

# Dixmier's Similarity Problem

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## Amenability and von Neumann's Problem



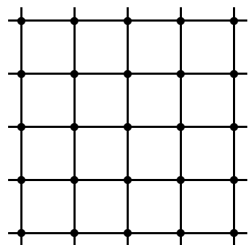
# Cayley Graph

graph  $\equiv$  countable set  $\Gamma$  & metric  $d: \Gamma \times \Gamma \rightarrow \mathbb{N} \cup \{\infty\}$ .

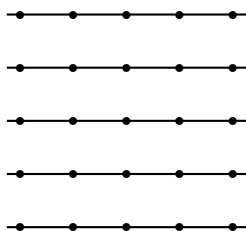
## Cayley graph $\text{Cayley}(\Gamma, \mathcal{S})$

$\Gamma$  countable discrete group,  $\mathcal{S} \subset \Gamma$  subset.

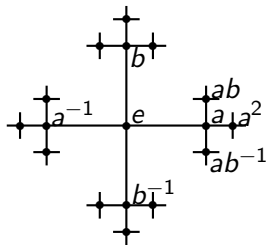
$$|x|_{\mathcal{S}} = \min\{n : \exists s_i \in \mathcal{S} \cup \mathcal{S}^{-1} \text{ s.t. } x = s_1 s_2 \cdots s_n\}.$$
$$d_{\mathcal{S}}(x, y) = |x^{-1}y|_{\mathcal{S}}.$$



$$\mathbb{Z}^2 = \langle (1,0), (0,1) \rangle$$



$$\mathbb{Z}^2, \mathcal{S} = \{(1,0)\}$$



$$\mathbb{F}_2 = \langle a, b \rangle$$

# Expansion Constant, Amenability

$N_r(A) := \{x \in \Gamma : d(x, A) \leq r\}$ ,  $r$ -neighborhood of  $A \subset \Gamma$

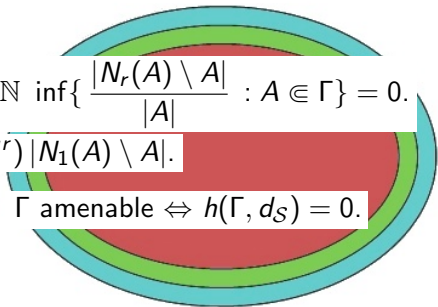
Expansion constant  $h(\Gamma) = \inf \left\{ \frac{|N_1(A) \setminus A|}{|A|} : \emptyset \neq A \subset \Gamma \text{ finite} \right\}$ .

## Definition

A graph  $\Gamma$  is amenable  $\iff h(\Gamma) = 0$ .

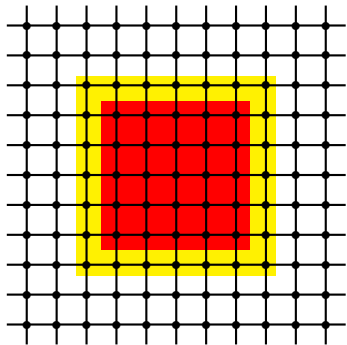
A discrete group  $\Gamma$  is amenable  $\iff h(\Gamma, d_S) = 0$  for all finite  $S \subset \Gamma$ .

- $|N_r(A)| \geq (1 + h(\Gamma))^r |A|$ .
- $\Gamma$  amenable,  $\deg(\Gamma) \leq d \Rightarrow \forall r \in \mathbb{N} \inf \left\{ \frac{|N_r(A) \setminus A|}{|A|} : A \in \Gamma \right\} = 0$ .  
 $\therefore |N_r(A) \setminus A| \leq (1 + d + \dots + d^r) |N_1(A) \setminus A|$ .
- If  $\Gamma = \langle S \rangle$  finitely generated, then  $\Gamma$  amenable  $\iff h(\Gamma, d_S) = 0$ .



# Examples of (Non-)Amenable Groups

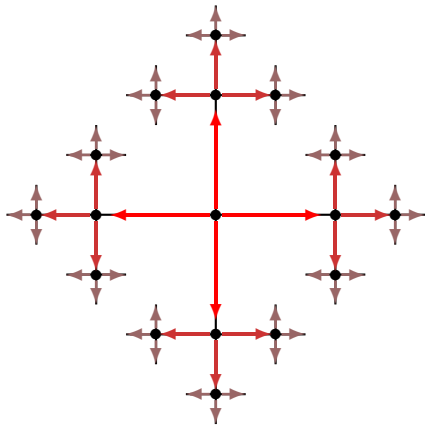
$\mathbb{Z}^d$  amenable



More generally, finite groups and solvable groups are amenable.

Moreover, subgroups, extensions, directed unions,...

$h(d\text{-regular tree}) = d - 2$



$\Gamma \supset \mathbb{F}_2 \implies \Gamma$  non-amenable.

# Pyramid Scheme

$\Gamma$  non-amenable **graph**  $\iff \exists$  successful pyramid scheme.

## Theorem (Benjamini–Schramm 1997)

$$h(\Gamma) \geq d \iff \exists \text{ spanning forest } F \subset \Gamma \text{ s.t. } h(F) = d.$$

Here, a *spanning forest* of a set  $\Gamma$  is a graph structure on  $\Gamma$  whose connected components are trees.

## von Neumann's Problem

$$\Gamma \text{ non-amenable **group** } \implies \mathbb{F}_2 \hookrightarrow \Gamma ?$$

## Tits Alternative (1972)

- $\Gamma$  f.g. linear group  $\implies$
- $\exists$  solvable subgroup  $\Lambda \leq \Gamma$  of finite index  
 $\rightsquigarrow \Gamma$  amenable
  - or**
  - $\mathbb{F}_2 \hookrightarrow \Gamma$

## von Neumann's Problem

$$\Gamma \text{ non-amenable} \implies \mathbb{F}_2 \hookrightarrow \Gamma ?$$

Free Burnside group:  $B(m, n) := \mathbb{F}_m / \langle\langle x^n : x \in \mathbb{F}_m \rangle\rangle$ .

Burnside's Problem (1902):  $|B(m, n)| < \infty$  for every  $m, n < \infty$  ?

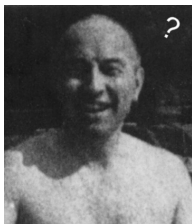
- Yes! for  $n = 2, 3, 4, 6$  (Burnside 1902, Sanov 1940, Hall 1958).
- Yes! for **linear** Burnside groups (Burnside 1905).
- No! for  $n \gg 1$  odd (Adian–Novikov 1968).

Obviously,  $\mathbb{F}_2 \not\hookrightarrow B(m, n)$ .  $\rightsquigarrow$  A counterexample to vN's problem?

## Theorem (Ol'shanskii 1980, Adian 1982)

Every free Burnside group  $B(m, n)$  is non-amenable, for  $n \gg 1$  odd.  
In particular, **No!** to von Neumann's problem.

## Uniformly Bounded Representations and Dixmier's Problem



# Sz.-Nagy's Theorem (1947)

$\mathbb{B}(\mathcal{H})$  = the space of bounded linear operators on a Hilbert space  $\mathcal{H}$ .

For  $T \in \mathbb{B}(\mathcal{H})$ ,  $\|T\| = \sup\{\|Th\| : h \in \mathcal{H}, \|h\| \leq 1\} < +\infty$ .

An isometric isomorphism on  $\mathcal{H}$  is called *unitary*:  $\|T\| = \|T^{-1}\| = 1$ .

## Theorem (Sz.-Nagy 1947)

$$\exists S \in \mathbb{B}(\mathcal{H})^{-1} \text{ s.t. } S^{-1}TS \text{ unitary} \iff \sup_{n \in \mathbb{Z}} \|T^n\| < \infty.$$

## Proof of ( $\Leftarrow$ ).

Let  $F_n := [-n, n] \cap \mathbb{Z}$  and define a new inner product  $\langle \cdot, \cdot \rangle_T$  on  $\mathcal{H}$  by

$$\langle h, k \rangle_T := \operatorname{Lim}_n \frac{1}{|F_n|} \sum_{n \in F_n} \langle T^n h, T^n k \rangle.$$

Then,  $T$  becomes unitary on  $(\mathcal{H}, \langle \cdot, \cdot \rangle_T)$ .

Let  $C = \sup_{n \in \mathbb{Z}} \|T^n\|$ . Since  $C^{-1}\|h\| \leq \|h\|_T \leq C\|h\|$ ,

$\exists S \in \mathbb{B}(\mathcal{H})$  s.t.  $\|S\| \leq C$ ,  $\|S^{-1}\| \leq C$  &  $S^{-1}TS$  unitary. □

# Uniformly Bounded Representations

## Definition

A representation  $\pi: \Gamma \rightarrow \mathbb{B}(\mathcal{H})$  is called *uniformly bounded* if

$$\|\pi\| := \sup_{g \in \Gamma} \|\pi(g)\| < \infty.$$

Recall: A group  $\Gamma$  is amenable  $\Leftrightarrow \exists F_n$  s.t.  $\frac{|F_n g \Delta F_n|}{|F_n|} \rightarrow 0$  for all  $g \in \Gamma$ .

## Theorem (Day, Dixmier 1950)

Every unif bdd representation  $\pi$  of an **amenable** group is *unitarizable* (aka similar to a unitary repn), i.e.,  $\exists S$  s.t.  $S\pi(\cdot)S^{-1}$  is unitary.

## Proof.

Define a new inner product  $\langle \cdot, \cdot \rangle_\pi$  on  $\mathcal{H}$  by

$$\langle h, k \rangle_\pi := \lim_n \frac{1}{|F_n|} \sum_{g \in F_n} \langle \pi(g)h, \pi(g)k \rangle.$$

$\vdots$

□

# Unitarizable Groups

## Definition

A group  $\Gamma$  is called *unitarizable* if every unif bdd repn of  $\Gamma$  is unitarizable.

## Dixmier's Problem (1950)

- 1 Are **all** groups unitarizable?
- 2 In case (1) is not true, does unitarizability characterize amenability?

Answer: No! for  $\Gamma = \mathrm{SL}(2, \mathbb{R})$  (Ehrenpreis–Mautner 1955).

## Corollary (by Induction)

$\mathbb{F}_2 \hookrightarrow \Gamma \implies \Gamma$  **not** unitarizable.

$\because \mathbb{F}_2 \supset \mathbb{F}_\infty \rightarrow \mathrm{SL}(2, \mathbb{R}) \xrightarrow{\pi} \mathbb{B}(\mathcal{H}).$

Leinert 1979, Mantero–Zappa, Pytlik–Szwarc, Bożejko–Fendler 1991, ...

$$\mathbb{F}_2 \hookrightarrow \Gamma \implies \Gamma \text{ non-amenable.}$$

von Neumann's Problem ✓

$$\Gamma \text{ non-amenable} \implies \mathbb{F}_2 \hookrightarrow \Gamma ?$$

$$\Gamma \text{ amenable} \implies \Gamma \text{ unitarizable.}$$

Dixmier's Problem

$$\Gamma \text{ unitarizable} \implies \Gamma \text{ amenable?}$$

We know:

- $\mathbb{F}_2 \hookrightarrow \Gamma \implies \Gamma \text{ not unitarizable.}$
- $\exists \text{ non-amenable } \Gamma \text{ s.t. } \mathbb{F}_2 \not\hookrightarrow \Gamma, \text{ e.g. } B(m, n).$

Pisier 2006, 2007: **strongly** unitarizable  $\implies$  amenable.

## Littlewood and Random Forests



# Derivations and Uniformly Bounded Representations

$\lambda: \Gamma \rightarrow \mathbb{B}(\ell_2\Gamma)$  the left regular representation:  $\lambda_x\delta_y = \delta_{xy}$ .

A map  $D: \Gamma \rightarrow \mathbb{B}(\ell_2\Gamma)$  is called a *derivation* if it satisfies the Leibniz rule:

$$D(xy) = \lambda(x)D(y) + D(x)\lambda(y).$$

$D$  derivation  $\iff \pi_D: \Gamma \ni x \mapsto \begin{bmatrix} \lambda(x) & D(x) \\ 0 & \lambda(x) \end{bmatrix} \in \mathbb{B}(\ell_2\Gamma \oplus \ell_2\Gamma)$  is a repr.

If  $D_T(x) = [T, \lambda(x)]$  is an inner derivation assoc. with  $T \in \mathbb{B}(\mathcal{H})$ , then

$$\pi_{D_T}(x) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{bmatrix} \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix}.$$

## Theorem

- $\pi_D$  uniformly bounded  $\iff D$  uniformly bounded.
- $\pi_D$  unitarizable  $\iff D$  inner.

# Littlewood and Pyramid Scheme

## Definition ( $p$ -Littlewood function; $p = 1$ )

$f: \Gamma \rightarrow \mathbb{C}$  belongs to  $T_1(\Gamma)$  if  $\exists A, B: \Gamma \times \Gamma \rightarrow \mathbb{C}$  s.t.

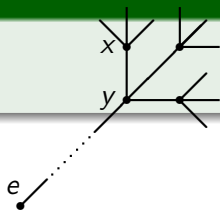
- $f(x^{-1}y) = A(x, y) + B(x, y)$  for all  $x, y \in \Gamma$ ,
- $\sup_x \sum_y |A(x, y)| + \sup_y \sum_x |B(x, y)| < \infty$ .

$$\|f\|_{T_1(\Gamma)} := \inf \{ \|A\|_{1,1} + \|B\|_{\infty,\infty} \}.$$

## Example

$$\Gamma = \mathbb{F}_d, 2 \leq d \leq \infty, \quad f(x) = \begin{cases} 1 & \text{if } |x| = 1 \\ 0 & \text{if } |x| \neq 1 \end{cases}.$$

$$A = \{(x, y) : |x^{-1}y| = 1, |x| > |y|\},$$
$$B = \{(x, y) : |x^{-1}y| = 1, |x| < |y|\}.$$



## Bożejko–Fendler's Construction (1991)

For  $f \in T_1(\Gamma)$  with  $f(x^{-1}y) = A(x, y) + B(x, y)$ , define  $D: \Gamma \rightarrow \mathbb{B}(\ell_2\Gamma)$

by 
$$D(g) := [A, \lambda(g)] = -[B, \lambda(g)].$$

One has  $\|D(g)\|_{1,1} \leq 2\|A\|_{1,1}$  &  $\|D(g)\|_{\infty,\infty} \leq 2\|B\|_{\infty,\infty}$ .

Consequently,  $\|D(g)\|_{2,2} \leq \|A\|_{1,1} + \|B\|_{\infty,\infty}$ .  $\rightsquigarrow D$  is unif bdd.

### Theorem

$D$  inner  $\implies f \in \ell_2\Gamma$ .

In particular, if  $\Gamma$  is unitarizable, then  $T_1(\Gamma) \subset \ell_2\Gamma$ .

### Corollary (Bożejko–Fendler 1991)

$\mathbb{F}_n \hookrightarrow \Gamma \implies \exists D: \Gamma \rightarrow \mathbb{B}(\ell_2\Gamma)$  non-inner unif bdd derivation.

# Random Forests

$\text{Forest}(\Gamma) := \{ \text{spanning forest of } \Gamma \}.$

Example (Free subgroup  $\mathbb{F}_n = \langle g_1, \dots, g_n \rangle \hookrightarrow \Gamma$ )

$\text{Cayley}(\Gamma, \{g_1, \dots, g_n\}) \in \text{Forest}(\Gamma)$ . Moreover, it is left  **$\Gamma$ -invariant**.

## Definition

A *random forest* on  $\Gamma$  means a  **$\Gamma$ -invariant** prob measure on  $\text{Forest}(\Gamma)$ .

Example (Free Minimal SF (Lyons–Peres–Schramm 2006))

$E :=$  Edge set of  $\text{Cayley}(\Gamma, \mathcal{S}) (= \Gamma \times \mathcal{S})$ .  $\Gamma \curvearrowright E$ .

$\Omega = [0, 1]^E$ ,  $\mu = m^{\otimes E}$ ,  $\Gamma \curvearrowright (\Omega, \mu)$  Bernoulli shift.

The push-out measure of  $\Theta: \Omega \rightarrow 2^E \cap \text{Forest}(\Gamma)$ ;

$\Theta(\omega) = \{e \in E : e \text{ is not the maximum of } \omega \text{ in any cycle containing } e\}.$

# Gaboriau–Lyons's Theorem

## Measure Group Theoretic Solution to vN's Problem

$$[\Lambda, \Gamma] := \{\alpha: \Lambda \rightarrow \Gamma \mid f(e) = e\} \subset \Gamma^\Lambda.$$

Then,  $\alpha \in [\Lambda, \Gamma]$  is a group homomorphism  $\iff \alpha(x) = \alpha(xs)\alpha(s)^{-1}$ .

So, we define  $\Lambda \curvearrowright [\Lambda, \Gamma]$  by  $(s \cdot \alpha)(x) := \alpha(xs)\alpha(s)^{-1}$ .

### Definition

A *random homomorphism* means a  $\Lambda$ -invariant prob measure on  $[\Lambda, \Gamma]$ .

### Theorem (Gaboriau–Lyons 2009)

$\Gamma$  non-amenable  $\iff \exists$  random embedding  $\mathbb{F}_2 \hookrightarrow \Gamma$ .

## Review 2

von Neumann's Problem ✓

$$\Gamma \text{ non-amenable} \implies \mathbb{F}_2 \hookrightarrow \Gamma ?$$

Dixmier's Problem

$$\Gamma \text{ unitarizable} \implies \Gamma \text{ amenable} ?$$

Theorem (Gaboriau–Lyons)

$$\Gamma \text{ non-amenable} \implies \exists \text{ random embedding } \mathbb{F}_2 \hookrightarrow \Gamma.$$

Theorem (Bożejko–Fendler)

$$\mathbb{F}_2 \hookrightarrow \Gamma \implies \exists D: \Gamma \rightarrow \mathbb{B}(\ell_2\Gamma) \text{ non-inner unif bdd derivation.}$$

$$\text{i.e., bounded cohomology group } H_b^1(\Gamma, \mathbb{B}(\ell_2\Gamma)) \neq 0.$$

# Now are our brows bound with Victorious Wreathes<sup>1</sup>

The *wreath product* of a group  $A$  by  $\Gamma$  is defined as

$$A \wr \Gamma := \left( \bigoplus_{\Gamma} A \right) \rtimes \Gamma.$$

Here  $((a_x)_x, g) \cdot ((b_x)_x, h) := ((a_x b_{g^{-1}x})_x, gh)$ .



Gaboriau–Lyons's theorem + Induction of  $D \in H_b^1(\mathbb{F}_2, \mathbb{B}(\ell_2 \mathbb{F}_2)) \rightsquigarrow$

## Theorem (Monod–Ozawa)

$\Gamma$  non-amenable  $\iff A \wr \Gamma$  not unitarizable, for  $\forall A$  infinite abelian.

## Corollary

Free Burnside groups  $B(m, n)$  are not unitarizable, for  $n \gg 1$  composite.

## Proof.

$$B(m, pq) \supset B(\infty, pq) \rightarrow \left( \bigoplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z} \right) \wr B(2, q). \quad \square$$

<sup>1</sup>Shakespeare, *Richard III*, 1:1.