestions on the records of Professor Kiyoshi Oka. The author expresses sincere gratitude to those three professors. Writing this book, the discussions with the members of the Monday seminar at the University of Tokyo were very helpful, and some colleagues kindly provided a number of references that the author did not know. The author is grateful to all of them. In the last year the author had oppotuities to give an intensive course of the contents of this book at Kanazawa University, Kyushu University and Tokyo Institute of Technology; in particular, the lecture at Kyushu University which was arranged by Professor Joe Kamimoto was very helpful. The author thanks him deeply. Last but not least the author would like to express his deepest thanks to Mr. Hiroya Oka and Professor Akira Takeuchi. Mr. H. Oka kindly agreed with printing some pictures of Professor Kiyoshi Oka at the end of this book, which were taken from some photo albums made by Professor Akira Takeuchi.

Komaba, Tokyo Junjiro Noguchi
Fall 2012

Added in the English Version

In the course of Grauert's proof of Oka's Theorem on Levi's Problem (Hartogs' Inverse Problem) L. Schwartz's finiteness theorem plays a key role (cf. Chap. 7), in the same way as in the Cartan–Serre Theorem (Chap. 8). The proof of L. Schwartz's finiteness theorem in the Japanese version is due to L. Bers [6], which is rather long and involved. Here in this English version, we give a very simple proof of L. Schwartz's finiteness theorem from J.-P. Demailly's notes [13].

During the preparation of the present English version, the author had the opporunity to give a series of lectures from March to May 2014 at the University of Roma II, "Tor Vergata" by kind invitation of Professor Filippo Bracci. Professor Joël Merker kindly invited the author to stay at University Paris Sud (Orsay) for a month from October to November 2014, where the author gave seminary talks on the contents of this book and had helpful discussions with him; he read through the manuscript with great care, and gave numerous useful remarks and comments. Translating Chap. 9, the author owes many suggestions and improvements of English expressions to Professor Alan Huckleberry. The author would like to express his sincere gratitude to Professors P. Bracci, J. Merker and A. Huckleberry.

Kamakura Junjiro Noguchi
Spring 2015

Conventions

(i) The set of natural numbers (positive integers) is denoted by $\mathbf{N}$, the set of integers by $\mathbf{Z}$, the set of rational numbers by $\mathbf{Q}$, the set of real numbers by $\mathbf{R}$, the set of complex numbers by $\mathbf{C}$, and the imaginary unit by i, as usual. The set of non-negative integers (resp. numbers) is denoted by $\mathbf{Z}^+$ (resp. $\mathbf{R}^+$).

(ii) For a complex number $z = x + iy \in \mathbf{C}$ we set $\Re z = x$ and $\Im z = y$.

(iii) Theorems, equations, etc., are numbered consecutively. Here an equation is numbered as (1.1.1) with parentheses; the first 1 stands for the chapter number and the second 1 for the section number.

(iv) *Monotone increasing* and *monotone decreasing* are used in the sense including the case of equality: e.g., a sequence of functions $\{\varphi_\nu(x)\}_{\nu=1}^{\infty}$ is said to be monotone increasing if for every point x of the defining domain $\varphi_\nu(x) \le \varphi_{\nu+1}(x)$ for all $\nu = 1, 2, \ldots$.

(v) A map $f : X \to Y$ between locally compact topological spaces is said to be *proper* if for every compact subset $K \subset Y$, the inverse image $f^{-1}K$ is also compact.

(vi) Manifolds are assumed to be connected, unless anything else is specified.

(vii) The symbol $\Subset$ stands for the relative compactness; e.g., $\Delta(a; r) \Subset U$ means that the topological closure $\overline{\Delta(a; r)}$ is compact in U.

(viii) The symbols $O(1), o(1)$, etc., follow after Landau's.

(ix) For a set S, $|S|$ denotes its cardinality.

(x) A map $f : X \to Y$ is said to be *injective* or an *injection* if $f(x_1) \neq f(x_2)$ for every distinct $x_1, x_2 \in X$, and to be *surjective* or a *surjection* if $f(X) = Y$. If f is injective and surjective, it is said to be bijective.

(xi) If a map $f : X \to Y$ between topological spaces X, Y is proper and the inverse image $f^{-1}\{y\}$ is always finite for all $y \in Y$, f is called a *finite map*. The restriction of f to a subset $E \subset X$ is denoted by $f|_E$.

(xii) A function f defined on an open subset $U \subset \mathbf{R}^m$ is said to be of C^k-class if f is k-times continuously differentiable. $C^k(U)$ denotes the set of all functions of C^k-class on U. $C_0^k(U)$ stands for the set of all $f \in C^k(U)$ with compact support.

(xiii) In general, for a differential form α we write $\alpha^k = \alpha \wedge \cdots \wedge \alpha$ (k-times).

(xiv) A polynomial in one variable with coefficients in a ring with $1(\neq 0)$ whose leading coefficient is 1 is called a monic polynomial.

(xv) A neighborhood is always assumed to be open, unless otherwise mentioned.

(xvi) A ring is commutative and contains $1 \neq 0$.

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