Quasi-homomorphism Rigidity with Noncommutative Targets

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Prologue: Zimmer's program

All groups are assumed second countable & locally compact.
All maps are assumed locally bounded & Borel measurable.

A "large" group should not act on a "small" manifold

Problem

Let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$.
Is every action of Γ on a (compact) mfld M of dimension $< n-1$ finite?

The known results are mostly for $M = S^1$ (a circle), or $M = \mathbb{R}$ (a line).

Theorem

- (Witte 1994)
  YES to the above Problem for finite index subgroups of $\mathrm{SL}(3,\mathbb{Z})$.
- **(Ghys, Burger–Monod 1999)**
  Let Γ be a lattice in $\mathrm{SL}(n \geq 3,\mathbb{R})$. Then, every action $\Gamma \curvearrowright S^1$ has at least one finite orbit, and every C^1-action $\Gamma \curvearrowright S^1$ is finite.
- (Navas 2002)
  Let Γ be a property (T) *group. Then, every C^2-action $\Gamma \curvearrowright S^1$ is finite.*

Lattices of $\mathrm{SL}(n \geq 3,\mathbb{R})$ have property (T) of Kazhdan.

Quasimorphisms

Let $\Gamma \curvearrowright S^1$. Each $g \in \Gamma$ has a lift $\tilde{g} \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ with $\tilde{\gamma}(0) \in [0,1)$. Then,

$$c(g,h) = (\widetilde{gh})^{-1}\tilde{g}\tilde{h} \in \{0,1\}$$

defines the Euler class e in the bounded cohomology $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{Z})$.

Theorem (Ghys 1987)

The Euler class $e \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{Z})$ determines $\Gamma \curvearrowright S^1$ up to semi-conjugacy.

Under certain assumption (e.g., $\mathrm{H}^2(\Gamma, \mathbb{R}) = 0$), the Euler cocycle c is a coboundary of a not-necessarily bounded map $q \colon \Gamma \to \mathbb{R}$.
The map q is a quasimorphism:

$$\sup_{g,h\in\Gamma} |q(gh) - \big(q(g) + q(h)\big)| < +\infty.$$

$\rightsquigarrow$ Want to show every quasimorphism on $\Gamma \leq \mathrm{SL}(3,\mathbb{R})$ is bounded.

Property (T) and what it is good for

All groups are assumed second countable & locally compact.
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Kazhdan's property (T)

Definition/Theorem (Kazhdan '67, Delorme '77, Guichardet '72)

G has property (T) if it satisfies one of the following equiv conditions:

- The trivial representation is isolated in the unitary dual of G.
- For every unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$, every cocycle $b\colon G \to \mathcal{H}$ is bounded. Here, a cocycle is a map b satisfying
$$\forall g, h \in G \quad b(gh) = b(g) + \pi(g)b(h).$$
Note: A cocycle b is bounded iff $\exists \xi \in \mathcal{H}$ s.t. $b(g) = \pi(g)\xi - \xi$.

Example

- Simple Lie groups of real rank ≥ 2 have property (T).
- A lattice Γ in G has property (T) iff G has property (T).
- $\mathrm{SL}(n, \mathbb{Z})$ has property (T) iff $n \geq 3$.
- Many hyperbolic groups, e.g. lattices in $\mathrm{Sp}(n, 1)$, have property (T).

Some consequences of Kazhdan's property (T)

Theorem (Kazhdan)

For a discrete group Γ *with property* (T)*, the following hold true.*

- Γ *is finitely generated.*
- Γ *has finite abelianization.*
- *For each* n*,* Γ *has only finitely many* n*-dimensional unitary reps, up to unitary equivalence.*

Sketchy proof of the last statement.

By property (T), $\exists$ a finite subset $E \subset \Gamma$ and $C > 0$ such that

$$\forall b \text{ cocycle} \qquad \sup_{g\in G} \|b(g)\| \leq C \max_{s\in E} \|b(s)\|.$$

For unitary reps $\pi, \sigma \colon \Gamma \to \mathcal{U}(n)$, consider the unitary rep $\pi \otimes \bar{\sigma}$ on HS_n defined by $X \mapsto \pi(g)X\sigma(g)^*$, and the cocycle $b(g) = \pi(g)\sigma(g)^* - I_n$.

$\rightsquigarrow$ If π and σ are close on E, then they are unitarily equivalent. $\square$

Property (TT) and what it is good for

 

All groups are assumed second countable & locally compact.
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Beef up Kazhdan's property (T)

Definition (Kazhdan, Delorme, Guichardet, Burger–Monod,)

A group G has *property* (TT) if every quasi-cocycle on G is bounded. Here, a quasi-cocycle is a map $b\colon G \to \mathcal{H}$, together with $\pi\colon G \to \mathcal{U}(\mathcal{H})$, which satisfies

- π is a representation, and
- b satisfies the ~~cocycle identity~~ rough cocycle inequality

$$\sout{b(gh) = b(g) + \pi(g)b(h).}$$

$$\sup_{g,h} \| b(gh) - \big(b(g) + \pi(g)b(h)\big)\| < +\infty.$$

A quasimorphism is a quasi-cocycle with the trivial representation.

Theorem (Buger–Monod 1999, 2002)

The group $\mathrm{SL}(n,\mathbb{R})$ *and its lattices have property* (TT) *for* $n \geq 3$.

Results of Burger and Monod

Theorem (Buger–Monod 1999, 2002)

The group $\mathrm{SL}(n,\mathbb{R})$ *and its lattices have property* (TT) *for* $n \geq 3$.

There are groups having property (T), but not (TT): Hyperbolic groups do not have property (TT), because they have proper quasi-cocycles. A cocycle $b\colon G \to \mathcal{H}$ is said to be proper if for any $C > 0$, the subset $\{g \in G : \|b(g)\| \leq C\}$ is relatively compact.

Corollary

Every quasimorphism on a lattice Γ *in* $\mathrm{SL}(n \geq 3,\mathbb{R})$ *is bounded.*

Corollary (Ghys, Burger–Monod)

Every action $\Gamma \curvearrowright S^1$ *has at least one finite orbit.*

Property (TTT) and what it is good for

All groups are assumed second countable & locally compact.
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Beef up Kazhdan's property (T) further

Definition (Kazhdan, Delorme, Guichardet, Burger–Monod, Oz.)

A group G has *property* (TTT) if every wq-cocycle on G is bounded. Here, a wq-cocycle is a map $b\colon G \to \mathcal{H}$, together with $\pi\colon G \to \mathcal{U}(\mathcal{H})$, which satisfies

- ~~π is a representation, and~~
- b satisfies the ~~cocycle identity~~ rough cocycle inequality

$$\sout{b(gh) = b(g) + \pi(g)b(h).}$$

$$\sup_{g,h} \| b(gh) - \big(b(g) + \pi(g)b(h)\big) \| < +\infty.$$

Theorem (Oz. 2009)

The group $\mathrm{SL}(n, \mathbb{R})$ *and its lattices have property* (TTT) *for* $n \geq 3$.

Quasi-homomorphisms

Definition

A map $q\colon G \to H$ is called a *quasi-homomorphism* if

$$\{q(gh)^{-1}q(g)q(h) : g, h \in G\}$$

is relatively compact in H.

If $b\colon H \to \mathcal{H}$ is a wq-cocycle and $q\colon G \to H$ is a quasi-homomorphism, then $b' = b \circ q$ is a wq-cocycle, because

$$b'(gh) = b(q(g)q(h)\square) \approx b'(g) + \pi'(g)b'(h).$$

↯ Even if π is multiplicative, $\pi' = \pi \circ q$ is not.

Definition

A group H is called *a-TTT-menable* if there is a proper wq-cocycle on H.

Examples: Abelian groups, solvable groups, amenable groups, a-T-menable (a.k.a. Haagerup) groups, hyperbolic groups...

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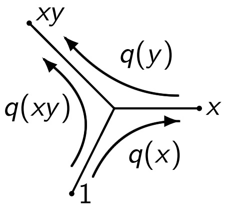
Corollary

If G has property (TTT) and H is a-TTT-menable, then every quasi-homomorphism from G into H has relatively compact image.

Examples of quasi-homomorphisms. $q(gh)^{-1}q(g)q(h)$

- $\widetilde{\mathrm{Homeo}}(S^1) = \{f \in \mathrm{Homeo}(\mathbb{R}) : f(x+1) = f(x)+1\}$ and $q\colon f \mapsto f(0) \in \mathbb{R}$.
  $\rightsquigarrow$ Application to $\Gamma \curvearrowright S^1$ (Burger–Monod, Ghys).

- $q\colon \mathbb{F}_2 = \langle a, b\rangle \to \mathbb{Z}$,
  $$q(w) = (\sharp \text{ of } ab \text{ occurs in } w) - (\sharp \text{ of } b^{-1}a^{-1} \text{ occurs in } w)$$



Generalizes to hyperbolic groups (Epstein–Fujiwara).

$\lightning$ Defect usually occurs around the joining area: $q(g)^{-1}q(gh)q(h)^{-1}$.
It's difficult to have quasi-homomorphisms with noncommutative targets.

ε-representations

Definition

For $\varepsilon > 0$, a (unitary) *ε-representation* of a group G on a Hilbert space $\mathcal{H}$ is a map $\pi\colon G \to \mathcal{U}(\mathcal{H})$ which satisfies

$$\sup_{g,h\in G} \|\pi(g)\pi(h) - \pi(gh)\| \leq \varepsilon.$$

Problem [S. M. Ulam, A collection of mathematical problems (1960).]

Is an ε-representation π close to a unitary representation?

Kazhdan (1982): YES! for amenable groups, and NO! in general.
$\rightsquigarrow$ NO! for any group which contains $\mathbb{F}_2$.

Example (From a quasimorphism to a quasi-character)

Let $q\colon \Gamma \to \mathbb{R}$ be a quasimorphism with $\sup |q(gh) - (q(g) + q(h))| \leq 1$. Then $\pi(g) := \exp\big(i\varepsilon q(g)\big)$ is an ε-character. For ε sufficiently small, π is close to a character iff q is a bounded distance from a homomorphism.

ε-representations and property (TTT)

Definition

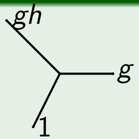
For $\varepsilon > 0$, a (unitary) *ε-representation* of a group G on a Hilbert space $\mathcal{H}$ is a map $\pi\colon G \to \mathcal{U}(\mathcal{H})$ which satisfies

$$\sup_{g,h\in G} \|\pi(g)\pi(h) - \pi(gh)\| \le \varepsilon.$$

An example of ε-representations (Rolli 2009)

Let $\mathbb{F}_2 = \langle a, b\rangle$ and $B(\varepsilon/3) = \{u \in \mathcal{U}(\mathcal{H}) : \|u - 1\| \le \varepsilon/3\}$.
Fix symmetric functions $\sigma_a, \sigma_b\colon \mathbb{Z} \to B(\varepsilon/3)$ and set

$$\pi(a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k}) = \sigma_a(m_1)\sigma_b(n_1)\cdots\sigma_a(m_k)\sigma_b(n_k).$$



Theorem (B.O.T.; Dimension dependent Ulam stability)

Let Γ be a property (TTT) group. Then, any ε-representation $\pi\colon \Gamma \to \mathcal{U}(d)$ with $\varepsilon < \kappa(d)$ is close to a unitary representation.

Ulam stability

Theorem (Kazhdan and Burger–Oz.–Thom)

- *If* Γ *is amenable, then every* ε*-repn is* 2ε*-close to a unitary repn.*
- *If* $\mathbb{F}_2 \hookrightarrow \Gamma$*, then for each* $\varepsilon > 0$*,* $\exists$ ε*-repn which is not close to any unitary repn.*
- *If* Γ *has property* (TT)*, then every* 1*-dim* ε*-repn is* $\delta(\varepsilon)$*-close to a unitary repn.*
- *If* Γ *has property* (TTT)*, then every* d*-dim* ε*-repn is* $\delta_d(\varepsilon)$*-close to a unitary repn.*
- *If* $\Gamma = \mathrm{SL}(n \geq 3, \mathbb{Z})$*, then every finite-dim* ε*-repn is* $\delta(\varepsilon)$*-close to a unitary repn. The same thing for certain* $\mathrm{SL}(2, A)$*.*

Are two ε-close unitary repns of Γ necessarily unitarily equivalent?
YES if Γ amenable (or unitarizable), and NO if $\mathbb{F}_2 \hookrightarrow \Gamma$.

Proof of Property (TTT)

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Relative property (TTT)

Definition

A subgroup $A \leq G$ has *relative property* (TTT) if every wq-cocycle on G is bounded on A.

Theorem

Let A be abelian and $G = G_0 \ltimes A$. Then, for $A \leq G$,

$$\textit{relative property (TTT)} \iff \textit{relative property (T)}$$

The proof is à la Burger, but goes with positive definite kernels

$$\theta_t(g,h) = \exp\big(-t\|\mathfrak{b}(g) - \mathfrak{b}(h)\|^2\big)$$

instead of positive type functions.

Bounded generation and property (TTT) for $\mathrm{SL}(n, \mathbb{K})$

Theorem

Let A be abelian and $G = G_0 \ltimes A$. Then, for $A \leq G$,

relative property (TTT) $\iff$ *relative property* (T)

Corollary

For $n \geq 3$, the group $\mathrm{SL}(n, \mathbb{K})$ has property (TTT).

Proof for $n = 3$.

By relative property (T) for $\mathbb{K}^2 \leq \mathrm{SL}(2, \mathbb{K}) \ltimes \mathbb{K}^2$, every wq-cocycle b on $\mathrm{SL}(3, \mathbb{K})$ is bounded on $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$, and on any other elementary matrices. Since every element of $\mathrm{SL}(3, \mathbb{K})$ is a product of at most 10 elementary matrices, the wq-cocycle b is bounded on $\mathrm{SL}(3, \mathbb{K})$. □

$$\sup_{g,h} \| b(gh) - \big(b(g) + \pi(g)b(h)\big) \| < +\infty.$$

Lattices

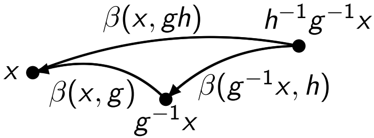
Let G be a property (TTT) group and $\Gamma \leq G$ be a *cocompact* lattice. Let $X = G/\Gamma$ and choose a section $\sigma\colon X \to G$.

Define the Borel cocycle $\beta\colon X \times G \to \Gamma$ by

$$\beta(x,g) = \sigma(x)^{-1} g \sigma(g^{-1}x).$$

It satisfies the cocycle identity:

$$\beta(x,gh) = \beta(x,g)\beta(g^{-1}x,h).$$



To prove that Γ has property (TTT), let a wq-cocycle $b\colon \Gamma \to \mathcal{H}$ be given, and $\tilde{b}\colon G \to L^2(X,\mathcal{H})$ be the induced wq-cocycle on G defined by

$$\tilde{b}(g)(x) = b(\beta(x,g)),$$

together with $\tilde{\pi}\colon G \to \mathcal{U}(L^2(X,\mathcal{H}))$, $\quad (\tilde{\pi}(g)\xi)(x) = \pi(\beta(x,g))\xi(g^{-1}x)$.

Problem

If we know $\tilde{b}$ is bounded on G, does it follow b is bounded on Γ?

Lattices and semi-length functions

Problem

If we know $\tilde{b}$ is bounded on G, does it follow b is bounded on Γ?

Burger–Monod: The answer is YES! if b is a quasi-cocycle, because the L^2-induction $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathcal{H}) \to \mathrm{H}^2_{\mathrm{cb}}(G, L^2(X, \mathcal{H}))$ is injective.

In general,
let $C := \sup \| b(gh) - \big(b(g) + \pi(g)b(h)\big)\|$ and $\ell(g) := \|b(g)\| + C$.
Then, ℓ is a semi-length function: $\ell(gh) \leq \ell(g) + \ell(h)$.
The induced semi-length function $L\colon G \to \mathbb{R}_{\geq 0}$ is given by

$$L(g) = \int_X \ell(\beta(x,g))\, dx.$$

The above problem generalizes to

Problem

If we know L is bounded on G, does it follow ℓ is bounded on Γ?

Semi-length functions and nonlinear cohomology?

Theorem

Let $G \curvearrowright X$ be a probability measure preserving action, and $\ell\colon X \times G \to \mathbb{R}_{\geq 0}$ be a groupoid semi-length function:

$$\ell(x, gh) \leq \ell(x, g) + \ell(g^{-1}x, h) \text{ a.e.}$$

If $\operatorname{ess-sup}_{g \in G} \int_X \ell(x, g)\, dx < +\infty$, *then* $\exists\, h \in L^1(X)$ *such that*

$$\ell(x, g) \leq h(x) + h(g^{-1}x) \text{ a.e.}$$

This theorem acts for the injectivity of $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathcal{H}) \to \mathrm{H}^2_{\mathrm{cb}}(G, L^2(X, \mathcal{H}))$.

Corollary

Let $\ell\colon \Gamma \to \mathbb{R}_{\geq 0}$ be a semi-length function and $L\colon G \to \mathbb{R}_{\geq 0}$ be the induced semi-length function. If L is bounded, then so is ℓ.
In particular, property (TTT) passes to a cocompact lattice.

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