4.2 Cartan's Merging Lemma

We consider a coherent sheaf $\mathscr{F} \to \Omega$ over a domain $\Omega \subset \mathbf{C}^n$. As finite local generator systems of $\mathscr{F}$ over adjoining closed subdomains E' and E'' of Ω are given, it is necessary to form a local finite generator system of $\mathscr{F}$ over $E' \cup E''$ by merging them. We begin with some elementary facts on matrices.

4.2.1 Matrix-valued Functions

We prepare several facts on matrices, matrix-valued functions, their series and infinite products, necessary in the later arguments.

In general, let $p \in \mathbf{N}$ and let A be a complex (p,p)-matrix. We may consider two norms for A:

$$\|A\|_\infty = \max_{i,j}\{|a_{ij}|\},$$
$$\|A\| = \max\{\|A\xi\|; \xi \in \mathbf{C}^p, \|\xi\| = 1\}.$$

By using $\xi = {}^t(0,\dots,0,1,0,\dots,0)$, we immediately get

$$\|A\|_\infty \le \|A\| \le p\|A\|_\infty.$$

Therefore, the convergences defined by the two norms are mutually equivalent. Since $\|A\|$ behaves better than $\|A\|_\infty$ for the product of matrices, we use $\|A\|$ in the sequel. We call $\|A\|$ the *operator norm* of A.

If $A = A(z)$ is a (p,p)-matrix valued function defined on a subset $E \subset \mathbf{C}^n$, we put

$$\|A\|_E = \sup\{\|A(z)\|; z \in E\}.$$

We denote the unit (p,p)-matrix by $\mathbf{1}_p$.

Proposition 4.2.1 *Let A be a (p,p)-matrix or a (p,p)-matrix valued function in E. Let B be another (p,p)-matrix. Then the following holds:*

(i) $\|A+B\| \le \|A\| + \|B\|$.
(ii) $\|AB\| \le \|A\| \cdot \|B\|$.
(iii) If $A = A(z)(z \in E)$ satisfies $\|A\|_E \le \varepsilon < 1$, then there exists the inverse $(\mathbf{1}_p - A(z))^{-1}$ and the following holds:

$$(\mathbf{1}_p - A(z))^{-1} = \mathbf{1}_p + A(z) + A(z)^2 + \cdots.$$

Here, the right-hand side converges uniformly on E, and $\|(\mathbf{1}_p - A)^{-1}\|_E \le \frac{1}{1-\varepsilon}$: In particular, for $\varepsilon = \frac{1}{2}$, $\|(\mathbf{1}_p - A)^{-1}\|_E \le 2$.
(iv) For $k = 0,1,\dots$, let positive numbers ε_k with $0 < \varepsilon_k < 1$ and (p,p)-matrix valued functions $A_k(z)$ $(z \in E)$ be given, so that $\|A_k\|_E \le \varepsilon_k$ and $\sum_{k=0}^\infty \varepsilon_k < \infty$.

Then the infinite products

$$\lim_{k\to\infty}(\mathbf{1}_p - A_0(z))\cdots(\mathbf{1}_p - A_k(z)),$$
$$\lim_{k\to\infty}(\mathbf{1}_p - A_k(z))\cdots(\mathbf{1}_p - A_0(z))$$

converge uniformly on E and the limits are invertible.

Proof (i), (ii): These are immediate from the definitions.

(iii): We deduce this from the following identity and inequality with $k \to \infty$:

$$(\mathbf{1}_p - A(z))(\mathbf{1}_p + A(z) + A(z)^2 + \cdots + A(z)^k) = \mathbf{1}_p - A(z)^{k+1},$$
$$\|\mathbf{1}_p + A(z) + A(z)^2 + \cdots + A(z)^k\|_E \le \sum_{j=0}^{k} \|A\|_E^j \le \sum_{j=0}^{k} \varepsilon^j = \frac{1-\varepsilon^{k+1}}{1-\varepsilon}.$$

(iv): The proofs of the both are similar; we show the second. Set

$$G_k(z) = (\mathbf{1}_p - A_k(z))\cdots(\mathbf{1}_p - A_0(z)) = \prod_{j=k}^{0}(\mathbf{1}_p - A_j(z)), \quad k = 0, 1, \ldots.$$

It suffices to show that $\{G_k\}_{k=0}^{\infty}$ is a uniform Cauchy sequence, and that $\{G_k^{-1}\}_{k=0}^{\infty}$ converges uniformly on E, too. We set $C_0 = \exp(\sum_{k=0}^{\infty}\varepsilon_k)$. Then,

$$\begin{aligned}\|G_k\|_E &\le \prod_{j=k}^{0}\|\mathbf{1}_p - A_j\|_E \le \prod_{j=0}^{k}(1+\|A_j\|_E) \le \prod_{j=0}^{k}(1+\varepsilon_j)\\ &= \exp\left(\sum_{j=0}^{k}\log(1+\varepsilon_j)\right) < \exp\left(\sum_{j=0}^{k}\varepsilon_j\right) < C_0.\end{aligned}$$

Let $l > k > 0$. It follows from the above equation that

$$\begin{aligned}&\|G_l - G_k\|_E\\ &\le \left\|(\mathbf{1}_p - A_l)(\mathbf{1}_p - A_{l-1})\cdots(\mathbf{1}_p - A_{k+1}) - \mathbf{1}_p\right\|_E \cdot \|G_k\|_E\\ &\le C_0\|-A_l - A_{l-1} - \cdots - A_{k+1} + A_l A_{l-1} + \cdots + (-1)^{l-k}A_l\cdots A_{k+1}\|_E\\ &\le C_0(\|A_l\|_E + \cdots + \|A_{k+1}\|_E + \|A_l\|_E\cdot\|A_{l-1}\|_E + \cdots + \|A_l\|_E\cdots\|A_{k+1}\|_E)\\ &= C_0\left(\prod_{j=l}^{k+1}(1+\|A_j\|_E) - 1\right) \le C_0\left(\prod_{j=k+1}^{l}(1+\varepsilon_j) - 1\right)\\ &\le C_0\left(\exp\left(\sum_{j=k+1}^{l}\varepsilon_j\right) - 1\right) \longrightarrow 0 \quad (l > k \to \infty).\end{aligned}$$

Thus $\{G_k\}$ is a uniform Cauchy sequence.

As for $G_k^{-1} = \prod_{j=0}^{k}(\mathbf{1}_p - A_j)^{-1}$, with setting $B_k = -A_k(\mathbf{1}_p - A_k)^{-1}$ we have

$$(\mathbf{1}_p - A_k)^{-1} = \mathbf{1}_p - B_k.$$

By making use of the consequence of (iii) we obtain

$$\|B_k\|_E \le \|A_k\|_E \cdot \|(\mathbf{1}_p - A_k)^{-1}\|_E \le \frac{\varepsilon_k}{1-\varepsilon_k}.$$

Put $0 < \theta := \max_k\{\varepsilon_k\} < 1$. Then it follows that

$$\|B_k\|_E \le \frac{\varepsilon_k}{1-\theta}.$$

Therefore, for every $k \gg 1$, B_k fulfills the condition that A_k satisfies. Hence, $\{G_k^{-1}\}_{k=0}^{\infty}$ converges uniformly on E. □

Assuming the existence of $(\mathbf{1}_p - S)^{-1}$ and $(\mathbf{1}_p - T)^{-1}$ for (p,p)-matrices S and T, we put

$$\begin{aligned} M(S,T) &= (\mathbf{1}_p - S)^{-1}(\mathbf{1}_p - S - T)(\mathbf{1}_p - T)^{-1}, \\ N(S,T) &= \mathbf{1}_p - M(S,T). \end{aligned} \tag{4.2.2}$$

Lemma 4.2.3 (Key) *Let S and T be (p,p)-matrices such that* $\max\{\|S\|,\|T\|\} \le \frac{1}{2}$. *Then*

$$\|N(S,T)\| \le 2^2(\max\{\|S\|,\|T\|\})^2.$$

Proof Noting

$$(\mathbf{1}_p - T)^{-1} = \mathbf{1}_p + T(\mathbf{1}_p - T)^{-1} = \mathbf{1}_p + T + T^2(\mathbf{1}_p - T)^{-1},$$

we see that

$$\begin{aligned} M(S,T) &= (\mathbf{1}_p - S)^{-1}(\mathbf{1}_p - S - T)(\mathbf{1}_p - T)^{-1} \\ &= (\mathbf{1}_p - (\mathbf{1}_p - S)^{-1}T)(\mathbf{1}_p - T)^{-1} \\ &= \mathbf{1}_p + T + T^2(\mathbf{1}_p - T)^{-1} \\ &\quad - (\mathbf{1}_p + S(\mathbf{1}_p - S)^{-1})T(\mathbf{1}_p + T(\mathbf{1}_p - T)^{-1}) \\ &= \mathbf{1}_p + T + T^2(\mathbf{1}_p - T)^{-1} \\ &\quad - T - T^2(\mathbf{1}_p - T)^{-1} - S(\mathbf{1}_p - S)^{-1}T(\mathbf{1}_p - T)^{-1} \\ &= \mathbf{1}_p - S(\mathbf{1}_p - S)^{-1}T(\mathbf{1}_p - T)^{-1}, \\ N(S,T) &= S(\mathbf{1}_p - S)^{-1}T(\mathbf{1}_p - T)^{-1}. \end{aligned}$$

Then the assumption implies that

$$\|N(S,T)\| \le \|S\| \cdot 2 \cdot \|T\| \cdot 2 \le 2^2(\max\{\|S\|,\|T\|\})^2.$$ □

4.2.2 Cartan's Matrix Decomposition

We here assume the following:

4.2.4 (Closed cubes) A *closed cube* or a *closed rectangle* is a closed subset of $\mathbf{C}^n$ bounded with all edges parallel to real or imaginary axes of the complex coordinates; here we include the case when the widths of some edges degenerate to zero.

Assume that two closed cubes $E', E'' \Subset \Omega$ are represented as follows. There are a closed cube $F \Subset \mathbf{C}^{n-1}$, and two closed rectangles $E'_n, E''_n \Subset \mathbf{C}$ sharing an edge ℓ, such that (cf. Fig. 4.2)

$$E' = F \times E'_n, \qquad E'' = F \times E''_n, \qquad \ell = E'_n \cap E''_n.$$

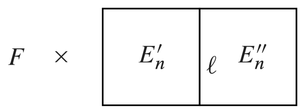

Fig. 4.2 Adjoining closed cubes

Let $GL(p;\mathbf{C})$ denote the general linear group of degree p, and let $\mathbf{1}_p$ denote the unit matrix of degree p. The following is due to H. Cartan [8].

Lemma 4.2.5 (Cartan's matrix decomposition) *Let the notation be as above. Then there is a neighborhood $V_0 \subset GL(p;\mathbf{C})$ of $\mathbf{1}_p$ such that for a matrix-valued holomorphic function $A : U \to V_0$ on a neighborhood U of $F \times \ell$, there is a matrix-valued holomorphic function $A' : U' \to GL(p;\mathbf{C})$ (resp. $A'' : U'' \to GL(p;\mathbf{C})$) on a neighborhood U' (resp. U'') of E' (resp. E'') satisfying $A = A' \cdot A''$ on a neighborhood of $F \times \ell$.*

Proof We widen each edge of F, E'_n, E''_n by the same length, $\delta > 0$ outward and denote the resulting closed cube and closed rectangles by $\tilde{F}, \tilde{E}'_{n(1)}, \tilde{E}''_{n(1)}$, respectively. Taking $\delta > 0$ sufficiently small, we have

$$F \times \ell \subset \tilde{F} \times \left(\tilde{E}'_{n(1)} \cap \tilde{E}''_{n(1)}\right) \Subset U.$$

Set the boundaries as in Fig. 4.3:

$$\begin{aligned} &\partial\left(\tilde{E}'_{n(1)} \cap \tilde{E}''_{n(1)}\right) = \gamma_{(1)} = \gamma'_{(1)} + \gamma''_{(1)}, \\ &\gamma'_{(1)} = \left(\partial \tilde{E}'_{n(1)}\right) \cap \tilde{E}''_{n(1)}, \quad \gamma''_{(1)} = \tilde{E}'_{n(1)} \cap \partial \tilde{E}''_{n(1)}. \end{aligned} \tag{4.2.6}$$

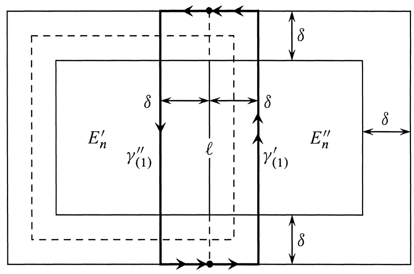

Fig. 4.3 δ-closed neighborhoods of the adjoining closed cubes

Similarly, keeping the inner $\frac{\delta}{2}$ of the width δ as E'_n is widened to $\tilde{E}'_{n(1)}$ we successively shrink inward by dividing in half the outer $\frac{\delta}{2}$. That is, $\tilde{E}'_{n(2)}$ denotes the closed cube shrunk inward by $\frac{\delta}{4}$ from $\tilde{E}'_{n(1)}$. Assuming $\tilde{E}'_{n(k)}$ determined, we denote by $\tilde{E}'_{n(k+1)}$ the closed cube shrunk inward by $\frac{\delta}{2^{k+1}}$ from $\tilde{E}'_{n(k)}$ (cf. Fig. 4.4). Since

$$\frac{\delta}{4}+\frac{\delta}{8}+\cdots=\frac{\delta}{2},$$

$$\bigcap_{k=1}^{\infty}\tilde{E}'_{n(k)}=\text{ the closed cube widened from } E'_n \text{ by } \frac{\delta}{2}.$$

We set $\tilde{E}''_{n(k)}$, similarly. As in (4.2.6) we write

$$\partial\left(\tilde{E}'_{n(k)}\cap\tilde{E}''_{n(k)}\right)=\gamma_{(k)}=\gamma'_{(k)}+\gamma''_{(k)}. \tag{4.2.7}$$

Let

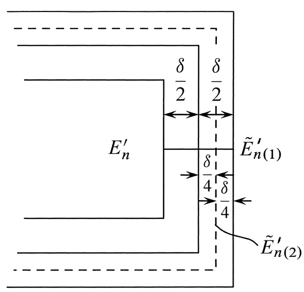

Fig. 4.4 Closed $\frac{\delta}{2^k}$-neighborhoods of closed cubes

(4.2.8) $$\tilde{E}'_{(k)} = \tilde{F} \times \tilde{E}'_{n(k)}, \qquad \tilde{E}''_{(k)} = \tilde{F} \times \tilde{E}''_{n(k)}$$

be the closed cube neighborhoods of E' and E'', respectively.

We write $A = \mathbf{1}_p - B_1$. By Cauchy's integral expression we have

(4.2.9) $$\begin{aligned} B_1(z', z_n) &= \frac{1}{2\pi i} \int_{\gamma_{(1)}} \frac{B_1(z', \zeta)}{\zeta - z_n} d\zeta \\ &= \frac{1}{2\pi i} \int_{\gamma'_{(1)}} \frac{B_1(z', \zeta)}{\zeta - z_n} d\zeta + \frac{1}{2\pi i} \int_{\gamma''_{(1)}} \frac{B_1(z', \zeta)}{\zeta - z_n} d\zeta \\ &= B'_1(z', z_n) + B''_1(z', z_n). \end{aligned}$$

Here, $B'_1(z', z_n)$ is holomorphic in $(z', z_n) \in \tilde{E}'_{(2)}$, and so is $B''_1(z', z_n)$ in $(z', z_n) \in \tilde{E}''_{(2)}$. Note that

(4.2.10) $$|z_n - \zeta| \geq \frac{\delta}{4}, \qquad {}^\forall (z', z_n) \in \tilde{E}'_{(2)}, \quad {}^\forall \zeta \in \gamma'_{(1)}.$$

Letting L be the length of the curve $\gamma'_{(1)}$, we get for $k \geq 1$

$$L = \text{the length of } \gamma''_{(1)} \geq \text{the length of } \gamma'_{(k)} = \text{the length of } \gamma''_{(k)}.$$

For $(z', z_n) \in \tilde{E}'_{(2)}$ it follows from (4.2.9) and (4.2.10) that

$$\|B'_1(z', z_n)\| \leq \frac{1}{2\pi} \cdot \frac{4}{\delta} L \cdot \max_{\gamma_{(1)}} \|B_1(z', \zeta)\|.$$

Therefore,

$$\|B'_1\|_{\tilde{E}'_{(2)}} \leq \frac{2L}{\pi\delta} \|B_1\|_{\tilde{E}'_{(1)} \cap \tilde{E}''_{(1)}}.$$

In the same way we get

$$\|B''_1\|_{\tilde{E}''_{(2)}} \leq \frac{2L}{\pi\delta} \|B_1\|_{\tilde{E}'_{(1)} \cap \tilde{E}''_{(1)}}.$$

Set

(4.2.11) $$\varepsilon_1 = \max\left\{ \|B'_1\|_{\tilde{E}'_{(2)}}, \|B''_1\|_{\tilde{E}''_{(2)}} \right\} \left(\leq \frac{2L}{\pi\delta} \|B_1\|_{\tilde{E}'_{(1)} \cap \tilde{E}''_{(1)}} \right).$$

We take $\delta > 0$, smaller if necessary, so that $\frac{\pi\delta}{2^5 L} \leq \frac{1}{2}$. Assume that

$$\|B_1\|_{\tilde{E}'_{(1)} \cap \tilde{E}''_{(1)}} \leq \frac{\pi^2\delta^2}{2^6 L^2}.$$

Then we have

$$\varepsilon_1 \le \frac{\pi\delta}{2^5 L} \le \frac{1}{2}, \tag{4.2.12}$$

$$\begin{aligned} A(z) = (\mathbf{1}_p - B_1(z)) &= (\mathbf{1}_p - B'_1(z))(\mathbf{1}_p - N(B'_1(z), B''_1(z))) \\ &\quad \cdot (\mathbf{1}_p - B''_1(z)), \quad z \in \tilde{E}'_{(2)} \cap \tilde{E}''_{(2)}. \end{aligned} \tag{4.2.13}$$

In the sequel, we proceed by induction. Assume that for $j = 1, \ldots, k$, (p, p)-matrix valued holomorphic functions

$$B'_j(z) \ (z \in \tilde{E}'_{(j+1)}), \quad B''_j(z) \ (z \in \tilde{E}''_{(j+1)}),$$

are determined, so that

$$\varepsilon_j := \max\left\{ \|B'_j\|_{\tilde{E}'_{(j+1)}}, \|B''_j\|_{\tilde{E}''_{(j+1)}} \right\} \le \frac{\pi\delta}{2^{j+4}L} \ \left(\le \frac{1}{2^j} \right), \quad 1 \le j \le k, \tag{4.2.14}$$

$$\begin{aligned} A(z) &= (\mathbf{1}_p - B'_1(z)) \cdots (\mathbf{1}_p - B'_k(z)) \cdot (\mathbf{1}_p - N(B'_k(z), B''_k(z))) \\ &\quad \cdot (\mathbf{1}_p - B''_k(z)) \cdots (\mathbf{1}_p - B''_1(z)), \quad z \in \tilde{E}'_{(k+1)} \cap \tilde{E}''_{(k+1)}; \end{aligned} \tag{4.2.15}$$

the case of $k = 1$ is due to (4.2.12) and (4.2.13).

We set (cf. (4.2.2))

$$\begin{aligned} B_{k+1}(z) &= N(B'_k(z), B''_k(z)), \quad z \in \tilde{E}'_{(k+2)} \cap \tilde{E}''_{(k+2)}, \\ B'_{k+1}(z', z_n) &= \frac{1}{2\pi i} \int_{\gamma'_{(k+1)}} \frac{B_{k+1}(z', \zeta)}{\zeta - z_n} d\zeta, \quad (z', z_n) \in \tilde{E}'_{(k+2)}, \\ B''_{k+1}(z', z_n) &= \frac{1}{2\pi i} \int_{\gamma''_{(k+1)}} \frac{B_{k+1}(z', \zeta)}{\zeta - z_n} d\zeta, \quad (z', z_n) \in \tilde{E}''_{(k+2)}. \end{aligned}$$

Here, note that $|\zeta - z_n| \ge \frac{\delta}{2^{k+2}}$ in the above integrands; we thus infer from (4.2.14) and Lemma 4.2.3 that

$$\begin{aligned} \varepsilon_{k+1} &\le \frac{L}{2\pi} \frac{2^{k+2}}{\delta} \|N(B'_k, B''_k)\|_{\tilde{E}'_{(k+1)} \cap \tilde{E}''_{(k+1)}} \\ &\le \frac{L}{2\pi} \frac{2^{k+2}}{\delta} 2^2 \varepsilon_k^2 \le \frac{1}{2} \varepsilon_k \le \frac{\pi\delta}{2^{k+5}L}, \\ \mathbf{1}_p - N(B'_k(z), B''_k(z)) &= (\mathbf{1}_p - B'_{k+1}(z))(\mathbf{1}_p - N(B'_{k+1}(z), B''_{k+1}(z))) \\ &\quad \cdot (\mathbf{1}_p - B''_{k+1}(z)), \quad z \in \tilde{E}'_{(k+2)} \cap \tilde{E}''_{(k+2)}. \end{aligned} \tag{4.2.16}$$

Thus, (4.2.14) and (4.2.15) hold for "$k+1$".

By (4.2.14) and Proposition 4.2.1 (iv) the infinite products

$$\begin{aligned} A'(z) &= \lim_{k\to\infty} (\mathbf{1}_p - B'_1(z)) \cdots (\mathbf{1}_p - B'_k(z)), \quad z \in \tilde{E}' := \bigcap_{k=1}^{\infty} \tilde{E}'_{(k)}, \\ A''(z) &= \lim_{k\to\infty} (\mathbf{1}_p - B''_k(z)) \cdots (\mathbf{1}_p - B''_1(z)), \quad z \in \tilde{E}'' := \bigcap_{k=1}^{\infty} \tilde{E}''_{(k)} \end{aligned}$$

converge uniformly on $\tilde{E}'$ and $\tilde{E}''$, respectively, and the limit $A'(z)$ (resp. $A''(z)$) is invertible and holomorphic in the interior of $\tilde{E}'$ (resp. $\tilde{E}''$).

For $z \in \tilde{E}' \cap \tilde{E}''$ we have by (4.2.14) and Lemma 4.2.3

$$\|N(B'_k(z), B''_k(z))\| \leq 2^2 \varepsilon_k^2 \leq \frac{1}{2^{2k-2}} \longrightarrow 0 \qquad (k \to \infty).$$

Therefore (4.2.15) yields $A(z) = A'(z)A''(z)$. □

Remark 4.2.17 (Estimate) In the above proof of Lemma 4.2.5 there are positive constants η, C and closed cube neighborhood $\tilde{E}'$ (resp. $\tilde{E}''$) of E' (resp. E''), dependent only on E', E'' and U such that

(i) $\tilde{E}' \cap \tilde{E}'' \subset U$;
(ii) if $A = \mathbf{1}_p - B$ with $\|B\|_U \leq \eta$, then there are $A' = \mathbf{1}_p - B'$ and $A'' = \mathbf{1}_p - B''$ satisfying

$$\mathbf{1}_p - B(z) = (\mathbf{1}_p - B'(z))(\mathbf{1}_p - B''(z)), \quad {}^\forall z \in \tilde{E}' \cap \tilde{E}'',$$
$$\max\{\|B'\|_{\tilde{E}'}, \|B''\|_{\tilde{E}''}\} \leq C\|B\|_U. \tag{4.2.18}$$

For the proof, repeat the above arguments together with (4.2.11) and (4.2.16).

4.2.3 Cartan's Merging Lemma

The following is Cartan's Merging Lemma in [8] (1940). In a footnote of the introduction of Oka VII, K. Oka describes a comment such that we owe a lot also to the theorems in [8].[1)]

Lemma 4.2.19 (Cartan's Merging Lemma) *Let $E' \subset U'$ and $E'' \subset U''$ be those in Lemma 4.2.5. Let $\mathscr{F} \to \Omega$ be a coherent sheaf.*

Assume that finitely many sections $\sigma'_j \in \Gamma(U', \mathscr{F})$, $1 \leq j \leq p'$, generate $\mathscr{F}$ over U', and similarly $\sigma''_k \in \Gamma(U'', \mathscr{F})$, $1 \leq k \leq p''$, generate $\mathscr{F}$ over U''. Furthermore, assume the existence of $a_{jk}, b_{kj} \in \mathscr{O}(U' \cap U'')$, $1 \leq j \leq p'$, $1 \leq k \leq p''$, such that

$$\sigma'_j = \sum_{k=1}^{p''} a_{jk}\sigma''_k, \quad \sigma''_k = \sum_{j=1}^{p'} b_{kj}\sigma'_j.$$

Then there are a neighborhood $W \supset E' \cup E''$ with $W \subset U' \cup U''$ and finitely many sections σ_l on W, $1 \leq l \leq p = p' + p''$, which generate $\mathscr{F}$ over W.

Proof We set column vectors and matrices as follows: $\sigma' = {}^t(\sigma'_1, \ldots, \sigma'_{p'})$, $\sigma'' = {}^t(\sigma''_1, \ldots, \sigma''_{p''})$, $A = (a_{jk})$, $B = (b_{kj})$. Then we have

[1)] In the original version of Oka VII (Iwanami) K. Oka wrote after the citation of [8], "dont nous devons beaucoup aussi aux théorèmes". In the version of Bull. Soc. Math. France, it is "Nous devons beaucoup aux théorèmes de ce Mémoire".

$$\sigma' = A\sigma'', \qquad \sigma'' = B\sigma'. \tag{4.2.20}$$

Adding 0 to σ' and σ'' to form vectors of the same degree p, we put

$$\tilde{\sigma}' = \left(\begin{array}{c} \sigma'_1 \\ \vdots \\ \sigma'_{p'} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right), \qquad \tilde{\sigma}'' = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline \sigma''_1 \\ \vdots \\ \sigma''_{p''} \end{array}\right).$$

We also put

$$\tilde{A} = \left(\begin{array}{c|c} \mathbf{1}_{p'} & A \\ \hline -B & \mathbf{1}_{p''} - BA \end{array}\right).$$

Since $BA\sigma'' = \sigma''$ by (4.2.20),

$$\tilde{\sigma}' = \tilde{A}\tilde{\sigma}''. \tag{4.2.21}$$

We take the following matrices consisting of the repetition of elementary transformations:

$$\begin{aligned} P &= \left(\begin{array}{c|c} \mathbf{1}_{p'} & A \\ \hline 0 & \mathbf{1}_{p''} \end{array}\right), \ P^{-1} = \left(\begin{array}{c|c} \mathbf{1}_{p'} & -A \\ \hline 0 & \mathbf{1}_{p''} \end{array}\right), \\ Q &= \left(\begin{array}{c|c} \mathbf{1}_{p'} & 0 \\ \hline B & \mathbf{1}_{p''} \end{array}\right), \ Q^{-1} = \left(\begin{array}{c|c} \mathbf{1}_{p'} & 0 \\ \hline -B & \mathbf{1}_{p''} \end{array}\right). \end{aligned} \tag{4.2.22}$$

Transforming $\tilde{A}$ from right and left, we get

$$Q\tilde{A}P^{-1} = \left(\begin{array}{c|c} \mathbf{1}_{p'} & 0 \\ \hline 0 & \mathbf{1}_{p''} \end{array}\right) = \mathbf{1}_p \quad (p = p' + p''). \tag{4.2.23}$$

Since $\tilde{A} = Q^{-1}P$, by setting $R = P^{-1}Q$ we have

$$\begin{aligned} R &= \left(\begin{array}{c|c} \mathbf{1}_{p'} & -A \\ \hline 0 & \mathbf{1}_{p''} \end{array}\right) \left(\begin{array}{c|c} \mathbf{1}_{p'} & 0 \\ \hline B & \mathbf{1}_{p''} \end{array}\right), \\ \tilde{A}R &= \mathbf{1}_p. \end{aligned} \tag{4.2.24}$$

Because of the form of R (cf. (4.2.24)), R is invertible for any choices of A and B. Since the elements a_{jk} (resp. b_{kh}) of A (resp. B) are holomorphic in a neighborhood of $E' \cap E'' = F \times \ell$, it follows from Corollary 1.2.22 that a_{jk} (resp. b_{kh}) are unifromly approximated in a neighborhood $W_0 (\Subset U' \cap U'')$ of $E' \cap E''$ by polynomials $\tilde{a}_{jk}$ (resp. $\tilde{b}_{kh}$). Let $\tilde{R}$ be the matrix formed by $\tilde{a}_{jk}, \tilde{b}_{kh}$ similarly as in (4.2.24). If the approximations are sufficient, we may have that with the neighborhood V_0 of $\mathbf{1}_p$ in Lemma 4.2.5

$$\hat{A}(z) = \tilde{A}(z)\,\tilde{R}(z) \in V_0, \qquad z \in W_0. \tag{4.2.25}$$

Then, Lemma 4.2.5 implies that there are a neighborhood W' (resp. W'') of E' (resp. E'') and an invertible (p,p)-matrix valued holomorphic function $\hat{A}'$ (resp. $\hat{A}''$) such that on $W' \cap W'' (\subset W_0)$

$$\hat{A} = \hat{A}'\,\hat{A}''. \tag{4.2.26}$$

It follows from this and (4.2.25) that $\tilde{A} = \hat{A}'\hat{A}''\tilde{R}^{-1}$. It is deduced from (4.2.21) that on $W' \cap W''$

$$\hat{A}'^{-1}\,\tilde{\sigma}' = \hat{A}''\,\tilde{R}^{-1}\,\tilde{\sigma}''. \tag{4.2.27}$$

Therefore, we may define $\tau_j \in \Gamma(W' \cup W'', \mathscr{F})$, $1 \leq j \leq p$, by

$$\begin{pmatrix} \tau_1 \\ \vdots \\ \tau_p \end{pmatrix} = \begin{cases} \hat{A}'^{-1}\,\tilde{\sigma}', & \text{on } W', \\ \hat{A}''\,\tilde{R}^{-1}\,\tilde{\sigma}'', & \text{on } W''. \end{cases}$$

Since $\hat{A}'^{-1}$ and $\hat{A}''\,\tilde{R}^{-1}$ are invertible, τ_j, $1 \leq j \leq p$, generate $\mathscr{F}$ over $W' \cup W''$. □

We call the above-obtained (τ_j) a locally finite generator system of $\mathscr{F}$ by *merging* (σ'_j) and (σ''_k).