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Basic Oka Theory in Several Complex Variables

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Preface

This book provides a comprehensive self-contained account of Oka theory, mainly concerned with the proofs of the Three Big Problems of approximation, Cousin and pseudonvexity (Hartogs, Levi) stated below, which were solved by Kiyoshi Oka and form the basics of complex analysis in several variables. It is the purpose to serve a textbook in course lectures just after complex function theory of one variable. The presentation is aimed to be readable, enjoyable and self-contained for those from beginners in Mathematics to researchers interested in complex analysis in several variables and complex geometry.

The nature of the present book should be featured by the following two points:

- We develop the theory by the method of Oka's Extension of holomorphic functions from a complex submanifold of a polydisk to the whole polydisk (Oka's Jôku-Ikô Principle);
- We represent Oka's original proofs, following his unpublished papers in 1943 and Oka IX (1953).

In those unpublished papers, historically in first, the pseudoconvexity problem (Hartogs' Inverse Problem, Levi's Problem) was solved not only for domains of $\mathbf{C}^n$ $(n \geq 2)$, but even for unramified domains over $\mathbf{C}^n$ (see [47], [41]).

We derive Oka's Extension Theorem of Jôku-Ikô from the coherence of the sheaf $\mathscr{O}_{\mathbf{C}^n}$ of holomorphic functions on $\mathbf{C}^n$ (Oka's First Coherence Theorem), which is proved by Weierstrass' Preparation Theorem; Weierstrass' Preparation Theorem is shown by the residue theorem in one variable. In this way we use only elementary techniques, yet reaching the core of the theory.

The basis of analytic function theory of several variables or complex analysis in several variables was founded till the first half of 1950's. Just after it new theories and generalizations were developed. Also the simplification of the theory has been done, but the step-gap of the entrance part has remained to be rather high for the beginners. The present book is aimed to provide a smooth introduction from the theory of one variable to that of several variables (cf. [36], [40], [1]).

The Three Big Problems which were summarized by Behnke–Thullen [3] in 1934 are stated as follows:

(P1) Approximation Problem (problem of developments) (Runge's Theorem in one variable).
(P2) Cousin Problem (I and II) (Mittag-Leffler's and Weierstrass' Theorems in one variable).
(P3) Pseudoconvexity Problem (Hartogs' Inverse Problem, Levi's Problem) (the natural boundary problem of analytic continuation).

K. Oka solved all these problem in Oka I—IX ([45], [46]), which are roughly classified into two groups:

(G1) Oka I—VI+IX.
(G2) Oka VII, VIII, IX.

Oka IX contains works belonging to both of the groups, and the Three Big Problems were solved by the first group (G1); in the second group (G2) he proved his Three Coherence Theorems, aiming a development beyond the original problems.

In the present book we restrict ourselves to the results of group (G1); this is the reason of the title with "Basic". Here we do not use:

- General theory of cohomologies with coefficients in sheaves;
- L^2-$\bar{\partial}$ method.

The solution of Pseudoconvexity Problem (P3) is the culmination of the works (G1); Oka's methods consist of

(i) (In VI, 1942; *univalent domains of* $\dim n = 2$) Cousin Problem & Weil's integral formula & Fredholm integral equation of the second kind;
(ii) (In VII–XI, 1943, unpublished (cf. [41]); *unramified multivalent domains over* $\mathbf{C}^n$ *of general* $n \geq 2$) Cousin Problem & "Primitive Coherence Theorem" & *Jôku-Ikô* [1)] & Fredholm integral equation of the second kind type;
(iii) (In IX, 1953, published; *unramified multivalent domains over* $\mathbf{C}^n$ *of general* $n \geq 2$) Cousin Problem & "Coherence Theorem" & *Jôku-Ikô* & Fredholm integral equation of the second kind type;

After all, K. Oka proved Pseudoconvexity Problem (P3) three times. It is noticed that in (ii) and (iii) above, Oka proved the Approximation Theorem and Cousin

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[1)] This is a term after K. Oka himself, and also called the *Jôku-Ikô Principle*, which is the guiding methodological principle of Oka theory. A direct translation might be "a transfer (=Ikô) to an upper space (=Jôku)". He found the principle in the study of Oka I (1936) and II (1937), and used it all through his works, till Oka IX (1953). It is an idea to solve the problem caused with the increased number of variables by increasing the number of variables more; one embeds the initial domains into simply shaped polydisks of higher dimensions, extends the problems over the polydisks, and then solves them by making use of the simplicity of polydisks.

In T. Nishino [34] the term was translated to the "Lifting Principle". As a matter of fact, the statement itself holds more generally for any subvarieties of Stein spaces, and so may be called an analytic extension or interpolation; then, however, the spirit of the wording will be lost, since the general case is proved through embeddings of analytic polyhedra into polydisks. So, here we prefer to use the original term as in [36].

Problem for unramified domains, multivalent in general, over $\mathbf{C}^n$ by a new method, which had been proved for univalent domains in his former papers (I—III). The content of Oka IX ((iii) above) is essentially the same as that of (ii) except for the part of "Coherence Theorem" obtained in (G2).

In Oka VI ((i) above), he mentioned the validity of the result for general dimension $n \geq 2$ in a modest phrase at the end of the paper:

> L'auteur pense que cette conclusion sera aussi indépendante des nombres de variables complexes.

In fact, the method of (i) was later generalized to univalent domains of general dimension n by S. Hitotsumatsu [25] 1949, H.J. Bremermann [7] 1954, and F. Norguet [42] 1954, independently. The method of (iii) was generalized for abstract complex spaces by T. Nishino [33] 1962 (cf. [34]), and A. Andreotti–R. Narasimhan [2] 1964.

In (ii) above, Oka formulated and proved a kind of "Primitive Coherence Theorem" with a certain condition, yet sufficient for the purpose, and he used some Fredholm integral equation of the second kind type. In Oka IX, he replaced the "Primitive Coherence Theorem" with his Coherence Theorems: the present book, hopefully, presents an easy comprehensive account of that theory.

H. Cartan once has written ([46], p. XII):

> ....
> Mais il faut avouer que les aspects techniques de ses démonstrations et le mode de présentation de ses résultats rendent difficile la tâche du lecteur, et que ce n'est qu'au prix d'un réel effort que l'on parvient à saisir la portée de ses résultats, qui est considérable. C'est pourquoi il est peut-être encore utile aujourd'hui, en hommage au grand créateur que fut Kiyoshi OKA, de présenter l'ensemble de son œuvre.
> ....

In English (by Noguchi),

> ....
> But we must admit that the technical aspects of his proofs and the mode of presentation of his results make it difficult to read, and that it is possible only at the cost of a real effort to grasp the scope of its results, which is considerable. This is why it is perhaps still useful today, for the homage of the great creator that was Kiyoshi OKA, to present the collection of his work.
> ....

It is interesting that, looking for an easier introduction of analytic function theory of several variables, we came back to Oka's original method.

To the best of the author's knowledge, there is no book nor monograph presenting Oka's original method except for Nishino [34], while there are many for the developments or other proofs obtained after Oka's works. The author hopes that the present book fulfills the gap even a little, and is useful to recognize Oka's original ideas.

It should be worthwhile for students and researchers to look into the original work of K. Oka, which may still contain some new ideas. Therefore the prerequisites of the present book are made minimum with assuming the contents from standard complex analysis in one variable (cf., e.g., [35]), which range from Cauchy's integral formula to Riemann's mapping theorem. We explain the necessary contents of topology, rings and modules; if they are not sufficient, it may suffice to confer any nearby books on those elementary materials. We avoided the general notion of manifolds.

Now we shortly describe the contents of the present book. In Chapter 1 we begin with the definition of holomorphic or analytic functions of several variables, and convergent power series. We then explain Hartogs' phenomenon, which was the starting point of analytic function theory of several variables. For the preparation of the chapters in the sequel we show the Runge approximation theorem on convex cylinder domains, the Cousin integral and analytic subsets.

Chapter 2 describes the notion of analytic sheaves and the coherence. Analytic sheaves will be defined just as sets or as collections of rings or modules without topology. We then show Weierstrass' Preparation Theorem by making use of the residue theorem in one variable. We then prove Oka's First Coherence Theorem, Cartan's Matrix Lemma and then, Oka's Syzygy Lemma, with which we finally derive Oka's Extension Theorem of Jôku-Ikô Principle.

Chapter 3 is devoted to the theory of domains of holomorphy and holomorphically convex domains. We prove foundational Cartan–Thullen's Theorem, which asserts the equivalence of those two domains in the univalent (schlicht) case. Then an analytic polyhedron is introduced, and the Oka–Weil Approximation Theorem is proved as a solution of the First Big Problem (P1) above by means of Jôku-Ikô. As a special case of one variable, we show Runge's Approximation Theorem.

Subsequently the Cousin Problem (the Second Big Problem (P2)) is dealt with. Here we formulate the *Continuous Cousin Problem* by which we unify the treatment of the Cousin I, II Problems and the problem of $\bar{\partial}$-equation for functions, where the *Oka Principle* is included. We then solve the Continuous Cousin Problem on holomorphically convex domains, equivalently on domains of holomorphy in the univalent case. We discuss the applications to the case of one variable, proving Mittag-Leffler's and Weierstrass' Theorems.

As an application of the Continuous Cousin Problem we prove the Hartogs extension of holomorphic functions over a compact subset of a domain of $\mathbf{C}^n$. By a similar method of the proof of the Continuous Cousin Problem, we solve the *interpolation problem* for complex submanifolds of univalent domains of holomorphy.

At the end of Chapter 3 we introduce the notion of multivalent domains over $\mathbf{C}^n$, which are here assumed to be unramified. We define the envelope of holomorphy of such a domain, and the notion of domains of holomorphy in the multivalent case. We introduce Stein domains in the multivalent case, and see that the results obtained for univalent domains of holomorphy remain to hold for multivalent Stein domains.

In Chapters 4 and 5 we deal with Pseudoconvexity Problem (P3) for domains over $\mathbf{C}^n$, where domains are assumed to be multivalent in general. Chapter 4 is devoted to the formulations and the reductions of Problem (P3). Firstly, we introduce the notion of plurisubharmonic functions. Using it, we prove Hartogs' separate analyticity theorem. Then, several kinds of pseudoconvexities of domains are defined. We discuss the equivalence and the relations of those pseudoconvexities, and formulate what is the pseudoconvexity problem. We then prove *Oka's Theorem of Boundary*

Distance Function. This serves the first important step toward the solution of Problem (P3). As an application we prove the *Tube Theorem* due to S. Bochner and K. Stein ($n = 2$).

In the last Chapter 5 we solve finally the Pseudoconvexity Problem (P3), the last of the Three Big Problems, which is formulated in the previous chapter. To begin with, we introduce the notion of a semi-normed space, a Baire space and a Fréchet space, and prove Banach's Open Mapping Theorem for them. We then show Oka's Extension Theorem of Jôku-Ikô Principle with estimate.

We give two proofs of the Steinness of strongly pseudoconvex domains (Levi's Problem); the first is K. Oka's and the second is the one due to H. Grauert; there is some similarity in the two proofs, which should be interesting for comparison.T. Nishino's book [34] presents the proof of Oka IX (1953) in more generalized form for complex spaces. In the proof a *Fredholm integral equation* of the second kind type is the key, and it is solved by a successive approximation and the convergence is obtained by the method of majorants. It is rather surprising to see such a difficult problem being solved by so elementary method.

The second method due to Grauert is the well-known "bumping method" combined with *L. Schwartz's Fredholm Theorem* [2)], of which short but complete proof is given (it is originally due to an idea of J.P. Demailly). For the purpose we introduce the first cohomology $H^1(\star, \mathscr{O})$.

With these preparations we finally prove *Oka's Pseudoconvexity Theorem* that pseudoconvex domains unramified over $\mathbf{C}^n$ are Stein.

At the end of each chapter some historical comments are put from the author's viewpoint and knowledges; it is expected to motivate the readers to confer other mentioned resources, but they are far from the completeness.

The present book is an outcome of the author [40], largely rewritten with a number of additions. It is not aimed to give the full exposition of the fundamentals of analytic function theory of several variables or complex analysis in several variables. But, the related topics are mentioned from place to place with references, which readers are suggested to confer. And some of them are presented in Exercises, which readers are hoped to solve by themselves. It is also recommended for readers to take a look into those books and monographs referred in various places.

It is of more than happiness of the author if readers get interested in the present subject through this book providing the elementary but self-contained proofs of the Three Big Problems which form the basics of several complex variables, and if the original works of Kiyoshi Oka, full of creative ideas, are enjoyed and recognized deeper.

While the author was writing the present book, he gave several talks at the weekly seminar on complex analysis and geometry, the University of Tokyo, Komaba, Tokyo (so-called Monday Morning Seminar), having a number of discussions with the members, which were very helpful and encouraging. In May 2017, he was kindly invited by Professor Sachiko Hamano of Osaka City University to give an intensive

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[2)] This term is due to A. Andreotti, according to a personal communication with A. Huckleberry (cf. Jahresber. Dtsch. Math.Ver. **115** (2013), 21-45).

one-week lectures based on the first draft of the book. In July of the same year, he gave a seminary talk on Oka's original method and related topics at Tor Vergata, Rome by an invitation of Professor Filippo Bracci. In March 2019, he talked on the book at Japan–Iceland Workshop, "Holomorphic Maps, Pluripotentials and Complex Geometry" by the invitation of Professor Masanori Adachi (Shizuoka University), and in May of the year he gave a series of talks by the invitation of Professor Steven Lu at Montreal; in July he gave a series of lectures based on this book at Workshop, "Summer Program on Complex Geometry and Several Complex Variables" at Shanghai Center for Mathematical Sciences, Fudan University, Shanghai by the invitation of Professor Min Ru (University of Houston). The author learned a number of references on the pseudoconvexity problem from Professor Makoto Abe (Hiroshima University), and had many useful and helpful discussions with Professors Y. Komori (Waseda Univerity, Tokyo) and J. Merker (Université Paris-Saclay). Professor V. Vâjâitu (Université des Sciences et Technologies de Lille) kindly suggested useful informations in Remarks at the end of Chap. 5. For all of them the author would like to express his sincere gratitude.

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