CHAPTER II

Measure Hyperbolic Manifolds

2.1 Holomorphic Line Bundles and Chern Forms

Let $M$ be an $m$-dimensional complex manifold. We call a triple $(L, \pi, M)$ a **holomorphic line bundle** if the following conditions are satisfied:

(2.1.1) (i) $L$ is an $(m+1)$-dimensional complex manifold.
(ii) $\pi: L \to M$ is a surjective holomorphic mapping.
(iii) For every $x \in M$, $\pi^{-1}(x)$ is a complex vector space of complex dimension 1.
(iv) For every $x \in M$, there exist an open neighborhood $U$ of $x$ and a biholomorphic mapping $\Phi: \pi^{-1}(U) \to U \times \mathbb{C}$, called a **local trivialization**, such that $p \circ \Phi = \pi$ and $q \circ \Phi|_{\pi^{-1}(y)}: \pi^{-1}(y) \to \mathbb{C}$ are linear isomorphisms for all $y \in U$, where $p: U \times \mathbb{C} \to U$ and $q: U \times \mathbb{C} \to \mathbb{C}$ denote the canonical projections.

If there occurs no confusion, we sometimes write $\pi: L \to M$ or $L$ instead of $(L, \pi, M)$. For $x \in M$, $\pi^{-1}(x)$ is sometimes denoted by $L_x$ and called the **fiber** of $L$ at $x$. Let $(L_j, \pi_j, M)$, $j = 1, 2$, be holomorphic line bundles over $M$. A holomorphic mapping $\Psi: L_1 \to L_2$ is called a **bundle homomorphism** if $\pi_2 \circ \Psi = \pi_1$ and $\Psi|_{L_1,x}: L_1,x \to L_2,x$ is a linear mapping for any $x \in M$ (cf. (2.1.1), (iii) and (iv)). If $\Psi$ is moreover a biholomorphic mapping, then $\Psi$ is called a **bundle isomorphism**. In this case, $L_1$ and $L_2$ are said to be **isomorphic**. A holomorphic line bundle is said to be **trivial** if it is isomorphic to the holomorphic line bundle $(M \times \mathbb{C}, \pi, M)$ with the natural projection $p: M \times \mathbb{C} \to M$, which is denoted by $1_M$.

Let $(L, \pi, M)$ be a holomorphic line bundle over $M$ and $U$ an open subset of $M$. A holomorphic mapping $s: U \to L$ is called a **holomorphic cross section** (defined over $U$) if $\pi \circ s(x) = x$ for $x \in U$. We write $\Gamma(U, L)$ for the set of all holomorphic
cross sections defined over $U$. We write $O$ for the mapping which sends each $x \in M$ to the zero vector $O_x$ of $L_x$. Clearly $O$ belongs to $\Gamma(M, L)$ and is called the zero section; we also denote by $O$ its image in $L$. For $s, t \in \Gamma(U, L)$ and $a, b \in \mathbb{C}$, set
\[
as + bt : x \in U \to as(x) + bt(x) \in L_x \subset L.\]
Then we have $as + bt \in \Gamma(U, L)$. Thus $\Gamma(U, L)$ is a complex vector space with $O|U$ as the zero element. If $s \in \Gamma(U, L)$ satisfies that $s(x) \neq O_x$ for any $x \in U$, we call $s$ a holomorphic local frame over $U$. Now write $O(U)$ for the complex vector space of all holomorphic functions on $U$. For $f \in O(U)$ and $s \in \Gamma(U, L)$, we define $fs \in \Gamma(U, L)$ by $(fs)(x) = f(x)s(x)$. Then the mapping
\[
f \in O(U) \to fs \in \Gamma(U, L)
\]
is linear for every $s \in \Gamma(U, L)$ and a linear isomorphism if $s$ is a holomorphic local frame over $U$; in this case, the restriction $L|U = \pi^{-1}(U)$ of $L$ over $U$ is isomorphic to $1_U$ by
\[
(x, a) \in U \times \mathbb{C} \to as(x) \in L|U,
\]
so that $L|U$ is trivial. Hence $L$ is trivial if and only if there exists a holomorphic local frame over $M$.

Let $\{U_\lambda\}$ be an open covering of $M$ and $s_\lambda \in \Gamma(U_\lambda, L)$ holomorphic local frames over $U_\lambda$. We call the pair $(U_\lambda, \ s_\lambda)$ a local trivialization covering of $L$. If $U_\lambda \cap U_\mu \neq \emptyset$, there exists uniquely a holomorphic function
\[
T_{\lambda\mu} : U_\lambda \cap U_\mu \to \mathbb{C}^*
\]
such that
\[
(2.1.2) \quad s_\lambda(x)T_{\lambda\mu}(x) = s_\mu(x)
\]
for $x \in U_\lambda \cap U_\mu$. Then $\{T_{\lambda\mu}\}$ satisfies the following, so called the cocycle condition:
\[
(2.1.3) \quad T_{\lambda\mu}T_{\mu\lambda} = 1, \quad \text{whenever} \ U_\lambda \cap U_\mu \neq \emptyset,
\quad T_{\lambda\mu}T_{\mu\nu}T_{\nu\lambda} = 1, \quad \text{whenever} \ U_\lambda \cap U_\mu \cap U_\nu \neq \emptyset.
\]
Now the pair $(\{U_\lambda\}, \ \{T_{\lambda\mu}\})$ is called the system of holomorphic transition functions subordinated to the local trivialization $(\{U_\lambda\}, \ \{s_\lambda\})$. Remark that there always exists a local trivialization of $L$ by (2.1.1), (iv).

The dual holomorphic line bundle $(L^*, \pi^*, M)$ of $(L, \pi, M)$ is defined as follows: For any $x \in M$, set
\[
L^*_x = \ "\text{the dual vector space of} \ L_x."
\]