Local Galois representations of Swan conductor one

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Abstract

We construct the local Galois representations over the complex field whose Swan conductors are one by using etale cohomology of Artin–Schreier sheaves on affine lines over finite fields. Then, we study the Galois representations, and give an explicit description of the local Langlands correspondences for simple supercuspidal representations. We discuss also a more natural realization of the Galois representations in the etale cohomology of Artin–Schreier varieties.

Introduction

Let K be a non-archimedean local field. Let n be a positive integer. The existence of the local Langlands correspondence for $GL_n(K)$, proved in [LRS93] and [HT01], is one of the fundamental results in the Langlands program. However, even in this fundamental case, an explicit construction of the local Langlands correspondence has not yet been obtained. One of the most striking results in this direction is Bushnell–Henniart's result for essentially tame representations in [BH05a], [BH05b] and [BH10]. On the other hand, we don't know much about the explicit construction outside essentially tame representations.

We discuss this problem for representations of Swan conductor 1. The irreducible supercuspidal representations of $GL_n(K)$ of Swan conductor 1 are equivalent to the simple supercuspidal representations in the sense of [AL16] (*cf.* [GR10], [RY14]). Such representations are called "epipelagic" in [BH14].

Let p be the characteristic of the residue field k of K. If n is prime to p, the simple supercuspidal representations of $GL_n(K)$ are essentially tame. Hence, this case is covered by Bushnell–Henniart's work. See also [AL16]. It is discussed in [Kal15] to generalize the construction of the local Langlands correspondence for essentially tame epipelagic representations to other reductive groups.

In this paper, we consider the case where p divides n. In this case, the simple supercuspidal representations of $GL_n(K)$ are not essentially tame. Moreover, if n is a power of p, the irreducible representations of the Weil group W_K of Swan conductor 1, which correspond to the simple supercuspidal representations via the local Langlands correspondence, cannot be induced from any proper subgroup. Such representations are called primitive (cf. [Koc77]). For simple supercuspidal representations, we have a straightforward characterization of the local Langlands correspondence given in [BH14]. Further, Bushnell–Henniart study the restriction to the wild inertia subgroup of the Langlands parameters for the simple supercuspidal representations explicitly. Actually, the restriction to the wild inertia subgroup already determines the original Langlands parameters up to character twists, but we need additional data, which appear in Bushnell–Henniart's characterization, to pin down the

2010 *Mathematics Subject Classification.* Primary: 11F80; Secondary: 11F70.

correct Langlands parameters. On the other hand, the construction of the irreducible representations of W_K of Swan conductor 1 is a non-trivial problem. What we will do in this paper is

- to construct the irreducible representations of W_K of Swan conductor 1 without appealing to the existence of the local Langlands correspondence, and
- to give a description of the Langlands parameters themselves for the simple supercuspidal representations.

Let ℓ be a prime number different from p. For the construction of the irreducible representations of W_K of Swan conductor 1, we use etale cohomology of an Artin–Schreier ℓ-adic sheaf on $\mathbb{A}^1_{k^{\mathrm{ac}}}$, where k^{ac} is an algebraic closure of k. It will be possible to avoid usage of geometry in the construction of the irreducible representations of W_K of Swan conductor 1. However, we prefer this approach, because

- we can use geometric tools such as the Lefschetz trace formula and the product formula of Deligne–Laumon to study the constructed representations, and
- the construction works also for ℓ-adic integral coefficients and mod ℓ coefficients.

A description of the local Langlands correspondence for the simple supercuspidal representations is discussed in [IT22] in the special case where $n = p = 2$. Even in the special case, our method in this paper is totally different from that in [IT22].

We explain the main result. We write $n = p^e n'$, where n' is prime to p. We fix a uniformizer ϖ of K and an isomorphism $\iota \colon \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$.

Let $\mathcal{L}_\psi$ be the Artin–Schreier $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{k^{\mathrm{ac}}}$ associated to a non-trivial character ψ of $\mathbb{F}_p$. Let $\pi \colon \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}}$ be the morphism defined by $\pi(y) = y^{p^e+1}$. Let $\zeta \in \mu_{q-1}(K)$, where $q = |k|$. We put $E_\zeta = K[X]/(X^{n'} - \zeta\varpi)$. Then we can define a natural action of W_{E_ζ} on $H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi)$. Using this action, we can associate a primitive representation $\tau_{n,\zeta,\chi,c}$ of W_{E_ζ} to $\zeta \in \mu_{q-1}(K)$, a character χ of $k^\times$ and $c \in \mathbb{C}^\times$. We construct an irreducible representation $\tau_{\zeta,\chi,c}$ of Swan conductor 1 as the induction of $\tau_{n,\zeta,\chi,c}$ to W_K.

We can associate a simple supercuspidal representation $\pi_{\zeta,\chi,c}$ of $GL_n(K)$ to the same triple (ζ, χ, c) by type theory. Any simple supercuspidal representation can be written in this form uniquely (*cf.* [IT18, Proposition 1.3]).

Theorem. *The representations $\tau_{\zeta,\chi,c}$ and $\pi_{\zeta,\chi,c}$ correspond via the local Langlands correspondence.*

In Section 1, we recall a general fact on representations of a semi-direct product of a Heisenberg group with a cyclic group. In Section 2, we give a construction of the irreducible representations of W_K of Swan conductor 1. To construct a representation of W_K which naturally fits a description of the local Langlands correspondence, we need a subtle character twist. Such a twist appears also in the essentially tame case in [BH10], where it is called a rectifier. Our twist can be considered as an analogue of the rectifier. We construct the representations of W_K using geometry, but we give also a representation theoretic characterization of the constructed representations. In Section 3, we give a construction of the simple supercuspidal representations of $GL_n(K)$ using the type theory.

In Section 4, we state the main theorem and recall a characterization of the local Langlands correspondence for simple supercuspidal representations given in [BH14]. The characterization consists of the three equalities of (i) the determinant and the central character, (ii) the refined Swan conductors, and (iii) the epsilon factors.

In Section 5, we recall some general facts on epsilon factors. In Section 6, we recall facts on Stiefel–Whitney classes, multiplicative discriminants and additive discriminants. We use these facts to calculate Langlands constants of wildly ramified extensions. In Section 7, we recall the product formula of Deligne–Laumon. In Section 8, we show the equality of the determinant and the central character using the product formula of Deligne–Laumon.

In Section 9, we construct a field extension T_{ζ}^{u} of E_{ζ} such that the restriction of $\tau_{n,\zeta,\chi,c}$ to $W_{T_{\zeta}^{\mathrm{u}}}$ is an induction of a character and $p \nmid [T_{\zeta}^{\mathrm{u}} : E_{\zeta}]$, which we call an imprimitive field. In Section 10, we show the equality of the refined Swan conductors. We see also that the constructed representations of W_K are actually of Swan conductor 1.

In Section 11, we show the equality of the epsilon factors. It is difficult to calculate the epsilon factors of irreducible representations of W_K of Swan conductor 1 directly, because primitive representations are involved. However, we know the equality of the epsilon factors up to p^e-th roots of unity if $n = p^e$, since we have already checked the conditions (i) and (ii) in the characterization. Using this fact and $p \nmid [T_{\zeta}^{\mathrm{u}} : E_{\zeta}]$, the problem is reduced to study an epsilon factor of a character. Next we reduce the problem to the case where the characteristic of K is p and $k = \mathbb{F}_p$. At this stage, it is possible to calculate the epsilon factor if $p \neq 2$. However, it is still difficult if $p = 2$, because the direct calculation of the epsilon factor involves an explicit study of the Artin reciprocity map for a wildly ramified extension with a non-trivial ramification filtration. This is a special phenomenon in the case where $p = 2$. We will avoid this difficulty by reducing the problem to the case where $e = 1$. In this case, we have already known the equality up to sign. Hence, it suffices to show the equality of non-zero real parts. This is easy, because the difficult study of the Artin reciprocity map involves only the imaginary part of the equality.

In Appendix A, we discuss a construction of irreducible representations of W_K of Swan conductor 1 in the cohomology of Artin–Schreier varieties. This geometric construction incorporates a twist by a "rectifier". We see that the "rectifier" parts come from the cohomology of Artin–Schreier varieties associated to quadratic forms. The Artin–Schreier varieties which we use have origins in studies of Lubin–Tate spaces in [IT17] and [IT21].

Acknowledgements

The authors would like to thank the handling editor Don Blasius and referees for helpful comments and suggestions. This work was supported by JSPS KAKENHI Grant Numbers 26707003, 20K03529, 21H00973.

Notation

For a finite abelian group A, let $A^{\vee}$ denote the character group $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{C}^{\times})$. For a non-archimedean local field K, let

- $\mathcal{O}_K$ denote the ring of integers of K,
- $\mathfrak{p}_K$ denote the maximal ideal of $\mathcal{O}_K$,
- v_K denote the normalized valuation of K which sends a uniformizer of K to 1,
- $\mathrm{ch}\, K$ denote the characteristic of K,
- G_K denote the absolute Galois group of K,
- W_K denote the Weil group of K,

- I_K denote the inertia subgroup of W_K,
- P_K denote the wild inertia subgroup of W_K,

and we put $U_K^m = 1 + \mathfrak{p}_K^m$ for any positive integer m.

1 Representations of finite groups

First, we recall a fact on representations of Heisenberg groups. Let G be a finite group with center Z. We assume the following:

(i) The group G/Z is an elementary abelian p-group.

(ii) For any $g \in G \setminus Z$, the map $c_g \colon G \to Z;\ g' \mapsto [g, g']$ is surjective.

Remark 1.1. *The map c_g in (ii) is a group homomorphism. Hence, Z is automatically an elementary abelian p-group.*

Let $\psi \in Z^\vee$ be a non-trivial character.

Proposition 1.2. *There is a unique irreducible representation ρ_ψ of G such that $\rho_\psi|_Z$ contains ψ. Moreover, we have $(\dim \rho_\psi)^2 = [G : Z]$ and we can construct ρ_ψ as follow: Take an abelian subgroup G_1 of G such that $Z \subset G_1$ and $2\dim_{\mathbb{F}_p}(G_1/Z) = \dim_{\mathbb{F}_p}(G/Z)$. Extend ψ to a character ψ_1 of G_1. Then $\rho_\psi = \mathrm{Ind}_{G_1}^G \psi_1$.*

Proof. The claims other than the construction of ρ_ψ is [BF83, (8.3.3) Proposition]. Note that if an abelian subgroup G_1 of G satisfies the conditions in the claim, then G_1/Z is a maximal totally isotropic subspace of G/Z under the pairing

$$(G/Z) \times (G/Z) \longrightarrow \mathbb{C}^\times;\ (gZ, g'Z) \mapsto \psi([g, g']).$$

Hence the construction follows from the proof of [BF83, (8.3.3) Proposition]. $\square$

Next, we consider representations of a semi-direct product of a Heisenberg group with a cyclic group. Let $A \subset \mathrm{Aut}(G)$ be a cyclic subgroup of order p^e+1 where $e = (\log_p[G : Z])/2$. We assume the following:

(iii) The group A acts on Z trivially.

(iv) For any non-trivial element $a \in A$, the action of a on G/Z fixes only the unit element.

We consider the semi-direct product $A \ltimes G$ by the action of A on G.

Lemma 1.3. *There is a unique irreducible representation ρ'_ψ of $A \ltimes G$ such that $\rho'_\psi|_G \simeq \rho_\psi$ and $\mathrm{tr}\, \rho'_\psi(a) = -1$ for every non-trivial element $a \in A$.*

Proof. The claim is proved in the proof of [BH06, 22.2 Lemma] if Z is cyclic and ψ is a faithful character. In fact, the same proof works also in our case. $\square$

Corollary 1.4. *There exists a unique representation ρ'_ψ of $A \ltimes G$ such that $\rho'_\psi|_Z \simeq \psi^{\oplus p^e}$ and $\mathrm{tr}\, \rho'_\psi(a) = -1$ for every non-trivial element $a \in A$. Further, the representation $\rho'_\psi|_G$ is irreducible.*

Proof. First we show the existence. We take the representation ρ'_ψ in Lemma 1.3. Then ρ'_ψ has a central character and the central character is equal to ψ by Proposition 1.2. This shows the existence.

We show the uniqueness and the irreducibility of $\rho'_\psi|_G$. Assume that ρ'_ψ satisfies the condition in the claim. Take an irreducible subrepresentation ρ_ψ of $\rho'_\psi|_G$. Then ρ_ψ satisfies the condition of Proposition 1.2. Hence, $\dim \rho_\psi = p^e$. Then we have $\rho_\psi = \rho'_\psi|_G$ and $\rho'_\psi|_G$ is irreducible. Such ρ_ψ is unique by Lemma 1.3. $\square$

2 Galois representations

2.1 Swan conductor

Let K be a non-archimedean local field with residue field k. Let p be the characteristic of k. Let f be the extension degree of k over $\mathbb{F}_p$. We put $q = p^f$.

Let

$$\mathrm{Art}_K \colon K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$$

be the Artin reciprocity map, which sends a uniformizer to a lift of the geometric Frobenius element.

Let τ be a finite dimensional irreducible continuous representation of W_K over $\mathbb{C}$. Let $\Psi \colon K \to \mathbb{C}^\times$ be a non-trivial additive character. Let $\varepsilon(\tau, s, \Psi)$ denote the Deligne–Langlands local constant of τ with respect to Ψ. We simply write $\varepsilon(\tau, \Psi)$ for $\varepsilon(\tau, 1/2, \Psi)$.

We define an unramified character $\omega_s \colon K^\times \to \mathbb{C}^\times$ by $\omega_s(\varpi) = q^{-s}$ for $s \in \mathbb{R}$, where ϖ is a uniformizer of K. We recall that

$$\varepsilon(\tau, s, \Psi) = \varepsilon(\tau \otimes \omega_s, 0, \Psi) \tag{2.1}$$

(*cf.* [Tat79, (3.6.4)]).

We define $\psi_0 \in \mathbb{F}_p^\vee$ by $\psi_0(1) = e^{2\pi\sqrt{-1}/p}$. We take an additive character $\psi_K \colon K \to \mathbb{C}^\times$ such that $\psi_K(x) = \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x}))$ for $x \in \mathcal{O}_K$. By [BH06, 29.4 Proposition], there exists an integer $\mathrm{sw}(\tau)$ such that

$$\varepsilon(\tau, s, \psi_K) = q^{-\mathrm{sw}(\tau)s}\varepsilon(\tau, 0, \psi_K).$$

We put $\mathrm{Sw}(\tau) = \max\{\mathrm{sw}(\tau), 0\}$, which we call the Swan conductor of τ.

2.2 Construction

In this subsection, we construct a group Q which acts on a curve C over an algebraic closure of k. By using this action of Q and Frobenius action, we construct a representation of a semi-direct product $Q \rtimes \mathbb{Z}$ in etale cohomology of C. Then we use the representation of $Q \rtimes \mathbb{Z}$ to construct a representation of a Weil group.

We fix an algebraic closure K^{ac} of K. Let k^{ac} be the residue field of K^{ac}. Let n be a positive integer. We write $n = p^e n'$ with $(p, n') = 1$. Throughout this paper, we assume that $e \geq 1$. Let

$$Q = \left\{ (a, b, c) \;\middle|\; a \in \mu_{p^e+1}(k^{\mathrm{ac}}),\ b, c \in k^{\mathrm{ac}},\ \ b^{p^{2e}} + b = 0,\ \ c^p - c + b^{p^e+1} = 0 \right\}$$

be the group whose multiplication is given by

$$(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = \left(a_1 a_2, b_1 + a_1 b_2, c_1 + c_2 + \sum_{i=0}^{e-1} \bigl(a_1 b_1^{p^e} b_2\bigr)^{p^i} \right).$$

Remark 2.1. *The construction of the group Q has its origin in a study of the automorphism of a curve C defined below. We can check that the above multiplication gives a group structure on Q directly, but it's also possible to show this by checking that the inclusion from Q to the automorphism group of C defined below is compatible with the multiplications.*

Note that $|Q| = p^{2e+1}(p^e + 1)$. Let $Q \rtimes \mathbb{Z}$ be a semidirect product, where $m \in \mathbb{Z}$ acts on Q by $(a, b, c) \mapsto (a^{p^{-m}}, b^{p^{-m}}, c^{p^{-m}})$. We put

$$\mathrm{Fr}(m) = ((1, 0, 0), m) \in Q \rtimes \mathbb{Z} \tag{2.2}$$

for $m \in \mathbb{Z}$.

Let C be the smooth affine curve over k^{ac} defined by

$$x^p - x = y^{p^e+1} \quad \text{in } \mathbb{A}^2_{k^{\mathrm{ac}}}.$$

We define a right action of $Q \rtimes \mathbb{Z}$ on C by

$$(x,y)((a,b,c),0) = \left(x + \sum_{i=0}^{e-1} (by)^{p^i} + c, a(y+b^{p^e}) \right),$$
$$(x,y)\,\mathrm{Fr}(1) = (x^p, y^p).$$

We consider the morphisms

$$h \colon \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}};\ x \mapsto x^p - x,$$
$$\pi \colon \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}};\ y \mapsto y^{p^e+1}.$$

Then we have the fiber product

$$\begin{tikzcd} C \arrow[r, "h'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathbb{A}^1_{k^{\mathrm{ac}}} \arrow[d, "\pi"] \\ \mathbb{A}^1_{k^{\mathrm{ac}}} \arrow[r, "h"'] & \mathbb{A}^1_{k^{\mathrm{ac}}} \end{tikzcd}$$

where π' and h' are the natural projections to the first and second coordinates respectively. Let $g = ((a,b,c),m) \in Q \rtimes \mathbb{Z}$. We consider the morphism

$$g_0 \colon \mathbb{A}^1_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}};\ y \mapsto \left(a(y+b^{p^e})\right)^{p^m}.$$

Let ℓ be a prime number different from p. Then we have a natural isomorphism

$$c_g \colon g_0^* h'_* \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} h'_* g^* \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} h'_* \overline{\mathbb{Q}}_\ell.$$

We take an isomorphism $\iota \colon \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. We sometimes view a character over $\mathbb{C}$ as a character over $\overline{\mathbb{Q}}_\ell$ by ι. Let $\psi \in \mathbb{F}_p^\vee$. We write $\mathcal{L}_\psi$ for the Artin–Schreier $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{k^{\mathrm{ac}}}$ associated to ψ, which is equal to $\mathfrak{F}(\psi)$ in the notation of [Del77, Sommes trig. 1.8 (i)]. Then we have a decomposition $h_* \overline{\mathbb{Q}}_\ell = \bigoplus_{\psi \in \mathbb{F}_p^\vee} \mathcal{L}_\psi$. This decomposition gives a canonical isomorphism

$$h'_* \overline{\mathbb{Q}}_\ell \cong \pi^* h_* \overline{\mathbb{Q}}_\ell \cong \bigoplus_{\psi \in \mathbb{F}_p^\vee} \pi^* \mathcal{L}_\psi. \tag{2.3}$$

The isomorphisms c_g and (2.3) induce $c_{g,\psi} \colon g_0^* \pi^* \mathcal{L}_\psi \to \pi^* \mathcal{L}_\psi$. We define a left action of $Q \rtimes \mathbb{Z}$ on $H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi)$ by

$$H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) \xrightarrow{g_0^*} H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, g_0^* \pi^* \mathcal{L}_\psi) \xrightarrow{c_{g,\psi}} H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi)$$

for $g \in Q \rtimes \mathbb{Z}$. Let τ_ψ be the representation of $Q \rtimes \mathbb{Z}$ over $\mathbb{C}$ defined by $H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi)$ and ι. For $\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee$, let $\mathcal{K}_\theta$ be the smooth Kummer $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}$ associated to θ. We view $\mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$ as a subgroup of Q by $(a,c) \mapsto (a,0,c)$.

Lemma 2.2. *We have a natural isomorphism*

$$H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^* \mathcal{L}_\psi) \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}} H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta),$$

which is compatible with the actions of $\mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$ where $(a,c) \in \mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$ acts on $H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta)$ by $\theta(a)\psi(c)$. Further, we have

$$\dim H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta) = 1$$

for any $\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}$.

Proof. By the projection formula, we have natural isomorphisms

$$\pi_*\pi^*\mathcal{L}_\psi \simeq \pi_*(\pi^*\mathcal{L}_\psi \otimes \overline{\mathbb{Q}}_\ell) \simeq \mathcal{L}_\psi \otimes \pi_*\overline{\mathbb{Q}}_\ell$$

on $\mathbb{A}^1_{k^{\mathrm{ac}}}$. Further, we have

$$\pi_*\overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee} \mathcal{K}_\theta$$

on $\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}$, since π is a finite etale $\mu_{p^e+1}(k^{\mathrm{ac}})$-covering over $\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}$. Therefore, we have

$$\pi_*\pi^*\mathcal{L}_\psi \simeq \mathcal{L}_\psi \otimes \pi_*\overline{\mathbb{Q}}_\ell \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee} \mathcal{L}_\psi \otimes \mathcal{K}_\theta \tag{2.4}$$

on $\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}$. Let $\{0\}$ denote the origin of $\mathbb{A}^1_{k^{\mathrm{ac}}}$. Let $i\colon \{0\} \to \mathbb{A}^1_{k^{\mathrm{ac}}}$ and $j\colon \mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}}$ be the natural immersions. From the exact sequence

$$0 \to j_!j^*\pi^*\mathcal{L}_\psi \to \pi^*\mathcal{L}_\psi \to i_*i^*\pi^*\mathcal{L}_\psi \to 0,$$

we have the exact sequence

$$0 \to H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \to H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi) \to H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi) \to 0, \tag{2.5}$$

since $H^0_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi) = 0$ and $H^1(\{0\}, i^*\pi^*\mathcal{L}_\psi) = 0$. Note that $H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \simeq \psi$. By (2.4), we have isomorphisms

$$H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi) \simeq H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi_*\pi^*\mathcal{L}_\psi) \simeq \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee} H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta). \tag{2.6}$$

We know that

$$\dim H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi \otimes \mathcal{K}_\theta) = 1 \tag{2.7}$$

for any $\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee$ by the proof of [IT17, Lemma 7.1] (*cf.* [IT23, (2.3)]). Since the composition of

$$H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \to H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \pi^*\mathcal{L}_\psi)$$

and (2.6) is compatible with the actions of $\mu_{p^e+1}(k^{\mathrm{ac}}) \times \mathbb{F}_p$, it factors through an isomorphism $H^0(\{0\}, i^*\pi^*\mathcal{L}_\psi) \simeq H^1_{\mathrm{c}}(\mathbb{G}_{\mathrm{m},k^{\mathrm{ac}}}, \mathcal{L}_\psi)$ by (2.7). Then the claim follows from (2.5), (2.6) and (2.7). $\square$

Let $\varrho\colon \mu_2(k) \hookrightarrow \mathbb{C}^\times$ be the non-trivial group homomorphism, if $p \neq 2$. We define a character $\theta_0 \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee$ by

$$\theta_0(a) = \begin{cases} \varrho(a^{(p^e+1)/2}) & \text{if } p \neq 2, \\ 1 & \text{if } p = 2 \end{cases} \tag{2.8}$$

for $a \in \mu_{p^e+1}(k^{\mathrm{ac}})$. For an integer m and a positive odd integer m', let $\left(\frac{m}{m'}\right)$ denote the Jacobi symbol. For an odd prime p, we set

$$\epsilon(p) = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4, \\ \sqrt{-1} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

We have $\epsilon(p)^2 = \left(\frac{-1}{p}\right)$. We define a representation τ_n of $Q \rtimes \mathbb{Z}$ as the twist of τ_{ψ_0} by the character

$$Q \rtimes \mathbb{Z} \to \mathbb{C}^\times;\ ((a,b,c),m) \mapsto \begin{cases} \theta_0(a)^n \Big(\big(-\epsilon(p)\big(\frac{-2n'}{p}\big)\big)^n p^{-\frac{1}{2}}\Big)^m & \text{if } p \neq 2, \\ \Big((-1)^{\frac{n(n-2)}{8}} p^{-\frac{1}{2}}\Big)^m & \text{if } p = 2. \end{cases} \tag{2.9}$$

The value of this character is related to a quadratic Gauss sum. A geometric origin of this character is given in (A.3). Let $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$. We take a uniformizer ϖ of K. We choose an element $\varphi'_\zeta \in K^{\mathrm{ac}}$ such that $\varphi'^{n'}_\zeta = \zeta\varpi$ and set $E_\zeta = K(\varphi'_\zeta)$. We choose elements $\alpha_\zeta, \beta_\zeta, \gamma_\zeta \in K^{\mathrm{ac}}$ such that

$$\alpha_\zeta^{p^e+1} = -\varphi'_\zeta, \quad \beta_\zeta^{p^{2e}} + \beta_\zeta = -\alpha_\zeta^{-1}, \quad \gamma_\zeta^p - \gamma_\zeta = \beta_\zeta^{p^e+1}. \tag{2.10}$$

For $\sigma \in W_{E_\zeta}$, we set

$$a_\sigma = \sigma(\alpha_\zeta)/(\alpha_\zeta), \quad b_\sigma = a_\sigma \sigma(\beta_\zeta) - \beta_\zeta, \quad c_\sigma = \sigma(\gamma_\zeta) - \gamma_\zeta + \sum_{i=0}^{e-1} (b_\sigma^{p^e}(\beta_\zeta + b_\sigma))^{p^i}. \tag{2.11}$$

Then we have $a_\sigma, b_\sigma, c_\sigma \in \mathcal{O}_{K^{\mathrm{ac}}}$. For $\sigma \in W_{E_\zeta}$, we put $n_\sigma = v_{E_\zeta}(\mathrm{Art}_{E_\zeta}^{-1}(\sigma))$. We have a homomorphism

$$\Theta_\zeta \colon W_{E_\zeta} \longrightarrow Q \rtimes \mathbb{Z};\ \sigma \mapsto \big((\bar{a}_\sigma, \bar{b}_\sigma, \bar{c}_\sigma), f n_\sigma\big). \tag{2.12}$$

Lemma 2.3. *The image of the homomorphism Θ_ζ is $Q \rtimes (f\mathbb{Z})$.*

Proof. It suffices to show that the image of $I_{E_\zeta} \subset W_{E_\zeta}$ under Θ_ζ is equal to $Q \subset Q \rtimes \mathbb{Z}$, since the homomorphism $W_{E_\zeta} \to f\mathbb{Z};\ \sigma \mapsto f n_\sigma$ is surjective. We put $N_\zeta = E_\zeta(\alpha_\zeta, \beta_\zeta, \gamma_\zeta)$. Then the kernel of Θ_ζ is equal to I_{N_ζ} by the definition. Hence we have an injection $I_{E_\zeta}/I_{N_\zeta} \hookrightarrow Q$. This injection is actually a bijection, since N_ζ is a totally ramified extension over E_ζ of degree $p^{2e+1}(p^e+1)$, which equals to $|Q|$. Therefore, we obtain the claim. $\square$

We write $\tau_{n,\zeta}$ for the representation of W_{E_ζ} given by Θ_ζ and τ_n. Recall that c is an element of $\mathbb{C}^\times$. Let $\phi_c \colon W_{E_\zeta} \to \mathbb{C}^\times$ be the character defined by $\phi_c(\sigma) = c^{n_\sigma}$. We have the isomorphism $\varphi'^{\mathbb{Z}}_\zeta \times \mathcal{O}_{E_\zeta}^\times \simeq E_\zeta{}^\times$ given by the multiplication. Let $\mathrm{Frob}_p \colon k^\times \to k^\times$ be the inverse of the p-th power map. We consider the following composition:

$$\lambda_\zeta \colon W_{E_\zeta}^{\mathrm{ab}} \simeq E_\zeta{}^\times \simeq \varphi'^{\mathbb{Z}}_\zeta \times \mathcal{O}_{E_\zeta}^\times \xrightarrow{\mathrm{pr}_2} \mathcal{O}_{E_\zeta}^\times \xrightarrow{\mathrm{can.}} k^\times \xrightarrow{\mathrm{Frob}_p^e} k^\times.$$

We put

$$\tau_{n,\zeta,\chi,c} = \tau_{n,\zeta} \otimes (\chi \circ \lambda_\zeta) \otimes \phi_c \quad \text{and} \quad \tau_{\zeta,\chi,c} = \mathrm{Ind}_{E_\zeta/K} \tau_{n,\zeta,\chi,c}. \tag{2.13}$$

We will see that $\tau_{\zeta,\chi,c}$ is an irreducible representation of Swan conductor 1 in Proposition 10.8. This Galois representation $\tau_{\zeta,\chi,c}$ is our main object in this paper. We will study several invariants associated to this, for example its determinant and epsilon factor.

2.3 Characterization

We put

$$Q_0 = \{(1,b,c) \in Q\}, \quad F = \{(1,0,c) \in Q \mid c \in \mathbb{F}_p\}.$$

We identify $\mathbb{F}_p$ with F by $c \mapsto (1,0,c)$.

Lemma 2.4. *For any $g = (1,b,c) \in Q_0$ with $b \neq 0$, the map $Q_0 \to F;\ g' \mapsto [g,g']$ is surjective.*

Proof. For $(1,b_1,c_1),(1,b_2,c_2) \in Q_0$, we have

$$[(1,b_1,c_1),(1,b_2,c_2)] = \left(1,0,\sum_{i=0}^{e-1}(b_1^{p^e}b_2 - b_1b_2^{p^e})^{p^i}\right).$$

If $b_1 \neq 0$, then

$$\left\{b \in k^{\mathrm{ac}} \mid b^{p^{2e}} + b = 0\right\} \to \mathbb{F}_{p^e};\ b_2 \to b_1^{p^e}b_2 - b_1b_2^{p^e}$$

is surjective. The claim follows from the surjectivity of $\mathrm{Tr}_{\mathbb{F}_{p^e}/\mathbb{F}_p}$. $\square$

By this lemma, we can apply the results from Section 1 to our situation with $G = Q_0$, $Z = F$ and $A = \mu_{p^e+1}(k^{\mathrm{ac}})$, where the action of $\mu_{p^e+1}(k^{\mathrm{ac}})$ on Q_0 is given by the embedding

$$\mu_{p^e+1}(k^{\mathrm{ac}}) \longrightarrow Q;\ a \mapsto (a,0,0)$$

and the conjugation. Let τ^0 denote the unique representation of Q characterized by

$$\tau^0|_F \simeq \psi_0^{\oplus p^e}, \quad \mathrm{Tr}\,\tau^0((a,0,0)) = -1 \tag{2.14}$$

for $a \in \mu_{p^e+1}(k^{\mathrm{ac}}) \setminus \{1\}$ (*cf.* Corollary 1.4).

We have a decomposition

$$\tau^0 = \bigoplus_{\theta \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}} L_\theta \tag{2.15}$$

such that $a \in \mu_{p^e+1}(k^{\mathrm{ac}})$ acts on L_θ by $\theta(a)$, since the both sides of (2.15) have the same character as representations of $\mu_{p^e+1}(k^{\mathrm{ac}})$. For a positive integer m dividing p^e+1, we consider $\mu_m(k^{\mathrm{ac}})^\vee$ as a subset of $\mu_{p^e+1}(k^{\mathrm{ac}})^\vee$ by the dual of the surjection

$$\mu_{p^e+1}(k^{\mathrm{ac}}) \to \mu_m(k^{\mathrm{ac}});\ x \to x^{\frac{p^e+1}{m}}.$$

We simply write Q for the subgroup $Q \times \{0\} \subset Q \rtimes \mathbb{Z}$.

Lemma 2.5. *We have $\tau_{\psi_0}|_Q \simeq \tau^0$.*

Proof. The representation $\tau_{\psi_0}|_Q$ satisfies the characterization (2.14) by Lemma 2.2. Hence $\tau_{\psi_0}|_Q$ is isomorphic to τ^0. $\square$

Corollary 2.6. *The representation $\tau_{\psi_0}|_{Q_0}$ is irreducible.*

Proof. This follows from Corollary 1.4, (2.14) and Lemma 2.5. $\square$

For any odd prime p, we have

$$\sum_{x \in \mathbb{F}_p^\times} \psi_0(x^2) = \sum_{x \in \mathbb{F}_p^\times} \left(\frac{x}{p}\right)\psi_0(x) = \epsilon(p)\sqrt{p} \tag{2.16}$$

by Gauss.

Lemma 2.7. *We have*

$$\operatorname{Tr}\tau_{\psi_0}\big(\mathrm{Fr}(1)\big)=\begin{cases}-\epsilon(p)\sqrt{p} & \text{if } p\neq 2,\\ 0 & \text{if } p=2.\end{cases}$$

Proof. By the Lefschetz trace formula, we have

$$\sum_{x\in\mathbb{A}^1(\mathbb{F}_p)}\operatorname{Tr}(\mathrm{Fr}_p,(\pi^*\mathcal{L}_{\psi_0})_x)=\sum_{i=0}^{2}(-1)^i\operatorname{Tr}\big(\mathrm{Fr}_p,H^i_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}},\pi^*\mathcal{L}_\psi)\big)$$

where Fr_p denotes the geometric p-th power Frobenius morphism. Since $H^i_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}},\pi^*\mathcal{L}_\psi)$ vanishes for $i=0,2$, we have

$$\begin{aligned}\operatorname{Tr}\tau_{\psi_0}\big(\mathrm{Fr}(1)\big)&=-\sum_{x\in\mathbb{A}^1(\mathbb{F}_p)}\operatorname{Tr}(\mathrm{Fr}_p,(\pi^*\mathcal{L}_{\psi_0})_x)\\&=-\sum_{x\in\mathbb{F}_p}\psi_0(x^{p^e+1})=-\sum_{x\in\mathbb{F}_p}\psi_0(x^2)=\begin{cases}-\epsilon(p)\sqrt{p} & \text{if } p\neq 2,\\ 0 & \text{if } p=2,\end{cases}\end{aligned}$$

where we use (2.16) in the last equality. $\square$

We assume $p=2$ in this paragraph. We take $b_0\in\mathbb{F}_{2^{2e}}$ such that $\operatorname{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0)=1$. Further, we put

$$c_0=b_0^{2^e}+\sum_{0\le i<j\le e-1}b_0^{2^{e+i}+2^j}. \tag{2.17}$$

Then we have

$$\begin{aligned}c_0^2-c_0&=b_0^{2^{e+1}}+b_0^{2^e}+\sum_{0\le i<j\le e-1}b_0^{2^{e+i+1}+2^{j+1}}+\sum_{0\le i<j\le e-1}b_0^{2^{e+i}+2^j}\\&=b_0^{2^{e+1}}+b_0^{2^e}+\sum_{i=0}^{e-2}b_0^{2^{e+i+1}+2^e}+\sum_{j=1}^{e-1}b_0^{2^e+2^j}\\&=b_0^{2^{e+1}}+b_0^{2^e}+b_0^{2^e}(1+b_0+b_0^{2^e})=b_0^{2^e+1},\end{aligned}\tag{2.18}$$

where we use $\operatorname{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0)=1$ at the third equality. We put

$$\boldsymbol{g}=\big((1,b_0,c_0),-1\big)\in Q\rtimes\mathbb{Z}.$$

Lemma 2.8. *We assume that $p=2$. Then we have* $\operatorname{Tr}\tau_{\psi_0}(\boldsymbol{g}^{-1})=-2$.

Proof. We note that

$$\boldsymbol{g}^{-1}=\mathrm{Fr}(1)\left(\left(1,b_0,c_0+\sum_{i=0}^{e-1}(b_0^{2^e+1})^{2^i}\right),0\right). \tag{2.19}$$

For $y\in k^{\mathrm{ac}}$ satisfying $y^2+b_0^{2^e}=y$, we take $x_y\in k^{\mathrm{ac}}$ such that $x_y^2-x_y=y^{2^e+1}$. We take $y_0\in k^{\mathrm{ac}}$ such that $y_0^2+b_0^{2^e}=y_0$. Then, by the Lefschetz trace formula and (2.19), we have

$$\begin{aligned}\operatorname{Tr}\tau_{\psi_0}(\boldsymbol{g}^{-1})&=-\sum_{y^2+b_0^{2^e}=y}\operatorname{Tr}(\boldsymbol{g}^{-1},(\pi^*\mathcal{L}_{\psi_0})_y)\\&=-\sum_{y^2+b_0^{2^e}=y}\psi_0\left(x_y^2-x_y+\sum_{i=0}^{e-1}(b_0y^2)^{2^i}+c_0+\sum_{i=0}^{e-1}(b_0^{2^e+1})^{2^i}\right)\\&=-\sum_{z\in\mathbb{F}_2}\psi_0\left((y_0+z)^{2^e+1}+\sum_{i=0}^{e-1}\big(b_0(y_0+z)\big)^{2^i}+c_0\right)=-2,\end{aligned}$$

where we change a variable by $y = y_0 + z$ at the second equality, and use

$$y_0^{2^e+1} + \sum_{i=0}^{e-1} (b_0 y_0)^{2^i} = y_0 \left(y_0 + \sum_{i=0}^{e-1} b_0^{2^{e+i}} \right) + \sum_{i=0}^{e-1} b_0^{2^i} \left(y_0 + \sum_{j=0}^{i-1} b_0^{2^{e+j}} \right) = c_0,$$

$$y_0^{2^e} + y_0 + \sum_{i=0}^{e-1} b_0^{2^i} = \sum_{i=0}^{e-1} (y_0^2 + y_0)^{2^i} + \sum_{i=0}^{e-1} b_0^{2^i} = \mathrm{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0) = 1$$

at the last equality. □

Proposition 2.9. *The representation τ_{ψ_0} is characterized by $\tau_{\psi_0}|_Q \simeq \tau^0$ and*

$$\begin{cases} \mathrm{Tr}\, \tau_{\psi_0}\big(\mathrm{Fr}(1)\big) = -\epsilon(p)\sqrt{p} & \text{if } p \neq 2, \\ \mathrm{Tr}\, \tau_{\psi_0}(\boldsymbol{g}^{-1}) = -2 & \text{if } p = 2. \end{cases}$$

In particular, τ_{ψ_0} does not depend on the choice of ℓ and ι.

Proof. This follows from Lemma 2.5, Lemma 2.7 and Lemma 2.8. □

3 Representations of general linear algebraic groups

3.1 Simple supercuspidal representation

Let π be an irreducible supercuspidal representation of $GL_n(K)$ over $\mathbb{C}$. Let $\varepsilon(\pi, s, \Psi)$ denote the Godement–Jacquet local constant of π with respect to the non-trivial character $\Psi\colon K \to \mathbb{C}^\times$. We simply write $\varepsilon(\pi, \Psi)$ for $\varepsilon(\pi, 1/2, \Psi)$. By [GJ72, Theorem 3.3 (4)], there exists an integer $\mathrm{sw}(\pi)$ such that

$$\varepsilon(\pi, s, \psi_K) = q^{-\mathrm{sw}(\pi)s} \varepsilon(\pi, 0, \psi_K).$$

We put $\mathrm{Sw}(\pi) = \max\{\mathrm{sw}(\pi), 0\}$, which we call the Swan conductor of π.

Definition 3.1. *An irreducible supercuspidal representation π of $GL_n(K)$ over $\mathbb{C}$ is called simple supercuspidal if* $\mathrm{Sw}(\pi) = 1$.

This definition is equivalent to [IT18, Definition 1.1] by [IT18, Proposition 1.3].

3.2 Construction

In the following, we construct a smooth representation $\pi_{\zeta,\chi,c}$ of $GL_n(K)$ for each triple $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$.

Let $B \subset M_n(k)$ be the subring consisting of upper triangular matrices. Let $\mathfrak{I} \subset M_n(\mathcal{O}_K)$ be the inverse image of B under the reduction map $M_n(\mathcal{O}_K) \to M_n(k)$. Then $\mathfrak{I}$ is a hereditary $\mathcal{O}_K$-order (cf. [BK93, (1.1)]). Let $\mathfrak{P}$ denote the Jacobson radical of the order $\mathfrak{I}$. We put $U_{\mathfrak{I}}^1 = 1 + \mathfrak{P} \subset GL_n(\mathcal{O}_K)$. We set

$$\varphi_\zeta = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ \zeta\varpi & \mathbf{0} \end{pmatrix} \in M_n(K) \quad \text{and} \quad L_\zeta = K(\varphi_\zeta).$$

Then, L_ζ is a totally ramified extension of K of degree n.

We put $\varphi_{\zeta,n} = n'\varphi_\zeta$ and

$$\epsilon_0 = \begin{cases} (n'+1)/2 & \text{if } p^e = 2, \\ 0 & \text{if } p^e \neq 2. \end{cases}$$

We define a character $\Lambda_{\zeta,\chi,c}\colon L_\zeta^\times U_{\mathfrak{J}}^1 \to \mathbb{C}^\times$ by

$$\Lambda_{\zeta,\chi,c}(\varphi_\zeta) = (-1)^{n-1+\epsilon_0 f} c, \quad \Lambda_{\zeta,\chi,c}(x) = \chi(\bar{x}) \quad \text{for } x \in \mathcal{O}_K^\times,$$
$$\Lambda_{\zeta,\chi,c}(x) = (\psi_K \circ \mathrm{tr})(\varphi_{\zeta,n}^{-1}(x-1)) \quad \text{for } x \in U_{\mathfrak{J}}^1,$$

where tr means the trace as an element of $M_n(K)$. We put

$$\pi_{\zeta,\chi,c} = \text{c-Ind}_{L_\zeta^\times U_{\mathfrak{J}}^1}^{GL_n(K)} \Lambda_{\zeta,\chi,c}.$$

Then, $\pi_{\zeta,\chi,c}$ is a simple supercuspidal representation of $GL_n(K)$, and every simple supercuspidal representation is isomorphic to $\pi_{\zeta,\chi,c}$ for a uniquely determined $(\zeta,\chi,c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \mathbb{C}^\times$ by [IT18, Proposition 1.3]. The representation $\pi_{\zeta,\chi,c}$ contains the m-simple stratum $\left[\mathfrak{J}, 1, 0, \varphi_{\zeta,n}^{-1}\right]$ in the sense of [BH14, 2.1].

Proposition 3.2. *$\varepsilon(\pi_{\zeta,\chi,c}, \psi_K) = (-1)^{n-1+\epsilon_0 f}\chi(n')c$.*

Proof. This follows from [BH99, 6.1 Lemma 2 and 6.3 Proposition 1]. □

4 Local Langlands correspondence

Our main theorem is the following:

Theorem 4.1. *The representations $\pi_{\zeta,\chi,c}$ and $\tau_{\zeta,\chi,c}$ correspond via the local Langlands correspondence.*

To prove this theorem, we recall a characterization of the local Langlands correspondence for epipelagic representations due to Bushnell–Henniart. Recall that $\Psi\colon K \to \mathbb{C}^\times$ is a non-trivial character. The following lemma is a special case of [DH81, Théorème].

Lemma 4.2. *([BH14, 2.3 Lemma]) Let τ be an irreducible smooth representation of W_K such that $\mathrm{sw}(\tau) \geq 1$. Then, there exists $\gamma_{\tau,\Psi} \in K^\times$ such that*

$$\varepsilon(\chi \otimes \tau, s, \Psi) = \chi(\gamma_{\tau,\Psi})^{-1} \varepsilon(\tau, s, \Psi)$$

for any tamely ramified character χ of W_K. This property determines the coset $\gamma_{\tau,\Psi} U_K^1$ uniquely.

Definition 4.3. *Let τ be an irreducible smooth representation of W_K such that $\mathrm{sw}(\tau) \geq 1$. We take $\gamma_{\tau,\Psi}$ as in Lemma 4.2. We put*

$$\mathrm{rsw}(\tau, \Psi) = \gamma_{\tau,\Psi}^{-1} \in K^\times / U_K^1,$$

which we call the refined Swan conductor of τ with respect to Ψ.

Remark 4.4. *By (2.1), we have $v_K\big(\mathrm{rsw}(\tau, \psi_K)\big) = \mathrm{Sw}(\tau)$ in Definition 4.3.*

Lemma 4.5. *Let π be an irreducible supercuspidal representation of $GL_n(K)$ such that $\mathrm{sw}(\pi) \geq 1$.*
1. *There exists $\gamma_{\pi,\Psi} \in K^{\times}$ such that*

$$\varepsilon(\chi \otimes \pi, s, \Psi) = \chi(\gamma_{\pi,\Psi})^{-1}\varepsilon(\pi, s, \Psi)$$

for any tamely ramified character χ of $K^{\times}$. This property determines the coset $\gamma_{\pi,\Psi}U_K^1$ uniquely.
2. *Let $[\mathfrak{A}, m, 0, \alpha]$ be a simple stratum contained in π. Then we have $\gamma_{\pi,\Psi} \equiv \det\alpha \mod U_K^1$.*

Proof. The first statement is [BH99, 1.4 Theorem (i)]. The second statement follows from [BH99, 1.4 Remark]. □

Definition 4.6. *Let π be an irreducible supercuspidal representation of $GL_n(K)$ such that $\mathrm{sw}(\pi) \geq 1$. We take $\gamma_{\pi,\Psi}$ as in Lemma 4.5. Then we put*

$$\mathrm{rsw}(\pi, \Psi) = \gamma_{\pi,\Psi}^{-1} \in K^{\times}/U_K^1,$$

which we call the refined Swan conductor of π with respect to Ψ.

Remark 4.7. *We have $v_K\big(\mathrm{rsw}(\pi, \psi_K)\big) = \mathrm{Sw}(\pi)$ in Definition 4.6.*

For an irreducible supercuspidal representation π of $GL_n(K)$, let ω_π denote the central character of π.

Proposition 4.8. *([BH14, 2.3 Proposition]) Let π be a simple supercuspidal representation of $GL_n(K)$. The Langlands parameter for π is characterized as the n-dimensional irreducible smooth representation τ of W_K satisfying*

(i) $\det\tau = \omega_\pi$,

(ii) $\mathrm{rsw}(\tau, \psi_K) = \mathrm{rsw}(\pi, \psi_K)$,

(iii) $\varepsilon(\tau, \psi_K) = \varepsilon(\pi, \psi_K)$.

We will show that $\tau_{\zeta,\chi,c}$ and $\pi_{\zeta,\chi,c}$ satisfy the conditions of Proposition 4.8 in Proposition 8.6, Proposition 10.5, Lemma 10.7 and Proposition 11.6.

5 General facts on epsilon factors

In this section, we recall some general facts on epsilon factors.

For a finite separable extension L over K, we put $\Psi_L = \Psi \circ \mathrm{Tr}_{L/K}$ and let

$$\lambda(L/K, \Psi) = \frac{\varepsilon(\mathrm{Ind}_{L/K}1, s, \Psi)}{\varepsilon(1, s, \Psi_L)}$$

denote the Langlands constant which is independent of s, where 1 is the trivial representation of W_L (cf. [BH06, 30.4]).

Proposition 5.1. *Let τ be a finite dimensional smooth representation of W_K such that $\tau|_{P_K}$ is irreducible and non-trivial. Let L be a tamely ramified finite extension of K. Then we have*

$$\varepsilon(\tau|_{W_L}, \Psi_L) = \lambda(L/K, \Psi)^{-\dim\tau}\delta_{L/K}(\mathrm{rsw}(\tau, \Psi))\varepsilon(\tau, \Psi)^{[L:K]}.$$

Proof. This is proved by the same arguments as [BH06, 48.3 Proposition]. □

Proposition 5.2. *Let τ be a finite dimensional smooth representation of W_K such that $\tau|_{P_K}$ does not contain the trivial character.*
1. *If ϕ is a tamely ramified character of W_K, then we have $\mathrm{rsw}(\tau \otimes \phi, \Psi) = \mathrm{rsw}(\tau, \Psi)$.*
2. *Let L be a tamely ramified finite extension of K. Then we have*

$$\mathrm{rsw}(\tau|_{W_L}, \Psi_L) = \mathrm{rsw}(\tau, \Psi) \mod U_L^1.$$

Proof. This is [BH06, 48.1 Theorem (2) and (3)]. □

For a non-trivial character ξ of $K^\times$, the level of ξ means the least integer $m \geq 0$ such that ξ is trivial on U_K^{m+1}.

Proposition 5.3. *Let ξ be a character of $K^\times$ of level $m \geq 1$. Assume that $\gamma \in K^\times$ satisfies*

$$\xi(1+x) = \Psi(\gamma x)$$

for $x \in \mathfrak{p}_K^{[m/2]+1}$.
1. *We have $\mathrm{rsw}(\xi, \Psi) = \gamma^{-1}$.*
2. *We have*

$$\varepsilon(\xi, \Psi) = q^{[(m+1)/2]-(m+1)/2} \sum_{y \in U_K^{[(m+1)/2]}/U_K^{[m/2]+1}} \xi(\gamma y)^{-1} \Psi(\gamma y).$$

Proof. The claim 1 follows from [BH06, 23.8 Stability theorem]. The claim 2 follows from [BH06, 23.5 Lemma 1, (23.6.2) and 23.6 Proposition]. □

For a finite Galois extension L of K, let $\psi_{L/K}$ denote the Herbrand function of L/K and $\mathrm{Gal}(L/K)_i$ denote the lower numbering i-th ramification subgroup of $\mathrm{Gal}(L/K)$ for $i \geq 0$ (*cf.* [Ser68, IV]). We use the following lemmas to calculate the refined Swan conductor of a character of a Weil group.

Lemma 5.4. *Let m be a positive integer dividing f. Let h be a positive integer that is prime to p and less than $p^m v_K(p)/(p^m - 1)$. Let L be a Galois extension of K defined by $x^{p^m} - x = 1/\varpi^h$. Then we have*

$$\mathrm{Gal}(L/K)_i = \begin{cases} \mathrm{Gal}(L/K) & \text{if } i \leq h, \\ \{1\} & \text{if } i > h \end{cases}$$

and

$$\psi_{L/K}(v) = \begin{cases} v & \text{if } v \leq h, \\ p^m(v-h)+h & \text{if } v > h. \end{cases}$$

Proof. Take an integer l such that $lh \equiv 1 \mod p^m$. Then we have

$$v_L\left(\frac{1}{x^l \varpi^{(lh-1)/p^m}}\right) = 1.$$

Hence, for $\sigma \in \mathrm{Gal}(L/K)$ and $i \geq 0$, we have $\sigma \in \mathrm{Gal}(L/K)_i$ if and only if

$$i+1 \leq v_L\left(\sigma\left(\frac{1}{x^l \varpi^{(lh-1)/p^m}}\right) - \frac{1}{x^l \varpi^{(lh-1)/p^m}}\right) = v_L(\sigma(x)^l - x^l) + hl + 1. \tag{5.1}$$

The right hand side of (5.1) is $h+1$ if $\sigma \neq 1$. Hence the first claim follows. The second claim follows from the first claim. □

Lemma 5.5. *Let L be a totally ramified finite abelian extension of K. Let $m \geq 1$.*
1. *We have*

$$\begin{aligned}&\mathrm{Nr}_{L/K}(U_L^{\psi_{L/K}(m)}) \subset U_K^m, \quad \mathrm{Nr}_{L/K}(U_L^{\psi_{L/K}(m)+1}) \subset U_K^{m+1},\\ &\mathrm{Art}_K(U_K^m) \subset \mathrm{Gal}(L/K)_{\psi_{L/K}(m)}.\end{aligned}$$

2. *We take $\alpha \in K$ and $\beta \in L$ such that $v_K(\alpha) = m$ and $v_L(\beta) = \psi_{L/K}(m)$. We put $P(z) = z^p - z$ for $z \in k$. Assume that*

$$\begin{tikzcd} U_L^{\psi_{L/K}(m)} \arrow[r, "\mathrm{Nr}_{L/K}"] \arrow[d, "p_{L,\beta}"] & U_K^m \arrow[d, "p_{K,\alpha}"] \\ k \arrow[r, "P"] & k \end{tikzcd}$$

is commutative, where

$$\begin{aligned}p_{K,\alpha}\colon U_K^m &\longrightarrow k;\ 1+\alpha x \mapsto \bar{x},\\ p_{L,\beta}\colon U_L^{\psi_{L/K}(m)} &\longrightarrow k;\ 1+\beta x \mapsto \bar{x}.\end{aligned}$$

Let ϖ_L be a uniformizer of L. Then we have

$$p_{L,\beta}\left(\frac{\mathrm{Art}_K(1+\alpha x)(\varpi_L)}{\varpi_L}\right) = \mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x})$$

for $x \in \mathcal{O}_K$.

Proof. The first claim follows from [Ser68, V, §3, Proposition 4 and XV, §2, Corollaire 3 of Théorème 1]. We note that our normalization of the Artin reciprocity map is inverse to that in [Ser68, XIII, §4]. Let $x \in \mathcal{O}_K$. By [Ser68, XV, §3, Proposition 4] and the construction of the isomorphism of [Ser68, XV, §2, Proposition 3], we have

$$p_{L,\beta}\left(\frac{\mathrm{Art}_K(1+\alpha x)(\varpi_L)}{\varpi_L}\right) = z_x^q - z_x,$$

where we take $z_x \in k^{\mathrm{ac}}$ such that $z_x^p - z_x = \bar{x}$. Then we have the second claim, since

$$z_x^q - z_x = \mathrm{Tr}_{k/\mathbb{F}_p}(z_x^p - z_x) = \mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x})$$

for such z_x. □

6 Stiefel–Whitney class and discriminant

6.1 Stiefel–Whitney class

Let $R(W_K, \mathbb{R})$ be the Grothendieck group of finite-dimensional representations of W_K over $\mathbb{R}$ with finite images. For $V \in R(W_K, \mathbb{R})$, we put $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and define $\varepsilon(V_{\mathbb{C}}, \Psi)$ by the additivity using the epsilon factors in subsection 2.1. For $V \in R(W_K, \mathbb{R})$, we define the i-th Stiefel–Whitney class $w_i(V) \in H^i(G_K, \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$ as in [Del76, (1.3)]. Let

$$\mathrm{cl}\colon H^2(G_K, \mathbb{Z}/2\mathbb{Z}) \to H^2(G_K, K^{\mathrm{ac},\times}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

where the first map is induced by $\mathbb{Z}/2\mathbb{Z} \to K^{\mathrm{ac},\times}; m \mapsto (-1)^m$ and the second isomorphism is the invariant map.

Theorem 6.1. *([Del76, (1.5) Théorème]) Assume that $V \in R(W_K, \mathbb{R})$ has dimension 0 and determinant 1. Then we have*

$$\varepsilon(V_{\mathbb{C}}, \Psi) = \exp\bigl(2\pi\sqrt{-1}\mathrm{cl}(w_2(V))\bigr).$$

In particular, we have $\varepsilon(V_{\mathbb{C}}, \Psi) = 1$ if $\mathrm{ch}\, K = 2$.

6.2 Discriminant

Let L be a finite separable extension of K. We put

$$\delta_{L/K} = \det(\mathrm{Ind}_{L/K} 1).$$

6.2.1 Multiplicative discriminant

We assume that $\mathrm{ch}\, K \neq 2$ in this subsubsection. We define $d_{L/K} \in K^{\times}/(K^{\times})^2$ as the discriminant of the quadratic form $\mathrm{Tr}_{L/K}(x^2)$ on L. For $a \in K^{\times}/(K^{\times})^2$, let $\{a\} \in H^1(G_K, \mathbb{Z}/2\mathbb{Z})$ and $\kappa_a \in \mathrm{Hom}(W_K, \{\pm 1\})$ be the elements corresponding to a under the natural isomorphisms

$$K^{\times}/(K^{\times})^2 \simeq H^1(G_K, \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}(W_K, \{\pm 1\}).$$

We have

$$\delta_{L/K} = \kappa_{d_{L/K}} \tag{6.1}$$

by [Bou81, V, §10, 2 Example 6)] (*cf.* [Ser84, 1.4]). For $a, b \in K^{\times}/(K^{\times})^2$, we put

$$\{a, b\} = \{a\} \cup \{b\} \in H^2(G_K, \mathbb{Z}/2\mathbb{Z}).$$

Proposition 6.2. *([AS10, Proposition 6.5]) Let m be the extension degree of L over K. We take a generator a of L over K. Let $f(x) \in K[x]$ be the minimal polynomial of a. We put $D = f'(a) \in L$. Then we have*

$$d_{L/K} = (-1)^{\binom{m}{2}} \mathrm{Nr}_{L/K}(D) \in K^{\times}/(K^{\times})^2,$$
$$w_2(\mathrm{Ind}_{L/K} \kappa_D) = \binom{m}{4}\{-1, -1\} + \{d_{L/K}, 2\} \in H^2(G_K, \mathbb{Z}/2\mathbb{Z}).$$

6.2.2 Additive discriminant

We put $P_m(x) = x^m - x$ for any positive integer m. We assume that $\mathrm{ch}\, K = 2$ in this subsubsection.

Definition 6.3. *([BM85, Définition 2.7]) Let m be the extension degree of L over K. Let $f(x) \in K[x]$ be the minimal polynomial of a generator of L over K. We have a decomposition $f(x) = \prod_{1 \leq i \leq m}(x - a_i)$ over the Galois closure of L over K. We put*

$$d^+_{L/K} = \sum_{1 \leq i < j \leq m} \frac{a_i a_j}{(a_i + a_j)^2} \in K/P_2(K),$$

which we call the additive discriminant of L over K.

Theorem 6.4. *([BM85, Théorème 2.7]) Let L' be the subextension of K^{ac} over K corresponding to $\mathrm{Ker}\, \delta_{L/K}$. Then the extension L' over K corresponds to $d^+_{L/K} \in K/P_2(K)$ by the Artin–Schreier theory.*

7 Product formula of Deligne–Laumon

We recall a statement of the product formula of Deligne–Laumon. In this paper, we need only the rank one case, which is proved in [Del73, Proposition 10.12.1], but we follow the notation in [Lau87].

7.1 Local factor

We consider a triple $(T, \mathcal{F}, \omega)$ which consists of the following:

- The affine scheme $T = \operatorname{Spec} \mathcal{O}_{K_T}$ where $\mathcal{O}_{K_T}$ is the ring of integers in a local field K_T of characteristic p whose residue field contains k.
- A constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on T.
- A non-zero meromorphic 1-form ω on T.

Then we can associate $\varepsilon_{\psi_0}(T, \mathcal{F}, \omega) \in \mathbb{C}^\times$ to the triple $(T, \mathcal{F}, \omega)$ as in [Lau87, Théorème (3.1.5.4)] using ι.

Assume that $K_T = k((t))$. Let $\eta = \operatorname{Spec} k((t))$ be the generic point of T with the natural inclusion $j \colon \eta \to T$. We define a character $\Psi_\omega \colon k((t)) \to \mathbb{C}^\times$ by

$$\Psi_\omega(a) = (\psi_0 \circ \operatorname{Tr}_{k/\mathbb{F}_p})(\operatorname{Res}(a\omega))$$

for $a \in k((t))$. Let $l(\Psi_\omega)$ be the level of Ψ_ω in the sense of [BH06, 1.7 Definition]. We fix an algebraic closure $k((t))^{\mathrm{ac}}$ of $k((t))$. For a rank 1 smooth $\overline{\mathbb{Q}}_\ell$-sheaf V on η corresponding to a character $\chi \colon G_{k((t))} \to \mathbb{C}^\times$ via ι, we have

$$\varepsilon_{\psi_0}(T, j_* V, \omega) = q^{-l(\Psi_\omega)/2} \varepsilon(\chi\omega_{-\frac{1}{2}}, \Psi_\omega) \tag{7.1}$$

by [Lau87, Théorème (3.1.5.4)(v)], [Tat79, (3.6.2)] and [BH06, 23.1 Proposition (3)].

7.2 Product formula

Let X be a geometrically connected proper smooth curve over k of genus g. Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X. Let $\operatorname{Frob}_q \in G_k$ be the geometric Frobenius element. We put

$$\varepsilon(X, \mathcal{F}) = \iota\left(\prod_{i=0}^{2} \det\bigl(-\operatorname{Frob}_q; H^i(X \otimes_k k^{\mathrm{ac}}, \mathcal{F})\bigr)^{(-1)^{i-1}} \right).$$

Let $\operatorname{rk}(\mathcal{F})$ be the generic rank of $\mathcal{F}$.

Theorem 7.1. *([Lau87, Théorème (3.2.1.1)]) Let ω be a non-zero meromorphic 1-form on X. Then we have*

$$\varepsilon(X, \mathcal{F}) = q^{\operatorname{rk}(\mathcal{F})(1-g)} \prod_{x \in |X|} \varepsilon_{\psi_0}(X_{(x)}, \mathcal{F}|_{X_{(x)}}, \omega|_{X_{(x)}}),$$

where $|X|$ is the set of closed points of X, and $X_{(x)}$ is the completion of X at x.

8 Determinant

In this section, we study $\det \tau_{\psi_0}$ to show the equality $\omega_{\pi_{\zeta,\chi,c}} = \det \tau_{\zeta,\chi,c}$ of the central character and the determinant. We use the product formula of Deligne–Laumon to study $\det \tau_{\psi_0}\bigl(\mathrm{Fr}(1)\bigr)$, where $\mathrm{Fr}(1)$ is defined in (2.2).

Lemma 8.1. *We have $Q^{\mathrm{ab}} = Q/Q_0$.*

Proof. By Lemma 2.4, we have $Q^{\mathrm{ab}} = (Q/F)^{\mathrm{ab}}$. For $(a,b,c) \in Q$, let (a,b) be the image of (a,b,c) in Q/F. Then we have

$$(a,0)(a,b)(a,0)^{-1}(a,b)^{-1} = (1,(a-1)b).$$

Hence, we obtain the claim. □

We view θ_0 defined in (2.8) as a character of Q by $(a,b,c) \mapsto \theta_0(a)$. Recall that τ^0 is the representation of Q defined in (2.14).

Lemma 8.2. *We have $\det \tau^0 = \theta_0$.*

Proof. By Lemma 8.1, it suffices to show $\det \tau^0 = \theta_0$ on $\mu_{p^e+1}(k^{\mathrm{ac}})$. By Lemma 2.2 and Lemma 2.5, we have

$$\det \tau^0(a) = \prod_{\chi \in \mu_{p^e+1}(k^{\mathrm{ac}})^\vee \setminus \{1\}} \chi(a)$$

for $a \in \mu_{p^e+1}(k^{\mathrm{ac}})$. Hence, the claim follows. □

For $a \in k^\times$, let $\left(\frac{a}{k}\right)$ denote the quadratic residue symbol of k defined by

$$\left(\frac{a}{k}\right) = \begin{cases} 1 & \text{if } a \text{ is square in } k, \\ -1 & \text{if } a \text{ is not square in } k. \end{cases}$$

Lemma 8.3. *Let m be a positive integer that is prime to p. We take an m-th root $\varpi^{1/m}$ of ϖ, and put $L = K(\varpi^{1/m})$.*
1. If m is odd, then $\delta_{L/K}$ is the unramified character satisfying $\delta_{L/K}(\varpi) = \left(\frac{q}{m}\right)$.
2. If m is even, we have $\delta_{L/K}(\varpi) = \left(\frac{-1}{q}\right)^{\frac{m}{2}}$ and $\delta_{L/K}(x) = \left(\frac{\bar{x}}{k}\right)$ for $x \in \mathcal{O}_K^\times$.

Proof. These are proved in [BF83, (10.1.6)] if $\mathrm{ch}\, K = 0$. Actually, the same proof works also in the positive characteristic case. □

Lemma 8.4. *Let m, m' be positive integers that are prime to p. We take an m-th root $\varpi^{1/m}$ of ϖ, and put $L = K(\varpi^{1/m})$. Let $\psi'_K \colon K \to \mathbb{C}^\times$ be a character such that $\psi'_K(x) = \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(m'\bar{x}))$ for $x \in \mathcal{O}_K$. Then we have*

$$\lambda(L/K, \psi'_K) = \begin{cases} \left(\frac{q}{m}\right) & \text{if } m \text{ is odd}, \\ -\left(-\epsilon(p)\left(\frac{2mm'}{p}\right)\left(\frac{-1}{p}\right)^{\frac{m}{2}-1}\right)^f & \text{if } m \text{ is even}. \end{cases}$$

Proof. If m is odd, we have

$$\lambda(L/K, \psi'_K) = \varepsilon(\delta_{L/K}, \psi'_K) = \left(\frac{q}{m}\right)$$

by [Hen84, Proposition 2] and Lemma 8.3.1.

Assume that m is even. Note that $p \neq 2$ in this case. Then we have

$$d_{L/K} = (-1)^{m/2} \operatorname{Nr}_{L/K}(m(\varpi^{1/m})^{m-1}) = -(-1)^{m/2}\varpi \in K^\times/(K^\times)^2 \tag{8.1}$$

by Proposition 6.2. For $\chi \in (\mathbb{F}_q^\times)^\vee$ and $\psi \in \mathbb{F}_q^\vee \setminus \{1\}$, we set

$$\tau(\chi, \psi) = -\sum_{x \in \mathbb{F}_q^\times} \chi^{-1}(x)\psi(x)$$

and have the Hasse–Davenport formula

$$\tau(\chi \circ \operatorname{Nr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}, \psi \circ \operatorname{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}) = \tau(\chi, \psi)^n. \tag{8.2}$$

Let

$$(\ ,\ )_K \colon K^\times/(K^\times)^2 \times K^\times/(K^\times)^2 \to \{\pm 1\}$$

denote the Hilbert symbol. By (6.1) and (8.1), we have

$$\delta_{L/K}(x) = \kappa_{d_{L/K}}(x) = (x, d_{L/K})_K = (x, \varpi)_K = \left(\frac{\bar{x}}{k}\right)$$

for $x \in \mathcal{O}_K^\times$. By [BH06, 23.5 Theorem], we have

$$\varepsilon(\delta_{L/K}, \psi'_K) = q^{-\frac{1}{2}} \sum_{x \in \mathcal{O}_K^\times/U_K^1} \delta_{L/K}(x)\psi'_K(x) = q^{-\frac{1}{2}} \sum_{x \in k^\times} \left(\frac{x}{k}\right)\psi_0(\operatorname{Tr}_{k/\mathbb{F}_p}(m'x)).$$

By applying (8.2) to the extension k over $\mathbb{F}_p$ and using (2.16), we have

$$q^{-\frac{1}{2}} \sum_{x \in k^\times} \left(\frac{x}{k}\right)\psi_0(\operatorname{Tr}_{k/\mathbb{F}_p}(m'x)) = -\left(-\epsilon(p)\left(\frac{m'}{p}\right)\right)^f.$$

Hence, we have

$$\lambda(L/K, \psi'_K) = \varepsilon(\delta_{L/K}, \psi'_K)\left(\frac{m}{q}\right)\left(\frac{-1}{q}\right)^{\frac{m}{2}-1}(d_{L/K}, 2)_K = -\left(-\epsilon(p)\left(\frac{2mm'}{p}\right)\left(\frac{-1}{p}\right)^{\frac{m}{2}-1}\right)^f$$

by [Sai95, Theorem II.2B] and [Tat79, (3.6.1)]. □

Lemma 8.5. *We have*

$$\det \tau_{\psi_0}\big(\operatorname{Fr}(1)\big) = \begin{cases} \left(-\epsilon(p)\left(\frac{2}{p}\right)\right)^f q^{\frac{p^e}{2}} & \text{if } p \neq 2, \\ q^{2^{e-1}} & \text{if } p = 2. \end{cases}$$

Proof. Let x be the standard coordinate of $\mathbb{A}_k^1$. Let j be the open immersion $\mathbb{A}_k^1 \hookrightarrow \mathbb{P}_k^1$. We put $t = 1/x$. As in Subsection 7.1, we put $T = \operatorname{Spec} k[[t]]$ and $\eta = \operatorname{Spec} k((t))$ with the open immersion $j \colon \eta \to T$.

We consider $k((s))$ as a subfield of $k((t))$ by $s = t^{p^e+1}$. Let $\tilde{\xi} \colon G_{k((s))} \to \mathbb{C}^\times$ be the Artin–Schreier character associated to $y^p - y = 1/s$ and ψ_0, which means the composite of

$$G_{k((s))} \longrightarrow \mathbb{F}_p;\ \sigma \mapsto \sigma(y) - y$$

and ψ_0^{-1} where y is an element of $k((t))^{\mathrm{ac}}$ such that $y^p - y = 1/s$.

We use the notation in Lemma 5.5. Note that $\psi_{k((s))(y)/k((s))}(1) = 1$ by Lemma 5.4. We can check that

$$\mathrm{Nr}_{k((s))(y)/k((s))}(1 + y^{-1}x) = 1 + s(x^p - x)$$

for $x \in k$. For $x \in \mathcal{O}_{k((s))}$, we have

$$\begin{aligned}\tilde{\xi}(1+sx) &= \psi_0^{-1}\Big(\mathrm{Art}_{k((s))}(1+sx)(y) - y\Big)\\ &= \psi_0^{-1}\left(-p_{k((s))(y),y^{-1}}\left(\frac{\mathrm{Art}_{k((s))}(1+sx)(y^{-1})}{y^{-1}}\right)\right) = \psi_0(\mathrm{Tr}_{k/\mathbb{F}_p}(\bar{x})),\end{aligned}$$

where we use Lemma 5.5 with $\alpha = s$, $\beta = \varpi_{k((s))} = y^{-1}$. Hence, we have $\mathrm{rsw}(\tilde{\xi}, \Psi_{s^{-1}ds}) = s$ by Proposition 5.3.1.

Let $\xi\colon G_{k((t))} \to \mathbb{C}^\times$ be the restriction of $\tilde{\xi}$ to $G_{k((t))}$. Then ξ is the Artin–Schreier character associated to $y^p - y = 1/t^{p^e+1}$ and ψ_0.

Let V_ξ be the smooth $\overline{\mathbb{Q}}_\ell$-sheaf on η corresponding to ξ via ι. Then we have $V_\xi \simeq \mathcal{L}_{\psi_0}|_\eta$ by [Del77, Définition 1.7 in Sommes trig.]. Let the notation be as in Lemma 2.2. We write ω for the meromorphic 1-form dx on $\mathbb{P}^1_k$. By [Lau87, Théorème (3.1.5.4)(v)], we have

$$\varepsilon_{\psi_0}\big(X_{(x)}, (j_!\pi^*\mathcal{L}_{\psi_0})|_{X_{(x)}}, \omega|_{X_{(x)}}\big) = 1$$

for any $x \in |\mathbb{A}^1_k|$ with $X = \mathbb{P}^1_k$ in the notation of Theorem 7.1. We simply write ω for $\omega|_T$. Then we have

$$\det \tau_{\psi_0}\big(\mathrm{Fr}(1)\big) = (-1)^{p^e}\varepsilon(\mathbb{P}^1_k, j_!\pi^*\mathcal{L}_{\psi_0}) = (-1)^p q\varepsilon_{\psi_0}(T, j_!V_\xi, \omega)$$

by Theorem 7.1. Since ξ is a ramified character, we have $j_!V_\xi \simeq j_*V_\xi$. Hence, we obtain

$$\varepsilon_{\psi_0}(T, j_!V_\xi, \omega) = \varepsilon_{\psi_0}(T, j_*V_\xi, \omega) = q^{-1}\varepsilon(\xi\omega_{-\frac{1}{2}}, \Psi_\omega)$$

by (7.1). Since $\omega = -t^{-2}dt$ on T, we have

$$\varepsilon(\xi\omega_{-\frac{1}{2}}, \Psi_\omega) = (\xi\omega_{-\frac{1}{2}})(-t^{-1})\varepsilon(\xi\omega_{-\frac{1}{2}}, \Psi_{t^{-1}dt})$$

by [BH06, 23.5 Lemma 1]. We have

$$\xi(-t^{-1}) = \xi(-t^{p^e}) = \xi(-t)^{p^e} = 1,$$

since $\mathrm{Nr}_{k((t))(y)/k((t))}(y) = 1/t^{p^e+1}$. Hence we obtain

$$(\xi\omega_{-\frac{1}{2}})(-t^{-1})\varepsilon(\xi\omega_{-\frac{1}{2}}, \Psi_{t^{-1}dt}) = q^{\frac{p^e}{2}}\varepsilon(\xi, \Psi_{t^{-1}dt})$$

by Lemma 4.2, since $\mathrm{rsw}(\xi, \Psi_{t^{-1}dt}) = s$ by $\mathrm{rsw}(\tilde{\xi}, \Psi_{s^{-1}ds}) = s$ and Proposition 5.2.2. By Proposition 5.3.2, we have $\varepsilon(\tilde{\xi}, \Psi_{s^{-1}ds}) = \tilde{\xi}(s) = 1$, since the level of $\tilde{\xi}$ is 1 and $\mathrm{Nr}_{k((s))(y)/k((s))}(y^{-1}) = s$. Hence, we obtain

$$\varepsilon(\xi, \Psi_{t^{-1}dt}) = \lambda(k((t))/k((s)), \Psi_{s^{-1}ds})^{-1}\delta_{k((t))/k((s))}\big(\mathrm{rsw}(\tilde{\xi}, \Psi_{s^{-1}ds})\big)$$

by Proposition 5.1. We have

$$\lambda(k((t))/k((s)), \Psi_{s^{-1}ds}) = \begin{cases} -\left(-\epsilon(p)\left(\frac{2}{p}\right)\left(\frac{-1}{p}\right)^{\frac{p^e-1}{2}}\right)^f & \text{if } p \neq 2,\\ \left(\frac{q}{p^e+1}\right) & \text{if } p = 2,\end{cases}$$

$$\delta_{k((t))/k((s))}\big(\mathrm{rsw}(\tilde{\xi}, \Psi_{s^{-1}ds})\big) = \begin{cases} \left(\frac{-1}{q}\right)^{\frac{p^e+1}{2}} & \text{if } p \neq 2,\\ \left(\frac{q}{p^e+1}\right) & \text{if } p = 2\end{cases}$$

by Lemma 8.4 and Lemma 8.3 respectively. The claim follows from the above equalities. □

We simply write τ_ζ for $\tau_{\zeta,1,1}$.

Proposition 8.6. *We have* $\omega_{\pi_{\zeta,\chi,c}} = \det \tau_{\zeta,\chi,c}$.

Proof. By (2.13) and [Gal65, (1)], we have

$$\det \tau_{\zeta,\chi,c} = \delta_{E_\zeta/K}^{p^e} (\det \tau_{n,\zeta,\chi,c})|_{K^\times}, \tag{8.3}$$

since $\delta_{E_\zeta/K} = \det(\mathrm{Ind}_{E_\zeta/K} 1)$ and the transfer homomorphism $W_K^{\mathrm{ab}} \to W_{E_\zeta}^{\mathrm{ab}}$ is compatible with the natural inclusion $K^\times \to E_\zeta^\times$ under the Artin reciprocity maps. Hence, we may assume $\chi = 1$ and $c = 1$ by twist (*cf.* (2.13)). Then it suffices to show $\det \tau_\zeta = 1$. We see that $\det \tau_\zeta$ is unramified by (2.9), Lemma 2.5, Lemma 8.2, Lemma 8.3 and (8.3).

If p and n' are odd, we have

$$\begin{aligned}
\det \tau_\zeta(\varpi) &= \left(\frac{q}{n'}\right)^{p^e} \left(-\epsilon(p) \left(\frac{2}{p}\right) p^{\frac{p^e}{2}} \right)^{fn'} \left(\left(-\epsilon(p) \left(\frac{-2n'}{p} \right) \right)^n p^{-\frac{1}{2}} \right)^{fn'p^e} \\
&= \left(\left(\frac{p}{n'}\right) \left(\frac{n'}{p}\right) \epsilon(p)^{p^e n - 1} \right)^{fn'} = \left(\left(\frac{p}{n'}\right) \left(\frac{n'}{p}\right) (-1)^{\frac{p-1}{2}\frac{n'-1}{2}} \right)^{fn'} = 1
\end{aligned}$$

by (8.3), Lemma 8.3.1 and Lemma 8.5. We see that $\det \tau_\zeta(\varpi) = 1$ similarly also in the other case using (8.3), Lemma 8.3 and Lemma 8.5. $\square$

9 Imprimitive field

In this section, we construct a field extension T_ζ^{u} of E_ζ such that $\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}$ is an induction of a character. We call T_ζ^{u} an imprimitive field of $\tau_{n,\zeta}$, since $\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}$ is not primitive.

9.1 Construction of character

In this subsection, we construct subgroups $R \subset Q' \subset Q \rtimes \mathbb{Z}$ and a character ϕ_n of R. In the next subsection, we will see that $\tau_n|_{Q'} \simeq \mathrm{Ind}_R^{Q'} \phi_n$. Our imprimitive field T_ζ^{u} will correspond to the subgroup $Q' \subset Q \rtimes \mathbb{Z}$.

Let e_0 be the positive integer such that $e_0 \in 2^{\mathbb{N}}$ and e/e_0 is odd.

Lemma 9.1. *Assume* $p \neq 2$. *Then we have* $\mathrm{Tr}\, \tau_{\psi_0}\big(\mathrm{Fr}(2e_0)\big) = p^{e_0}$.

Proof. For $a \in k^{\mathrm{ac}}$ and $b \in \mathbb{F}_{p^{2e_0}}$ such that $a^p - a = b^{p^e+1}$, we have that

$$a^{p^{2e_0}} - a = \mathrm{Tr}_{\mathbb{F}_{p^{2e_0}}/\mathbb{F}_p}(b^{p^e+1}). \tag{9.1}$$

By (9.1) and the Lefschetz trace formula, we see that

$$\begin{aligned}
\mathrm{Tr}\, \tau_{\psi_0}\big(\mathrm{Fr}(2e_0)\big) &= -\sum_{b \in \mathbb{F}_{p^{2e_0}}} (\psi_0 \circ \mathrm{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p})\big(\mathrm{Tr}_{\mathbb{F}_{p^{2e_0}}/\mathbb{F}_{p^{e_0}}}(b^{p^{e_0}+1})\big) \\
&= -1 - (p^{e_0}+1) \sum_{x \in \mathbb{F}_{p^{e_0}}^\times} (\psi_0 \circ \mathrm{Tr}_{\mathbb{F}_{p^{e_0}}/\mathbb{F}_p})(x) = p^{e_0}
\end{aligned}$$

using $(p^e+1, p^{2e_0}-1) = p^{e_0}+1$. $\square$

Corollary 9.2. *Assume* $p \neq 2$. *Then we have* $\mathrm{Tr}\, \tau_n\big(\mathrm{Fr}(2e_0)\big) = (-1)^{ne_0(p-1)/2}$.

Proof. This follows from (2.9) and Lemma 9.1. $\square$

Let n_0 be the biggest integer such that 2^{n_0} divides $p^{e_0}+1$. We take $r \in k^{\mathrm{ac}}$ such that $r^{2^{n_0}} = -1$. We define a subgroup R_0 of Q_0 by

$$R_0 = \{(1,b,c) \in Q_0 \mid b^{p^e} - rb = 0\}.$$

Lemma 9.3. 1. *If $p \neq 2$, the action of $2e_0\mathbb{Z} \subset \mathbb{Z}$ on Q stabilizes R_0.*
2. *If $p = 2$, the action of $\boldsymbol{g}$ on $Q \rtimes \mathbb{Z}$ by conjugation stabilizes R_0.*

Proof. The first claim follows from $r^{p^{2e_0}-1} = 1$. We can see the second claim easily using (2.19). $\square$

We put

$$Q' = \begin{cases} Q_0 \rtimes (2e_0\mathbb{Z}) & \text{if } p \neq 2, \\ Q_0 \rtimes \mathbb{Z} & \text{if } p = 2, \end{cases} \qquad R = \begin{cases} R_0 \rtimes (2e_0\mathbb{Z}) & \text{if } p \neq 2, \\ R_0 \cdot \langle \boldsymbol{g} \rangle & \text{if } p = 2 \end{cases}$$

as subgroups of $Q \rtimes \mathbb{Z}$, which are well-defined by Lemma 9.3. We are going to construct a character ϕ_n of R in this subsection. Then, we will show that $\tau_n|_{Q'} \simeq \mathrm{Ind}_R^{Q'} \phi_n$ in the next subsection.

First, we consider the case where p is odd. We define a homomorphism $\phi_n \colon R \to \mathbb{C}^\times$ by

$$\begin{aligned} \phi_n\bigl(((1,b,c),0)\bigr) &= \psi_0\left(c - \frac{1}{2}\sum_{i=0}^{e-1}(rb^2)^{p^i}\right) \quad \text{for } \ (1,b,c) \in R_0, \\ \phi_n\bigl(\mathrm{Fr}(2e_0)\bigr) &= (-1)^{ne_0\frac{p-1}{2}}. \end{aligned} \tag{9.2}$$

Then ϕ_n extends the character ψ_0 of F.

Next, we consider the case where $p = 2$. We define an abelian group R_0' as

$$R_0' = \left\{(b,c) \;\middle|\; b \in \mathbb{F}_2,\ c \in \mathbb{F}_{2^{2e}},\ c^{2^e} - c = b\right\}$$

with the multiplication given by

$$(b_1,c_1)\cdot(b_2,c_2) = (b_1+b_2, c_1+c_2+b_1b_2).$$

We define $\phi \colon R_0 \to R_0'$ by

$$\phi((1,b,c)) = \left(\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b), c + \sum_{0 \le i < j \le e-1} b^{2^i+2^j}\right) \quad \text{for } \ (1,b,c) \in R_0,$$

which is a homomorphism by

$$\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b)\,\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b') = \mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(bb') + \sum_{0 \le i < j \le e-1} (b^{2^i}b'^{2^j} + b'^{2^i}b^{2^j}) \tag{9.3}$$

for $b, b' \in \mathbb{F}_{2^e}$. Let $b_0 \in \mathbb{F}_{2^{2e}}$ be as before Lemma 2.8. Let F' be the kernel of the homomorphism

$$\mathbb{F}_{2^e} \to \mathbb{F}_2;\ c \mapsto \mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}\bigl((b_0 + b_0^{2^e})c\bigr).$$

We put $R_0'' = R_0'/F'$, where we consider F' as a subgroup of R_0' by $c \mapsto (0,c)$. Then R_0'' is a cyclic group of order 4. We write $\bar{g}(b,c)$ for the image of $(b,c) \in R_0'$ under the projection $R_0' \to R_0''$. Let $\phi' \colon R_0 \to R_0''$ be the composite of ϕ and the projection $R_0' \to R_0''$. We put

$$s = \sum_{i=0}^{e-1} b_0^{2^i}, \quad t = \mathrm{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_{2^e}}(b_0). \tag{9.4}$$

We have $s^2 + s = t$ and $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(t) = \mathrm{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0) = 1$. We have

$$\left(1, s^2 + \sum_{0 \le i < j \le e-1} t^{2^i + 2^j}\right) \in R_0',$$

which is of order 4. The element $\bar{g}\bigl(1, s^2 + \sum_{0 \le i < j \le e-1} t^{2^i+2^j}\bigr)$ is a generator of R_0'', because $2\bar{g}\bigl(1, s^2 + \sum_{0 \le i < j \le e-1} t^{2^i+2^j}\bigr) = \bar{g}(0,1) \neq 0$. Let $\tilde{\psi}_0 \colon R_0'' \to \mathbb{C}^\times$ be the faithful character satisfying

$$\tilde{\psi}_0\left(\bar{g}\left(1, s^2 + \sum_{0 \le i < j \le e-1} t^{2^i+2^j}\right)\right) = -\sqrt{-1}.$$

We define a homomorphism $\phi_n \colon R \to \mathbb{C}^\times$ by

$$\begin{aligned} \phi_n\bigl(((1,b,c),0)\bigr) &= (\tilde{\psi}_0 \circ \phi')((1,b,c)) \quad \text{for} \quad (1,b,c) \in R_0, \\ \phi_n(\boldsymbol{g}) &= (-1)^{\frac{n(n-2)}{8}} \frac{-1+\sqrt{-1}}{\sqrt{2}}, \end{aligned} \tag{9.5}$$

which is a character of order 8. Then ϕ_n extends the character ψ_0 of F.

9.2 Induction of character

Lemma 9.4. *We have $\tau_n|_{Q'} \simeq \mathrm{Ind}_R^{Q'} \phi_n$.*

Proof. We write $\tilde{\psi}_n$ for $\phi_n|_{R_0}$. We know that $\tau_n|_{Q_0} \cong \mathrm{Ind}_{R_0}^{Q_0} \tilde{\psi}_n$ by Proposition 1.2, since R_0 is an abelian group such that $2\dim_{\mathbb{F}_p}(R_0/F) = \dim_{\mathbb{F}_p}(Q_0/F)$.

First, we consider the case where p is odd. The claim for general f follows from the claim for $f=1$ by the restriction. Hence, we may assume that $f=1$.

If $\tilde{\psi} \in R_0^\vee$ satisfies $\tilde{\psi}|_F = \psi_0$, then we have $\tau_n|_{Q_0} \cong \mathrm{Ind}_{R_0}^{Q_0} \tilde{\psi}$ by Proposition 1.2, and obtain an injective homomorphism $\tilde{\psi} \hookrightarrow \tau_n|_{R_0}$ as representations of R_0 by Frobenius reciprocity. Hence we have a decomposition

$$\tau_n|_{R_0} = \bigoplus_{\tilde{\psi} \in R_0^\vee,\, \tilde{\psi}|_F = \psi_0} \tilde{\psi}, \tag{9.6}$$

since the number of $\tilde{\psi} \in R_0^\vee$ such that $\tilde{\psi}|_F = \psi_0$ is p^e.

We put $\overline{R}_0 = \{b \in k^{\mathrm{ac}} \mid b^{p^e} - rb = 0\}$. The $\tilde{\psi}_n$-component in (9.6) is the unique component that is stable by the action of $((1,0,0),2e_0)$, since the homomorphism

$$\overline{R}_0 \to \overline{R}_0;\ b \mapsto b^{p^{2e_0}} - b$$

is an isomorphism. Hence, we have a non-trivial homomorphism $\phi_n \to \tau_n|_R$ by Corollary 9.2. Then we have a non-trivial homomorphism $\mathrm{Ind}_R^{Q'} \phi_n \to \tau_n|_{Q'}$ by Frobenius reciprocity.

The representation $\tau_n|_{Q'}$ is irreducible by Corollary 2.6. Then we obtain the claim, since $[Q' : R] = p^e$.

Next we consider the case where $p = 2$. Then it suffices to show that

$$\mathrm{Tr}(\mathrm{Ind}_R^{Q'} \phi_n)(\boldsymbol{g}^{-1}) = -(-1)^{\frac{n(n-2)}{8}}\sqrt{2}$$

by (2.9) and Proposition 2.9. We have a decomposition

$$(\mathrm{Ind}_R^{Q'} \phi_n)|_{R_0} = \bigoplus_{\phi \in R_0^\vee,\, \phi|_F = \psi_0} \phi. \tag{9.7}$$

Let $\tilde{\psi}'_n$ be the twist of $\tilde{\psi}_n$ by the character

$$R_0 \to \overline{\mathbb{Q}}_\ell^\times;\ (1, b, c) \mapsto \psi_0\bigl(\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(b)\bigr).$$

Then only the $\tilde{\psi}_n$-component and the $\tilde{\psi}'_n$-component in (9.6) are stable by the action of $((1, b_0, c_0), 1)$, since the image of the homomorphism

$$\mathbb{F}_{2^e} \to \mathbb{F}_{2^e};\ b \mapsto b^2 - b$$

is equal to $\mathrm{Ker}\,\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}$. The action of $\mathrm{Fr}(e)$ permutes the $\tilde{\psi}_n$-component and the $\tilde{\psi}'_n$-component. Hence, $\boldsymbol{g}$ acts on the $\tilde{\psi}'_n$-component by $\phi_n(\boldsymbol{g})$ times

$$\phi_n\Bigl(\mathrm{Fr}(e)^{-1}\boldsymbol{g}\,\mathrm{Fr}(e)\boldsymbol{g}^{-1}\Bigr) = \phi_n\Biggl(\biggl(\Bigl(1, t, c_0 + c_0^{2^e} + \sum_{i=0}^{e-1}(b_0^{2^e+1} + b_0^{2^{e+1}})^{2^i}\Bigr), 0\biggr)\Biggr) = \sqrt{-1}.$$

Hence we have

$$\mathrm{Tr}(\mathrm{Ind}_R^{Q'} \phi_n)(\boldsymbol{g}^{-1}) = \bigl(1 - \sqrt{-1}\bigr)\phi_n(\boldsymbol{g}^{-1}) = -(-1)^{\frac{n(n-2)}{8}}\sqrt{2}.$$

□

We use the notations in (2.10). We set $T_\zeta = E_\zeta(\alpha_\zeta)$, $M_\zeta = T_\zeta(\beta_\zeta)$ and $N_\zeta = M_\zeta(\gamma_\zeta)$. Let f_0 be the positive integer such that $f_0 \in 2^{\mathbb{N}}$ and f/f_0 is odd. We put

$$N = \begin{cases} 2e_0/f_0 & \text{if } p \neq 2 \text{ and } f_0 \mid 2e_0, \\ 1 & \text{otherwise.} \end{cases}$$

Let K^{ur} be the maximal unramified extension of K in K^{ac}. Let $K^{\mathrm{u}} \subset K^{\mathrm{ur}}$ be the unramified extension of degree N over K. Let k_N be the residue field of K^{u}. For a finite field extension L of K in K^{ac}, we write L^{u} for the composite field of L and K^{u} in K^{ac}. For $a \in k^{\mathrm{ac}}$, we write $\hat{a} \in \mathcal{O}_{K^{\mathrm{ur}}}$ for the Teichmüller lift of a. We put

$$\delta'_\zeta = \begin{cases} \beta_\zeta^{p^e} - \hat{r}\beta_\zeta & \text{if } p \neq 2, \\ \beta_\zeta^{2^e} - \beta_\zeta + \sum_{i=0}^{e-1} \hat{b}_0^{2^i} & \text{if } p = 2, \end{cases} \quad \epsilon_1 = \begin{cases} 0 & \text{if } p \neq 2, \\ 1 & \text{if } p = 2. \end{cases} \tag{9.8}$$

Then we have $\delta_\zeta'^{p^e} - \hat{r}^{-1}\delta'_\zeta \equiv -\alpha_\zeta^{-1} + \epsilon_1 \mod \mathfrak{p}_{T_\zeta^{\mathrm{u}}(\delta'_\zeta)}$. We take $\delta_\zeta \in T_\zeta^{\mathrm{u}}(\delta'_\zeta)$ such that

$$\delta_\zeta^{p^e} - \hat{r}^{-1}\delta_\zeta = -\alpha_\zeta^{-1} + \epsilon_1, \quad \delta_\zeta \equiv \delta'_\zeta \mod \mathfrak{p}_{T_\zeta^{\mathrm{u}}(\delta'_\zeta)}. \tag{9.9}$$

We put $M_\zeta'^{\mathrm{u}} = T_\zeta^{\mathrm{u}}(\delta_\zeta)$. The image of $\Theta_\zeta|_{W_{M_\zeta'^{\mathrm{u}}}}$ is contained in R. Let $\xi_{n,\zeta} \colon W_{M_\zeta'^{\mathrm{u}}} \to \mathbb{C}^\times$ be the composite of the restrictions $\Theta_\zeta|_{W_{M_\zeta'^{\mathrm{u}}}}$ and $\phi_n|_R$. By the local class field theory, we regard $\xi_{n,\zeta}$ as a character of $M_\zeta'^{\mathrm{u}\times}$.

Proposition 9.5. *We have* $\tau_{n,\zeta}|_{W_{T^{\mathrm{u}}_\zeta}} \simeq \mathrm{Ind}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}\xi_{n,\zeta}$.

Proof. This follows from Lemma 9.4. □

Remark 9.6. *Our imprimitive field is different from that in [BH14, 5.1]. In our case,* T^{u}_ζ *need not be normal over* K*. This choice is technically important in our proof of the main result.*

9.3 Study of character

In this subsection, we study the character $\xi_{n,\zeta}$ in detail.

Assume that $\mathrm{ch}\, K = p$ and $f = 1$ in this subsection. We will use results in this subsection to compute the epsilon factor of $\xi_{n,\zeta}$ later after a reduction to the case where $\mathrm{ch}\, K = p$ and $f = 1$. By (2.10), (9.8), (9.9) and $\mathrm{ch}\, K = p$, we have that $\delta_\zeta = \delta'_\zeta$.

9.3.1 Odd case

Assume $p \neq 2$ in this subsubsection. We put

$$\theta_\zeta = \gamma_\zeta + \frac{1}{2}\sum_{i=0}^{e-1}(r\beta_\zeta^2)^{p^i}. \tag{9.10}$$

Since $r^{p^{e_0}+1} = -1$ and $(p^e+1)/(p^{e_0}+1)$ is an odd integer, we have $r^{p^e+1} = -1$. Then we have

$$\theta_\zeta^p - \theta_\zeta = \beta_\zeta^{p^e+1} - \frac{1}{2r}(\beta_\zeta^{2p^e} + r^2\beta_\zeta^2) = -\frac{1}{2r}(\beta_\zeta^{2p^e} - 2r\beta_\zeta^{p^e+1} + r^2\beta_\zeta^2) = -\frac{1}{2r}\delta_\zeta^2. \tag{9.11}$$

We put $N'^{\mathrm{u}}_\zeta = M'^{\mathrm{u}}_\zeta(\theta_\zeta)$. Let $\xi'_{n,\zeta}$ be the twist of $\xi_{n,\zeta}$ by the unramified character

$$W_{M'^{\mathrm{u}}_\zeta} \to \mathbb{C}^\times;\ \sigma \mapsto \sqrt{-1}^{\,nn_\sigma\frac{p-1}{2}},$$

where n_σ is as before (2.12).

Lemma 9.7. *If* $p \neq 2$*, then* $\xi'_{n,\zeta}$ *factors through* $\mathrm{Gal}(N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta)$.

Proof. Let $\sigma \in \mathrm{Ker}\,\xi'_{n,\zeta}$. Recall that $a_\sigma, b_\sigma, c_\sigma$ are defined in (2.11). Then we have $(\bar{a}_\sigma, \bar{b}_\sigma, \bar{c}_\sigma) \in R_0$ and

$$\bar{c}_\sigma - \frac{1}{2}\sum_{i=0}^{e-1}(r\bar{b}_\sigma^2)^{p^i} = 0$$

by (9.2). Hence, we see that

$$\begin{aligned}\sigma(\theta_\zeta) - \theta_\zeta &= c_\sigma - \sum_{i=0}^{e-1}\bigl(rb_\sigma(\beta_\zeta + b_\sigma)\bigr)^{p^i} + \frac{1}{2}\sum_{i=0}^{e-1}\Bigl(r\bigl((\beta_\zeta + b_\sigma)^2 - \beta_\zeta^2\bigr)\Bigr)^{p^i} \\ &= c_\sigma - \frac{1}{2}\sum_{i=0}^{e-1}(rb_\sigma^2)^{p^i} \equiv 0 \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}\end{aligned}$$

by (2.11). Therefore, we obtain the claim by $\sigma(\delta_\zeta) = \delta_\zeta$ and (9.11). □

9.3.2 Even case

Assume $p=2$ in this subsubsection. Let $\xi'_{n,\zeta}$ be the twist of $\xi_{n,\zeta}$ by the character

$$W_{M'^{\mathrm{u}}_\zeta} \to \mathbb{C}^\times;\ \sigma \mapsto \left((-1)^{\frac{n(n-2)}{8}} \frac{-1+\sqrt{-1}}{\sqrt{2}}\right)^{n_\sigma}. \tag{9.12}$$

We take $b_1, b_2 \in k^{\mathrm{ac}}$ such that

$$b_1^2 - b_1 = s, \quad b_2^2 - b_2 = t\Big(b_1^2 + \sum_{i=0}^{e-1} (b_1 s)^{2^i}\Big). \tag{9.13}$$

We put

$$\eta_\zeta = \sum_{i=0}^{e-1} \beta_\zeta^{2^i} + b_1, \quad \gamma'_\zeta = \gamma_\zeta + \sum_{0\le i<j\le e-1} \beta_\zeta^{2^i+2^j}, \tag{9.14}$$

$$\theta'_\zeta = \sum_{i=0}^{e-1} (t\gamma'_\zeta)^{2^i} + \sum_{0\le i\le j\le e-2} t^{2^i} (\delta_\zeta \eta_\zeta)^{2^j} + \sum_{0\le j<i\le e-1} t^{2^i} (b_1\delta_\zeta + s\eta_\zeta)^{2^j} + b_1^2 \eta_\zeta + b_2. \tag{9.15}$$

Lemma 9.8. *We have $\eta_\zeta^2 - \eta_\zeta = \delta_\zeta$ and $\theta'^2_\zeta - \theta'_\zeta = (\delta_\zeta\eta_\zeta)^{2^{e-1}}$.*

Proof. We can check the first claim easily. We show the second claim. We use P_m in Subsubsection 6.2.2. We have

$$P_2(\gamma'_\zeta) = (\beta_\zeta^{2^e} - \beta_\zeta) \sum_{i=0}^{e-1} \beta_\zeta^{2^i} + \beta_\zeta^2 = (\delta_\zeta - s)(\eta_\zeta - b_1) + \beta_\zeta^2. \tag{9.16}$$

Hence, we have

$$P_{2^e}(\gamma'_\zeta) = \sum_{i=0}^{e-1} \big((\delta_\zeta - s)(\eta_\zeta - b_1)\big)^{2^i} + (\eta_\zeta - b_1)^2. \tag{9.17}$$

By $b_1^4 + b_1 = s^2 + s = t$ and $\eta_\zeta^2 - \eta_\zeta = \delta_\zeta$, we have

$$(b_1^2\eta_\zeta)^2 + b_1^2\eta_\zeta = t\eta_\zeta^2 + b_1\eta_\zeta^2 + b_1^2\eta_\zeta = t\eta_\zeta^2 + b_1(\eta_\zeta^2 + \eta_\zeta) + s\eta_\zeta = t\eta_\zeta^2 + b_1\delta_\zeta + s\eta_\zeta.$$

Hence, by using $\sum_{i=1}^{e-1} t^{2^i} = 1 - t$ and $t \in \mathbb{F}_{2^e}$, we have

$$\begin{aligned}
\theta'^2_\zeta - \theta'_\zeta &= tP_{2^e}(\gamma'_\zeta) + t\sum_{i=0}^{e-1} (\delta_\zeta\eta_\zeta + b_1\delta_\zeta + s\eta_\zeta)^{2^i} + (\delta_\zeta\eta_\zeta)^{2^{e-1}} + t\eta_\zeta^2 + b_2^2 - b_2 \\
&= t\left(\sum_{i=0}^{e-1} (b_1 s)^{2^i} + \eta_\zeta^2 + b_1^2\right) + (\delta_\zeta\eta_\zeta)^{2^{e-1}} + t\eta_\zeta^2 + b_2^2 - b_2 = (\delta_\zeta\eta_\zeta)^{2^{e-1}},
\end{aligned}$$

where we use (9.17) at the second equality and (9.13) at the third one. □

We take $\theta_\zeta \in K^{\mathrm{ac}}$ such that $\theta'_\zeta = \theta_\zeta^{2^{2e-1}}$. Then we have $\theta_\zeta^2 - \theta_\zeta = \delta_\zeta\eta_\zeta$. We put $N'^{\mathrm{u}}_\zeta = M'^{\mathrm{u}}_\zeta(\eta_\zeta, \theta_\zeta)$, which is a cyclic extension of M'^{u}_ζ of order 4 by Lemma 9.8.

Lemma 9.9. *The character $\xi'_{n,\zeta}$ factors through* $\mathrm{Gal}(N'^{\mathrm{u}}_\zeta / M'^{\mathrm{u}}_\zeta)$.

Proof. Let $\sigma \in \operatorname{Ker} \xi'_{n,\zeta}$. We take $\sigma_1, \sigma_2 \in \operatorname{Ker} \xi'_{n,\zeta}$ such that $\sigma = \sigma_1 \sigma_2^{-n_\sigma}$, $\sigma_1 \in I_{M'^{\mathrm{u}}_\zeta}$ and $\Theta_\zeta(\sigma_2) = ((1, b_0, c_0), -1)$. Then we have $(\bar{a}_{\sigma_1}, \bar{b}_{\sigma_1}, \bar{c}_{\sigma_1}) \in R_0$, $\operatorname{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(\bar{b}_{\sigma_1}) = 0$ and

$$\operatorname{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}\biggl(t\Bigl(\bar{c}_{\sigma_1} + \sum_{0 \le i < j \le e-1} \bar{b}_{\sigma_1}^{2^i + 2^j} \Bigr) \biggr) = 0 \tag{9.18}$$

by (9.5). It suffices to show that $\sigma_i(\eta_\zeta) = \eta_\zeta$ and $\sigma_i(\theta'_\zeta) = \theta'_\zeta$ for $i = 1, 2$.

We have

$$\begin{aligned} \sigma_1(\eta_\zeta) - \eta_\zeta &\equiv \sum_{i=0}^{e-1} b_{\sigma_1}^{2^i} \equiv 0 \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}, \\ \sigma_2(\eta_\zeta) - \eta_\zeta &\equiv \sum_{i=0}^{e-1} b_0^{2^i} + b_1^2 - b_1 \equiv 0 \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta} \end{aligned}$$

by $\operatorname{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(\bar{b}_{\sigma_1}) = 0$ and $b_1^2 - b_1 = s$. By Lemma 9.8, we have $\sigma_i(\eta_\zeta) - \eta_\zeta \in \mathbb{F}_2$ for $i = 1, 2$. Hence, we have $\sigma_i(\eta_\zeta) = \eta_\zeta$ for $i = 1, 2$. We have

$$\sigma_1(\theta'_\zeta) - \theta'_\zeta = \sum_{i=0}^{e-1} \bigl(t(\sigma_1(\gamma'_\zeta) - \gamma'_\zeta) \bigr)^{2^i}.$$

Further, we have

$$\begin{aligned} \sigma_1(\gamma'_\zeta) - \gamma'_\zeta &\equiv c_{\sigma_1} + \sum_{i=0}^{e-1} (b_{\sigma_1})^{2^{i+1}} + \sum_{i=0}^{e-1} b_{\sigma_1}^{2^i} \sum_{i=0}^{e-1} \beta_\zeta^{2^i} + \sum_{0 \le i < j \le e-1} b_{\sigma_1}^{2^i + 2^j} \\ &\equiv c_{\sigma_1} + \sum_{0 \le i < j \le e-1} b_{\sigma_1}^{2^i + 2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}, \end{aligned}$$

where we use (2.11) and $\bar{b}_{\sigma_1} \in \mathbb{F}_{2^e}$ at the first equality, and use $\operatorname{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(\bar{b}_{\sigma_1}) = 0$ at the second one. This implies $\sigma_1(\theta'_\zeta) \equiv \theta'_\zeta \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$ by (9.18). By a similar argument as above using Lemma 9.8, we obtain $\sigma_1(\theta'_\zeta) = \theta'_\zeta$.

It remains to show $\sigma_2(\theta'_\zeta) = \theta'_\zeta$. Using (9.16) and $\operatorname{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(t) = 1$, we see that

$$\sum_{i=0}^{e-1} (t\gamma'_\zeta)^{2^i} = \gamma'_\zeta + \sum_{1 \le i \le j \le e-1} t^{2^j} \beta_\zeta^{2^i} + \sum_{0 \le i < j \le e-1} t^{2^j} \bigl((\delta_\zeta - s)(\eta_\zeta - b_1) \bigr)^{2^i}. \tag{9.19}$$

We put

$$\gamma''_\zeta = \gamma'_\zeta + \sum_{1 \le i \le j \le e-1} t^{2^j} \beta_\zeta^{2^i}.$$

By $c_0^2 + c_0 = b_0^{2^e + 1}$ and $t = b_0 + b_0^{2^e}$ (*cf.* (2.18), (9.4)), we have

$$\sigma_2(\gamma_\zeta) - \gamma_\zeta \equiv c_0 + \sum_{i=0}^{e-1} (b_0^{2^e}(\beta_\zeta + b_0))^{2^i} \equiv c_0^{2^e} + \sum_{i=0}^{e-1} ((b_0 + t)\beta_\zeta)^{2^i} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

Then we have

$$\sigma_2(\gamma'_\zeta) - \gamma'_\zeta \equiv c_0^{2^e} + s(\eta_\zeta - b_1) + \sum_{i=0}^{e-1} (t\beta_\zeta)^{2^i} + \sum_{0 \le i < j \le e-1} b_0^{2^i + 2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$$

by (9.4) and (9.14). Hence, we have

$$\begin{aligned}\sigma_2(\gamma''_\zeta) - \gamma''_\zeta &\equiv \sigma_2(\gamma'_\zeta) - \gamma'_\zeta + \sum_{1 \le i \le j \le e-1} t^{2^{j+1}} (\beta_\zeta + b_0)^{2^i} - \sum_{1 \le i \le j \le e-1} t^{2^j} \beta_\zeta^{2^i} \\ &\equiv \sigma_2(\gamma'_\zeta) - \gamma'_\zeta + t(\eta_\zeta - b_1) + \sum_{i=0}^{e-1} (t\beta_\zeta)^{2^i} + \sum_{1 \le i < j \le e} b_0^{2^i} t^{2^j} \\ &\equiv c_0^{2^e} + s^2(\eta_\zeta - b_1) + \sum_{0 \le i < j \le e-1} b_0^{2^i + 2^j} + \sum_{1 \le i < j \le e} b_0^{2^i} t^{2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}\end{aligned}$$

where we use (9.14) and $t \in \mathbb{F}_{2^e}$ at the second equality and $s^2 + s = t$ at the last equality. We can check that

$$c_0^{2^e} + \sum_{0 \le i < j \le e-1} b_0^{2^i + 2^j} + \sum_{1 \le i < j \le e} b_0^{2^i} t^{2^j} = st$$

by (2.17), (9.4) and $\mathrm{Tr}_{\mathbb{F}_{2^{2e}}/\mathbb{F}_2}(b_0) = 1$. As a result, we obtain

$$\sigma_2(\gamma''_\zeta) - \gamma''_\zeta \equiv s^2 \eta_\zeta + b_1 s^2 + st \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

Hence, by (9.15) and (9.19), we have

$$\sigma_2(\theta'_\zeta) - \theta'_\zeta \equiv \sum_{i=0}^{2^{e-1}+2^{e-2}} d_i \eta_\zeta^i \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$$

for some $d_i \in k^{\mathrm{ac}}$. We have

$$d_0 = b_1 s^2 + st + t \sum_{j=1}^{e-1} (b_1 s)^{2^j} + b_1 s \sum_{l=1}^{e-1} t^{2^l} + b_2^2 - b_2 = 0.$$

This implies $\sigma_2(\theta'_\zeta) = \theta'_\zeta$, since we know that $\sigma_2(\theta'_\zeta) - \theta'_\zeta \in \mathbb{F}_2$ by Lemma 9.8. □

10 Refined Swan conductor

Let $\widetilde{K} \subset K^{\mathrm{ur}}$ be the unramified extension of K^{u} generated by $\mu_{p^{4pe}-1}(K^{\mathrm{ur}})$. For a finite field extension L of K in K^{ac}, we write $\widetilde{L}$ for the composite field of L and $\widetilde{K}$ in K^{ac}. We write $\widetilde{M'_\zeta}$ for $\widetilde{M'^{\mathrm{u}}_\zeta}$. Then $\widetilde{N}_\zeta$ is a Galois extension of $\widetilde{M'_\zeta}$. By (9.8) and (9.9), we can take $\beta'_\zeta \in \widetilde{M}_\zeta$ such that

$$\beta'^{p^e}_\zeta - \hat{r}\beta'_\zeta = \delta_\zeta, \quad \beta'_\zeta \equiv \beta_\zeta \mod \mathfrak{p}_{\widetilde{M}_\zeta}, \tag{10.1}$$

since there is $x \in \mathbb{F}_{2^{4e}}$ such that $x^{2^e} - x = \sum_{i=0}^{e-1} b_0^{2^i}$ if $p = 2$. Then we have $\widetilde{M}_\zeta = \widetilde{M'_\zeta}(\beta'_\zeta)$ by Krasner's lemma.

Lemma 10.1. 1. *We have*

$$\psi_{\widetilde{N}_\zeta/\widetilde{M'_\zeta}}(v) = \begin{cases} v & \text{if } v \le 1, \\ p^e(v-1)+1 & \text{if } 1 < v \le 2, \\ p^{e+1}(v-2)+p^e+1 & \text{if } 2 < v. \end{cases} \tag{10.2}$$

2. *We have*

$$\mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M'_\zeta})_i = \begin{cases} \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M'_\zeta}) & \text{if } i \le 1, \\ \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta) & \text{if } 2 \le i \le p^e+1, \\ \{1\} & \text{if } p^e+2 \le i. \end{cases}$$

Proof. We have

$$\psi_{\widetilde{M}_\zeta/\widetilde{M}'_\zeta}(v) = \begin{cases} v & \text{if } v \le 1, \\ p^e(v-1)+1 & \text{if } v > 1, \end{cases}$$
$$\psi_{\widetilde{N}_\zeta/\widetilde{M}_\zeta}(v) = \begin{cases} v & \text{if } v \le p^e+1, \\ p(v-p^e-1)+p^e+1 & \text{if } v > p^e+1 \end{cases}$$

by (2.10), (10.1) and Lemma 5.4 noting that $\hat{r}$ has a (p^e-1)-st root in $\widetilde{M}'_\zeta$. Hence, the claim 1 follows from $\psi_{\widetilde{N}_\zeta/\widetilde{M}'_\zeta} = \psi_{\widetilde{N}_\zeta/\widetilde{M}_\zeta} \circ \psi_{\widetilde{M}_\zeta/\widetilde{M}'_\zeta}$. The claim 2 follows from the claim 1 and $\mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}'_\zeta)_{p^e+1} \supset \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta)_{p^e+1} = \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta)$. □

We set $\varpi_{\widetilde{M}'_\zeta} = \delta_\zeta^{-1}$, $\varpi_{\widetilde{M}_\zeta} = \beta_\zeta^{-1}$ and $\varpi_{\widetilde{N}_\zeta} = (\gamma_\zeta \varpi_{\widetilde{M}_\zeta}^{p^{e-1}})^{-1}$. Then the elements $\varpi_{\widetilde{M}'_\zeta}$, $\varpi_{\widetilde{M}_\zeta}$ and $\varpi_{\widetilde{N}_\zeta}$ are uniformizers of $\widetilde{M}'_\zeta$, $\widetilde{M}_\zeta$ and $\widetilde{N}_\zeta$ respectively. Let $\widetilde{k}$ be the residue field of $\widetilde{K}$.

Lemma 10.2. *We have a commutative diagram*

$$\begin{array}{ccc} U_{\widetilde{N}_\zeta}^{p^e+1} & \xrightarrow{\mathrm{Nr}_{\widetilde{N}_\zeta/\widetilde{M}'_\zeta}} & U_{\widetilde{M}'_\zeta}^{2} \\ \downarrow & & \downarrow \\ \widetilde{k} & \xrightarrow{P} & \widetilde{k} \end{array}$$

where the map P is given by $x \mapsto x^p - x$ and the vertical maps are given by

$$p_{\widetilde{N}_\zeta, -\gamma_\zeta^{-1}} \colon U_{\widetilde{N}_\zeta}^{p^e+1} \longrightarrow \widetilde{k};\ 1 - x\gamma_\zeta^{-1} \mapsto \bar{x},$$
$$p_{\widetilde{M}'_\zeta, \hat{r}\varpi^2_{\widetilde{M}'_\zeta}} \colon U_{\widetilde{M}'_\zeta}^{2} \longrightarrow \widetilde{k};\ 1 + x\hat{r}\varpi^2_{\widetilde{M}'_\zeta} \mapsto \bar{x}.$$

Proof. The norm maps $\mathrm{Nr}_{\widetilde{N}_\zeta/\widetilde{M}_\zeta}$ and $\mathrm{Nr}_{\widetilde{M}_\zeta/\widetilde{M}'_\zeta}$ induce

$$U_{\widetilde{N}_\zeta}^{p^e+1}/U_{\widetilde{N}_\zeta}^{p^e+2} \to U_{\widetilde{M}_\zeta}^{p^e+1}/U_{\widetilde{M}_\zeta}^{p^e+2};\ 1 - u\gamma_\zeta^{-1} \mapsto 1 - (u^p - u)\varpi_{\widetilde{M}_\zeta}^{p^e+1},$$
$$U_{\widetilde{M}_\zeta}^{p^e+1}/U_{\widetilde{M}_\zeta}^{p^e+2} \to U_{\widetilde{M}'_\zeta}^{2}/U_{\widetilde{M}'_\zeta}^{3};\ 1 - u\varpi_{\widetilde{M}_\zeta}^{p^e+1} = 1 - u\beta_\zeta'^{-1}\varpi_{\widetilde{M}'_\zeta} \mapsto 1 + u\hat{r}\varpi^2_{\widetilde{M}'_\zeta}$$

respectively by Lemma 5.5.1 and calculations of the norms. Hence, the claim follows. □

For any finite extension M of K, we write ψ_M for the composite $\psi_K \circ \mathrm{Tr}_{M/K}$.

Lemma 10.3. *We have* $\mathrm{rsw}(\xi_{n,\zeta}|_{W_{M'^{\mathrm{u}}_\zeta}}, \psi_{M'^{\mathrm{u}}_\zeta}) = -n'\delta_\zeta^{-(p^e+1)} \mod U^1_{M'^{\mathrm{u}}_\zeta}$.

Proof. We put $\widetilde{\xi}_{n,\zeta} = \xi_{n,\zeta}|_{W_{\widetilde{M}'_\zeta}}$, and regard it as a character of $\widetilde{M}'^{\times}_\zeta$. By (2.12), Lemma 5.5.1 and Lemma 10.1, the restriction of $\widetilde{\xi}_{n,\zeta}$ to $U^2_{\widetilde{M}'_\zeta}$ is given by the composition

$$U^2_{\widetilde{M}'_\zeta} \xrightarrow{\mathrm{Art}_{\widetilde{M}'_\zeta}} \mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta) \simeq \mathbb{F}_p \xrightarrow{\psi_0} \overline{\mathbb{Q}}_\ell^\times,$$

where the isomorphism $\mathrm{Gal}(\widetilde{N}_\zeta/\widetilde{M}_\zeta) \simeq \mathbb{F}_p$ is given by $\sigma \mapsto \overline{\sigma(\gamma_\zeta) - \gamma_\zeta}$. We define $p_{\widetilde{N}_\zeta,-\gamma_\zeta^{-1}}$ as in Lemma 10.2. For $u \in \mathcal{O}_{\widetilde{M}'_\zeta}$, we put $\sigma_u = \mathrm{Art}_{\widetilde{M}'_\zeta}(1 + u\hat{r}\varpi^2_{\widetilde{M}'_\zeta})$ and then have

$$\begin{aligned}\widetilde{\xi}_{n,\zeta}(1 + u\hat{r}\varpi^2_{\widetilde{M}'_\zeta}) &= \psi_0\Big(\overline{\sigma_u(\gamma_\zeta) - \gamma_\zeta}\Big) = \psi_0\left(p_{\widetilde{N}_\zeta,-\gamma_\zeta^{-1}}\left(\frac{\gamma_\zeta}{\sigma_u(\gamma_\zeta)}\right)\right) \\ &= \psi_0\left(p_{\widetilde{N}_\zeta,-\gamma_\zeta^{-1}}\left(\frac{\sigma_u(\varpi_{\widetilde{N}_\zeta})}{\varpi_{\widetilde{N}_\zeta}}\right)\right) = \psi_0 \circ \mathrm{Tr}_{\widetilde{k}/\mathbb{F}_p}(\bar{u}),\end{aligned} \tag{10.3}$$

where we use Lemma 5.5.2 and Lemma 10.2 at the last equality. Since we have

$$\overline{\mathrm{Tr}_{\widetilde{M}'_\zeta/\widetilde{T}_\zeta}(\delta_\zeta^{p^e-1}u)} = -r^{-1}\bar{u}$$

for $u \in \mathcal{O}_{\widetilde{M}'_\zeta}$, we obtain

$$\widetilde{\xi}_{n,\zeta}(1+x) = \psi_{\widetilde{M}'_\zeta}(-n'^{-1}\delta_\zeta^{p^e+1}x)$$

for $x \in \mathfrak{p}^2_{\widetilde{M}'_\zeta}$ by (10.3). This implies

$$\xi_{n,\zeta}(1+x) = \psi_{M'^{\mathrm{u}}_\zeta}(-n'^{-1}\delta_\zeta^{p^e+1}x) \tag{10.4}$$

for $x \in \mathfrak{p}^2_{M'^{\mathrm{u}}_\zeta}$, because $\mathrm{Tr}_{\widetilde{k}/k_N} \colon \widetilde{k} \to k_N$ is surjective. The claim follows from (10.4) and Proposition 5.3.1. □

Lemma 10.4. *We have* $\mathrm{rsw}(\tau_{n,\zeta,\chi,c}, \psi_{E_\zeta}) = n'\varphi'_\zeta \mod U^1_{E_\zeta}$.

Proof. By Proposition 5.2.1, we may assume that $\chi = 1$ and $c = 1$. By Proposition 9.5 and Lemma 10.3, we have

$$\mathrm{rsw}(\tau_{n,\zeta}|_{W_{T^{\mathrm{u}}_\zeta}}, \psi_{T^{\mathrm{u}}_\zeta}) = \mathrm{Nr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}\big(\mathrm{rsw}(\xi_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta})\big) = n'\varphi'_\zeta \mod U^1_{T^{\mathrm{u}}_\zeta}. \tag{10.5}$$

Since T^{u}_ζ is a tamely ramified extension of E_ζ, we have

$$\mathrm{rsw}(\tau_{n,\zeta}, \psi_{E_\zeta}) = \mathrm{rsw}(\tau_{n,\zeta}|_{W_{T^{\mathrm{u}}_\zeta}}, \psi_{T^{\mathrm{u}}_\zeta}) \mod U^1_{T^{\mathrm{u}}_\zeta} \tag{10.6}$$

by Proposition 5.2.2. The claim follows from (10.5) and (10.6). □

Proposition 10.5. *We have* $\mathrm{rsw}(\tau_{\zeta,\chi,c}, \psi_K) = \mathrm{rsw}(\pi_{\zeta,\chi,c}, \psi_K)$.

Proof. By $\tau_{\zeta,\chi,c} = \mathrm{Ind}_{E_\zeta/K}\tau_{n,\zeta,\chi,c}$, we have

$$\mathrm{rsw}(\tau_{\zeta,\chi,c}, \psi_K) = \mathrm{Nr}_{E_\zeta/K}\big(\mathrm{rsw}(\tau_{n,\zeta,\chi,c}, \psi_{E_\zeta})\big). \tag{10.7}$$

Hence, the claim follows from Lemma 4.5 and Lemma 10.4. □

Lemma 10.6. *We have* $\mathrm{Sw}(\tau_{\zeta,\chi,c}) = 1$.

Proof. This follows from Lemma 10.4 and (10.7). □

Lemma 10.7. *The representation $\tau_{\zeta,\chi,c}$ is irreducible.*

Proof. We know that the restriction of $\tau_{n,\zeta,\chi,c}$ to the wild inertia subgroup of W_{E_ζ} is irreducible by Corollary 2.6. Assume that $\tau_{\zeta,\chi,c}$ is not irreducible. Then we have an irreducible factor τ' of $\tau_{\zeta,\chi,c}$ such that $\mathrm{Sw}(\tau') = 0$, by Lemma 10.6 and the additivity of Sw. Then, the restriction of τ' to the wild inertia subgroup of W_K is trivial by $\mathrm{Sw}(\tau') = 0$. On the other hand, we have an injective homomorphism $\tau_{n,\zeta,\chi,c} \to \tau'|_{W_{E_\zeta}}$ by Frobenius reciprocity. This is a contradiction. □

Proposition 10.8. *The representation $\tau_{\zeta,\chi,c}$ is irreducible of Swan conductor* 1.

Proof. This follows from Lemma 10.6 and Lemma 10.7. □

11 Epsilon factor

11.1 Reduction to special cases

In this subsection, we show the equality $\varepsilon(\tau_{\zeta,\chi,c},\psi_K)=\varepsilon(\pi_{\zeta,\chi,c},\psi_K)$ of epsilon factors assuming some results in the special case where $n=p^e$, $\operatorname{ch}K=p$ and $f=1$. The results in the special case will be proved in the next subsection.

Lemma 11.1. *We have*

$$\lambda(E_\zeta/K,\psi_K)=\begin{cases}\left(\frac{q}{n'}\right) & \text{if } n' \text{ is odd,}\\ -\left(-\epsilon(p)\left(\frac{2n'}{p}\right)\left(\frac{-1}{p}\right)^{\frac{n'}{2}-1}\right)^f & \text{if } n' \text{ is even,}\end{cases}$$

$$\lambda(T_\zeta^{\mathrm{u}}/E_\zeta,\psi_{E_\zeta})=\begin{cases}-(-1)^{\frac{(p-1)fN}{4}} & \text{if } p\neq 2,\\ \left(\frac{q}{p^e+1}\right) & \text{if } p=2.\end{cases}$$

Proof. We have

$$\lambda(T_\zeta^{\mathrm{u}}/E_\zeta,\psi_{E_\zeta})=\lambda(T_\zeta^{\mathrm{u}}/E_\zeta^{\mathrm{u}},\psi_{E_\zeta^{\mathrm{u}}})\lambda(E_\zeta^{\mathrm{u}}/E_\zeta,\psi_{E_\zeta})^{p^e+1}=\lambda(T_\zeta^{\mathrm{u}}/E_\zeta^{\mathrm{u}},\psi_{E_\zeta^{\mathrm{u}}}).$$

If $p\neq 2$, we have

$$\lambda(T_\zeta^{\mathrm{u}}/E_\zeta^{\mathrm{u}},\psi_{E_\zeta^{\mathrm{u}}})=-\left(-\epsilon(p)\left(\frac{2n'}{p}\right)\left(\frac{-1}{p}\right)^{\frac{p^e-1}{2}}\right)^{fN}=-(-1)^{\frac{(p-1)fN}{4}}$$

by Lemma 8.4, since fN is even. The other assertions immediately follow from Lemma 8.4. $\square$

Lemma 11.2. *We have*

$$\lambda(M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}},\psi_{T_\zeta^{\mathrm{u}}})=\begin{cases}(-1)^f & \text{if } p=2 \text{ and } e\leq 2,\\ \left(\frac{r}{k_N}\right) & \text{otherwise.}\end{cases}$$

Proof. Let $K_{(0)}$ and $K_{(p)}$ be non-archimedean local fields of characteristic 0 and p respectively. Assume that the residue fields of $K_{(0)}$ and $K_{(p)}$ are isomorphic to k. We take uniformizers $\varpi_{(0)}$ and $\varpi_{(p)}$ of $K_{(0)}$ and $K_{(p)}$ respectively. We define $T_{\zeta,(0)}^{\mathrm{u}}$ similarly as T_ζ^{u} starting from $K_{(0)}$. We use similar notations also for other objects in the characteristic zero side and the positive characteristic side. We have the isomorphism

$$\mathcal{O}_{T_{\zeta,(p)}^{\mathrm{u}}}/\mathfrak{p}_{T_{\zeta,(p)}^{\mathrm{u}}}^2\xrightarrow{\sim}\mathcal{O}_{T_{\zeta,(0)}^{\mathrm{u}}}/\mathfrak{p}_{T_{\zeta,(0)}^{\mathrm{u}}}^2;\ \xi_0+\xi_1\varpi_{T_{\zeta,(p)}^{\mathrm{u}}}\mapsto\hat{\xi}_0+\hat{\xi}_1\varpi_{T_{\zeta,(0)}^{\mathrm{u}}}$$

of algebras, where $\xi_0,\xi_1\in k$. Hence, it suffices to show the claim in one of the characteristic zero side and the positive characteristic side by [Del84, Proposition 3.7.1], since $\mathrm{Gal}(M_{\zeta,(p)}'^{\mathrm{u}}/T_{\zeta,(p)}^{\mathrm{u}})^2=1$ and $\mathrm{Gal}(M_{\zeta,(0)}'^{\mathrm{u}}/T_{\zeta,(0)}^{\mathrm{u}})^2=1$, where we use upper numbering filtration of Galois groups.

First, we consider the case where $p\neq 2$ and $\operatorname{ch}K=p$. Then, we have $d_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}=\hat{r}$ by Proposition 6.2 and the fact that fN is even. Hence, $\delta_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}$ is unramified by (6.1). Hence, we have

$$\lambda(M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}},\psi_{T_\zeta^{\mathrm{u}}})=\varepsilon(\delta_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}},\psi_{T_\zeta^{\mathrm{u}}})^{p^e}=\left(\frac{r}{k_N}\right)$$

by [Hen84, Proposition 2], [BH06, 23.5 Proposition] and (6.1).

We consider the case where $p=2$. Assume that $e \geq 3$ and $\operatorname{ch} K = 0$. We have $D = 2^e \delta_\zeta^{2^e-1}+1$ in the notation of Proposition 6.2 with $(L,K,a)=(M_\zeta'^{\mathrm{u}}, T_\zeta^{\mathrm{u}}, \delta_\zeta)$. Then, we have $D \in (M_\zeta'^{\mathrm{u}\times})^2$. Hence, we have $\kappa_D = 1$, $d_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}} = 1$ and

$$w_2(\operatorname{Ind}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}} 1) = 1$$

by Proposition 6.2 and $\binom{p^e}{4} \equiv 0 \mod 2$. Therefore we have

$$\lambda(M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}, \psi_{T_\zeta^{\mathrm{u}}}) = \varepsilon(\operatorname{Ind}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}} 1, \psi_{T_\zeta^{\mathrm{u}}}) = \varepsilon(1^{\oplus p^e}, \psi_{T_\zeta^{\mathrm{u}}}) = 1$$

by Theorem 6.1.

Assume that $e=2$ and $\operatorname{ch} K = 2$. Then we see that $d^+_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}} = 1$ by Definition 6.3. Hence, $\delta_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}$ is the unramified character satisfying $\delta_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}(\varpi_{T_\zeta^{\mathrm{u}}}) = (-1)^f$ by Theorem 6.4. Then we see that

$$\lambda(M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}, \psi_{T_\zeta^{\mathrm{u}}}) = \varepsilon(\operatorname{Ind}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}} 1, \psi_{T_\zeta^{\mathrm{u}}}) = \varepsilon(\delta_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}} \oplus 1^{\oplus 3}, \psi_{T_\zeta^{\mathrm{u}}}) = (-1)^f,$$

where we use Theorem 6.1 at the second equality.

Assume that $e=1$ and $\operatorname{ch} K = 2$. Let $\kappa_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}$ be the quadratic character associated to the extension $M_\zeta'^{\mathrm{u}}$ over T_ζ^{u}. Then we have

$$\lambda(M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}, \psi_{T_\zeta^{\mathrm{u}}}) = \varepsilon(\kappa_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}, \psi_{T_\zeta^{\mathrm{u}}})$$

by Theorem 6.1 similarly as above. We can check that the norm map $\operatorname{Nr}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}$ induces

$$U^1_{M_\zeta'^{\mathrm{u}}}/U^2_{M_\zeta'^{\mathrm{u}}} \to U^1_{T_\zeta^{\mathrm{u}}}/U^2_{T_\zeta^{\mathrm{u}}};\ 1+u\delta_\zeta^{-1} \mapsto 1+(u^2-u)\alpha_\zeta.$$

Then, by Lemma 5.5, we have

$$\begin{aligned} \kappa_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}(1+\alpha_\zeta x) &= \psi_0\Big(\operatorname{Art}_{T_\zeta^{\mathrm{u}}}(1+\alpha_\zeta x)(\delta_\zeta) - \delta_\zeta\Big) \\ &= \psi_0\left(p_{M_\zeta'^{\mathrm{u}},\delta_\zeta^{-1}}\left(\frac{\operatorname{Art}_{T_\zeta^{\mathrm{u}}}(1+\alpha_\zeta x)(\delta_\zeta^{-1})}{\delta_\zeta^{-1}}\right)\right) = \psi_0(\operatorname{Tr}_{k/\mathbb{F}_p}(\bar{x})) \qquad (11.1) \end{aligned}$$

for $x \in \mathcal{O}_{T_\zeta^{\mathrm{u}}}$ noting that $k_N = k$. Hence, we have $\operatorname{rsw}(\kappa_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}, \psi_{T_\zeta^{\mathrm{u}}}) = \alpha_\zeta$ by Proposition 5.3.1. By Proposition 5.3.2, we have

$$\varepsilon(\kappa_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}, \psi_{T_\zeta^{\mathrm{u}}}) = \kappa_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}(\alpha_\zeta) = \kappa_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}(1+\alpha_\zeta) = (-1)^f,$$

where we use $\operatorname{Nr}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}(\delta_\zeta) = \alpha_\zeta^{-1}+1$ and (11.1) at the last equality.

□

Lemma 11.3. *We have*

$$\operatorname{Tr}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}(\delta_\zeta^i) = \begin{cases} 0 & \text{if } 1 \leq i \leq p^e-2, \\ \hat{r}^{-1}(p^e-1) & \text{if } i = p^e-1. \end{cases}$$

Proof. Vanishing for $1 \leq i \leq p^e-2$ follows from (9.9). We have also

$$\operatorname{Tr}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}(\delta_\zeta^{p^e-1}) = \operatorname{Tr}_{M_\zeta'^{\mathrm{u}}/T_\zeta^{\mathrm{u}}}\Big(\hat{r}^{-1} + \delta_\zeta^{-1}(-\alpha_\zeta^{-1}+\epsilon_1)\Big) = \hat{r}^{-1}(p^e-1)$$

by (9.9). □

Lemma 11.4. *We have*

$$\delta_{T^{\mathrm{u}}_\zeta/E_\zeta}\big(\mathrm{rsw}(\tau_{n,\zeta},\psi_{E_\zeta})\big)=\begin{cases}1 & \text{if } p\neq 2,\\ \left(\frac{q}{p^e+1}\right) & \text{if } p=2.\end{cases}$$

Proof. If $p=2$, the claim follows from Lemma 8.3.1 and Lemma 10.4, since T^{u}_ζ is totally ramified over E_ζ.

Assume that $p\neq 2$. Then we have $d_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta}=(-1)^{(p^e+1)/2}\varphi'_\zeta$ by Proposition 6.2. Hence, we have $\delta_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta}\big((-1)^{(p^e-1)/2}\varphi'_\zeta\big)=1$ by Lemma 8.3.2. Therefore, we have

$$\delta_{T^{\mathrm{u}}_\zeta/E_\zeta}\big(\mathrm{rsw}(\tau_{n,\zeta},\psi_{E_\zeta})\big)=\delta_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta}(n'\varphi'_\zeta)=\delta_{T^{\mathrm{u}}_\zeta/E^{\mathrm{u}}_\zeta}(n'(-1)^{\frac{p^e-1}{2}})=\left(\frac{n'(-1)^{\frac{p^e-1}{2}}}{q^N}\right)=1$$

by [Gal65, (1)], Lemma 8.3.2, Lemma 10.4 and the fact that fN is even. $\square$

Lemma 11.5. *Assume that $n=p^e$. Then we have $\varepsilon(\tau_{\zeta,\chi,c},\psi_K)\equiv\varepsilon(\pi_{\zeta,\chi,c},\psi_K)\mod \mu_{p^e}(\mathbb{C})$.*

Proof. Let π be the representation of $GL_n(K)$ corresponding to $\tau_{\zeta,\chi,c}$ by the local Langlands correspondence. By the proof of [BH14, 2.2 Proposition], Proposition 8.6 and Proposition 10.5, we have $\pi\simeq\text{c-Ind}_{L^\times_\zeta U^1_{\mathfrak{J}}}^{GL_n(K)}\Lambda$ for a character $\Lambda\colon L^\times_\zeta U^1_{\mathfrak{J}}\to\mathbb{C}^\times$ which coincides with $\Lambda_{\zeta,\chi,c}$ on $K^\times U^1_{\mathfrak{J}}$. Then, the claim follows from [BH14, 2.2 Lemma (1)], because $L^\times_\zeta U^1_{\mathfrak{J}}/(K^\times U^1_{\mathfrak{J}})$ is the cyclic group of order p^e. $\square$

Proposition 11.6. *We have $\varepsilon(\tau_{\zeta,\chi,c},\psi_K)=\varepsilon(\pi_{\zeta,\chi,c},\psi_K)$.*

Proof. By Proposition 3.2 and $\tau_{\zeta,\chi,c}\simeq\mathrm{Ind}_{E_\zeta/K}\tau_{n,\zeta,\chi,c}$, it suffices to show that

$$\lambda(E_\zeta/K,\psi_K)^{p^e}\varepsilon(\tau_{n,\zeta,\chi,c},\psi_{E_\zeta})=(-1)^{n-1+\epsilon_0 f}\chi(n')c.$$

By Lemma 10.4, we may assume $\chi=1$ and $c=1$. Hence, it suffices to show

$$\lambda(E_\zeta/K,\psi_K)^{p^e}\varepsilon(\tau_{n,\zeta},\psi_{E_\zeta})=(-1)^{n-1+\epsilon_0 f}. \tag{11.2}$$

Assuming that (11.2) is proved for $n=p^e$, we show (11.2) for general n. Let $\tau'_{n,\zeta}$ denote the representation of W_{E_ζ} given by Θ_ζ in (2.12) and τ_{p^e}. We put $\psi'_{E_\zeta}=n'^{-1}\psi_{E_\zeta}$. Applying the result for $n=p^e$ to E_ζ,φ'_ζ in place of K,ϖ, we have $\varepsilon(\tau'_{n,\zeta},\psi'_{E_\zeta})=(-1)^{p^e-1+\epsilon'_0 f}$, where ϵ'_0 denote ϵ_0 for $n=p^e$. Since $\det\tau'_{n,\zeta}$ is unramified as in the proof of Proposition 8.6, we have

$$\varepsilon(\tau'_{n,\zeta},\psi_{E_\zeta})=\det\tau'_{n,\zeta}(n')\varepsilon(\tau'_{n,\zeta},\psi'_{E_\zeta})=(-1)^{p^e-1+\epsilon'_0 f}. \tag{11.3}$$

We note that the inflation of the character in (2.9) by Θ_ζ factors through

$$W_{E_\zeta}\to\{\pm1\}\times\mathbb{Z};\ \sigma\mapsto\left(a_\sigma^{\frac{p^e+1}{2}},fn_\sigma\right).$$

If $p\neq 2$, we have $(n'\varphi'_\zeta,-\varphi'_\zeta)_{E_\zeta}=\left(\frac{n'}{q}\right)$, where

$$(\ ,\ )_{E_\zeta}\colon E^\times_\zeta/(E^\times_\zeta)^2\times E^\times_\zeta/(E^\times_\zeta)^2\to\{\pm1\}$$

denotes the Hilbert symbol. Hence, we have

$$\frac{\varepsilon(\tau_{n,\zeta},\psi_{E_\zeta})}{\varepsilon(\tau'_{n,\zeta},\psi_{E_\zeta})}=\begin{cases}\left(\frac{n'}{q}\right)^{n-p^e}\left(\left(\frac{n'}{p}\right)^n\left(-\epsilon(p)\left(\frac{-2}{p}\right)\right)^{n-p^e}\right)^f & \text{if } p\neq 2,\\ (-1)^{\left(\frac{n(n-2)}{8}-\frac{2^e(2^e-2)}{8}\right)f} & \text{if } p=2\end{cases} \tag{11.4}$$

by (2.9), Lemma 4.2 and Lemma 10.4. Then we have (11.2) by Lemma 11.1, (11.3) and (11.4).

Therefore, we may assume that $n = p^e$. By Lemma 11.1 and Lemma 11.5, it suffices to show that

$$\varepsilon(\tau_{n,\zeta}, \psi_{E_\zeta})^{N(p^e+1)} = \begin{cases} 1 & \text{if } p \neq 2, \\ (-1)^{1+\epsilon_0 f} & \text{if } p = 2. \end{cases}$$

By Proposition 5.1, we have

$$\varepsilon(\tau_{n,\zeta}, \psi_{E_\zeta})^{N(p^e+1)} = \delta_{T_\zeta^{\mathrm{u}}/E_\zeta}\big(\mathrm{rsw}(\tau_{n,\zeta}, \psi_{E_\zeta})\big)^{-1} \lambda(T_\zeta^{\mathrm{u}}/E_\zeta, \psi_{E_\zeta})^{p^e} \varepsilon(\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}, \psi_{T_\zeta^{\mathrm{u}}}).$$

By this, Lemma 11.1 and Lemma 11.4, it suffices to show that

$$\varepsilon(\tau_{n,\zeta}|_{W_{T_\zeta^{\mathrm{u}}}}, \psi_{T_\zeta^{\mathrm{u}}}) = \begin{cases} -(-1)^{\frac{(p-1)fN}{4}} & \text{if } p \neq 2, \\ (-1)^{1+\epsilon_0 f}\left(\frac{q}{p^e+1}\right) & \text{if } p = 2. \end{cases}$$

This follows from Lemma 11.2 and Proposition 11.7. □

We set $\varpi_{M_\zeta'^{\mathrm{u}}} = \delta_\zeta^{-1}$.

Proposition 11.7. *Assume that $n = p^e$. Then we have*

$$\varepsilon(\xi_{n,\zeta}, \psi_{M_\zeta'^{\mathrm{u}}}) = \begin{cases} -(-1)^{\frac{(p-1)fN}{4}}\left(\frac{r}{k_N}\right) & \textit{if } p \neq 2, \\ (-1)^{1+\epsilon_0 f} & \textit{if } p = 2. \end{cases}$$

Proof. First, we reduce the problem to the positive characteristic case. Assume that $\mathrm{ch}\, K = 0$. Take a positive characteristic local field $K_{(p)}$ whose residue field is isomorphic to k. We define $M_{\zeta,(p)}'^{\mathrm{u}}$ similarly as $M_\zeta'^{\mathrm{u}}$ starting from $K_{(p)}$. We use similar notations also for other objects in the positive characteristic side. Then we have the isomorphism

$$\mathcal{O}_{M_{\zeta,(p)}'^{\mathrm{u}}}/\mathfrak{p}^3_{M_{\zeta,(p)}'^{\mathrm{u}}} \xrightarrow{\sim} \mathcal{O}_{M_\zeta'^{\mathrm{u}}}/\mathfrak{p}^3_{M_\zeta'^{\mathrm{u}}};\ \xi_0 + \xi_1\varpi_{M_{\zeta,(p)}'^{\mathrm{u}}} + \xi_2\varpi^2_{M_{\zeta,(p)}'^{\mathrm{u}}} \mapsto \hat{\xi}_0 + \hat{\xi}_1\varpi_{M_\zeta'^{\mathrm{u}}} + \hat{\xi}_2\varpi^2_{M_\zeta'^{\mathrm{u}}}$$

of algebras, where $\xi_1, \xi_2, \xi_3 \in k$. Hence, the problem is reduced to the positive characteristic case by [Del84, Proposition 3.7.1].

We may assume $K = \mathbb{F}_q((t))$. We put $K_{\langle 1\rangle} = \mathbb{F}_p((t))$. We define $M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}$ similarly as $M_\zeta'^{\mathrm{u}}$ starting from $K_{\langle 1\rangle}$. We use similar notations also for other objects in the $K_{\langle 1\rangle}$-case. We put $f' = [M_\zeta'^{\mathrm{u}} : M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}]$. We have

$$\delta_{M_\zeta'^{\mathrm{u}}/M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}}(\mathrm{rsw}(\xi_{n,\zeta,\langle 1\rangle}, \psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}})) = (-1)^{f'-1}$$

by Lemma 10.3. We have $\lambda(M_\zeta'^{\mathrm{u}}/M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}, \psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}}) = 1$, since the level of $\psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}}$ is $2 - p^e$ by Lemma 11.3. Then, we obtain

$$\varepsilon(\xi_{n,\zeta}, \psi_{M_\zeta'^{\mathrm{u}}}) = (-1)^{f'-1}\varepsilon(\xi_{n,\zeta,\langle 1\rangle}, \psi_{M_{\zeta,\langle 1\rangle}'^{\mathrm{u}}})^{f'} \tag{11.5}$$

by Proposition 5.1. By (11.5), the problem is reduced to the case where $f = 1$. In this case, the claim follows from Lemma 11.11 and Lemma 11.16. □

11.2 Special cases

We assume that $n = p^e$, $\mathrm{ch}\, K = p$ and $f = 1$ in this subsection.

11.2.1 Odd case

Assume that $p \neq 2$ in this subsubsection.

Lemma 11.8. *We have* $\psi_{M'^{\mathrm{u}}_\zeta}\bigl(-\delta_\zeta^{p^e+1}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})\bigr)=1$ *for* $x \in k_N$.

Proof. For $x \in k_N$, we have

$$\psi_{M'^{\mathrm{u}}_\zeta}\bigl(-\delta_\zeta^{p^e+1}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})\bigr)=\psi_{M'^{\mathrm{u}}_\zeta}\bigl(-(r^{-1}\delta_\zeta-\alpha_\zeta^{-1})(\delta_\zeta+x)\bigr)=\psi_{M'^{\mathrm{u}}_\zeta}(-r^{-1}\delta_\zeta^2),$$

because $\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta)=0$ and $[M'^{\mathrm{u}}_\zeta : T^{\mathrm{u}}_\zeta]=p^e$. If $p^e \neq 3$, then we have the claim, because $\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta^2)=0$.

We assume that $p^e=3$. Then we have

$$\psi_{M'^{\mathrm{u}}_\zeta}\bigl(-r^{-1}\delta_\zeta^2\bigr)=\psi_{T^{\mathrm{u}}_\zeta}\bigl(-2r^{-2}\bigr)=\psi_0\bigl(\mathrm{Tr}_{k_N/\mathbb{F}_p}(-2r^{-2})\bigr)=\psi_0\bigl(-N\,\mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(r^{-2})\bigr)=1$$

by $\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/T^{\mathrm{u}}_\zeta}(\delta_\zeta^2)=2r^{-1}$ and $r^4=-1$. □

Let θ_ζ be as in (9.10).

Lemma 11.9. *We have*

$$\mathrm{Nr}_{N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta}(1+x\theta_\zeta^{\frac{p-1}{2}}\varpi_{M'^{\mathrm{u}}_\zeta})\equiv 1+(-2r)^{\frac{1-p}{2}}x^p\varpi_{M'^{\mathrm{u}}_\zeta}+\frac{x^2}{2}\varpi_{M'^{\mathrm{u}}_\zeta}^2 \mod \mathfrak{p}_{M'^{\mathrm{u}}_\zeta}^3$$

for $x \in k_N$.

Proof. We put $T=1+x\theta_\zeta^{\frac{p-1}{2}}\varpi_{M'^{\mathrm{u}}_\zeta}$. By $\theta_\zeta^p-\theta_\zeta=(-2r)^{-1}\delta_\zeta^2$ in (9.11), we have

$$\theta_\zeta=-\frac{1}{2r}\delta_\zeta^2\Bigl(\bigl(x^{-1}(T-1)\delta_\zeta\bigr)^2-1\Bigr)^{-1}.$$

Substituting this to $x^{-1}(T-1)\delta_\zeta=\theta_\zeta^{\frac{p-1}{2}}$, we have

$$\bigl(T^2-2T+1-x^2\varpi_{M'^{\mathrm{u}}_\zeta}^2\bigr)^{\frac{p-1}{2}}(T-1)-(-2r)^{\frac{1-p}{2}}x^p\varpi_{M'^{\mathrm{u}}_\zeta}=0.$$

The claim follows from this. □

Lemma 11.10. *We have*

$$\sum_{x\in k_N}\xi_{n,\zeta}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})^{-1}=-((-1)^{\frac{p-1}{2}}p)^{e_0}\Bigl(\frac{r}{k_N}\Bigr).$$

Proof. Let $\xi'_{n,\zeta}$ be as in subsection 9.3. We note that the left hand side of the claim does not change even if we replace $\xi_{n,\zeta}$ by $\xi'_{n,\zeta}$. We have

$$\begin{aligned}\sum_{x\in k_N}\xi'_{n,\zeta}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})^{-1}&=\sum_{x\in k_N}\xi'_{n,\zeta}\bigl(1+(-2r)^{\frac{1-p}{2}}x^p\varpi_{M'^{\mathrm{u}}_\zeta}\bigr)^{-1}\\&=\sum_{x\in k_N}\xi'_{n,\zeta}\Bigl(1-\frac{x^2}{2}\varpi_{M'^{\mathrm{u}}_\zeta}^2\Bigr)^{-1}=\sum_{x\in k_N}\psi_{M'^{\mathrm{u}}_\zeta}\Bigl(-\frac{x^2}{2}\delta_\zeta^{p^e-1}\Bigr),\end{aligned}\tag{11.6}$$

where we use Lemma 9.7 and Lemma 11.9 at the second equality and (10.4) at the last equality. The last expression in (11.6) is equal to

$$\sum_{x\in k_N}\psi_{T^{\mathrm{u}}_\zeta}\bigl(-(2r)^{-1}(p^e-1)x^2\bigr)=\sum_{x\in k_N}\psi_0(\mathrm{Tr}_{k_N/\mathbb{F}_p}(rx^2))=-\bigl((-1)^{\frac{p-1}{2}}p\bigr)^{e_0}\Bigl(\frac{r}{k_N}\Bigr)$$

by (2.16), (8.2), Lemma 11.3 and $N=2e_0$. □

Lemma 11.11. *We have* $\varepsilon(\xi_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta}) = -(-1)^{\frac{(p-1)e_0}{2}}\left(\frac{r}{k_N}\right)$.

Proof. We have

$$\begin{aligned}\varepsilon(\xi_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta}) &= p^{-e_0} \sum_{x \in k_N} \xi_{n,\zeta}\big(-\delta_\zeta^{p^e+1}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})\big)^{-1} \psi_{M'^{\mathrm{u}}_\zeta}\big(-\delta_\zeta^{p^e+1}(1+x\varpi_{M'^{\mathrm{u}}_\zeta})\big) \\ &= -(-1)^{\frac{(p-1)e_0}{2}} \left(\frac{r}{k_N}\right) \xi_{n,\zeta}(-\delta_\zeta^{p^e+1})^{-1}\end{aligned}$$

by Proposition 5.3.2, Lemma 11.8 and Lemma 11.10. We have

$$\xi_{n,\zeta}(-\delta_\zeta^{p^e+1}) = \xi'_{n,\zeta}(-\delta_\zeta^{p^e+1})(-1)^{\frac{p-1}{2}\frac{p^e+1}{2}N} = \xi'_{n,\zeta}(-\delta_\zeta^{p^e+1}) = \xi'_{n,\zeta}\big(-(-2r)^{(p^e+1)\frac{1-p}{2}}\big) = 1,$$

where we use

$$\mathrm{Nr}_{N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta}(\theta_\zeta^{\frac{p-1}{2}} \varpi_{M'^{\mathrm{u}}_\zeta}) = (-2r)^{\frac{1-p}{2}} \varpi_{M'^{\mathrm{u}}_\zeta}$$

at the third equality and $k_N^\times \subset \mathrm{Nr}_{N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta}((N'^{\mathrm{u}}_\zeta)^\times)$ at the last equality. Thus, we have the claim. □

11.2.2 Even case

Assume that $p = 2$ in this subsubsection.

Lemma 11.12. *We have* $\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/K}(\delta_\zeta^{2^e+1}) = 0$ *and*

$$\mathrm{Tr}_{M'^{\mathrm{u}}_\zeta/K}(\delta_\zeta^{2^e}) = \begin{cases} 1 & \text{if } e = 1, \\ 0 & \text{if } e \geq 2. \end{cases}$$

Proof. These follow from $\delta_\zeta^{2^e} - \delta_\zeta = \alpha_\zeta^{-1} + 1$. □

Lemma 11.13. *We have* $\mathrm{Nr}_{N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta}(\theta_\zeta \delta_\zeta^{-1}) = \delta_\zeta^{-1}$.

Proof. We have $\mathrm{Nr}_{N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta}(\theta_\zeta) = \delta_\zeta^3$ by $\theta_\zeta^2 - \theta_\zeta = \delta_\zeta \eta_\zeta$ and $\eta_\zeta^2 - \eta_\zeta = \delta_\zeta$. The claim follows from this. □

Let $\sigma_0 \in \mathrm{Gal}(N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta)$ be a generator of $\mathrm{Gal}(N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta)$ determined by $\sigma_0(\eta_\zeta) - \eta_\zeta = 1$ and $\sigma_0(\theta_\zeta) - \theta_\zeta = \eta_\zeta$.

Lemma 11.14. *Let* $\iota_{n,\zeta} \colon \mathrm{Gal}(N'^{\mathrm{u}}_\zeta/M'^{\mathrm{u}}_\zeta) \to \mathbb{C}^\times$ *be the homomorphism induced by* $\xi'_{n,\zeta}$ *(cf. Lemma 9.9). Then we have* $\iota_{n,\zeta}(\sigma_0) = -\sqrt{-1}$.

Proof. Let s, t be as in (9.4). We take $\sigma \in I_{M'^{\mathrm{u}}_\zeta}$ such that $\Theta_\zeta(\sigma) = \big((1, t, s^2), 0\big)$. Recall that

$$\phi'\left((1, t, s^2)\right) = \bar{g}\left(1, s^2 + \sum_{0 \leq i < j \leq e-1} t^{2^i + 2^j}\right) \in R''_0$$

is a generator. Then it suffices to show that $\sigma(\eta_\zeta) - \eta_\zeta = 1$ and $\sigma(\theta_\zeta) - \theta_\zeta = \eta_\zeta$. We can check the first equality easily. To show the second equality, it suffices to show that $\sigma(\theta'_\zeta) - \theta'_\zeta = \eta_\zeta^{2^{e-1}}$. By (2.11), we have

$$\sigma(\gamma_\zeta) - \gamma_\zeta \equiv s^2 + \sum_{i=0}^{e-1} (t\beta_\zeta + t^2)^{2^i} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

By $t = \sigma(\beta_\zeta) - \beta_\zeta$, $\mathrm{Tr}_{\mathbb{F}_{2^e}/\mathbb{F}_2}(t) = 1$ and (9.14), we have

$$\sigma(\gamma'_\zeta) - \gamma'_\zeta = \eta_\zeta - b_1 + s^2 + \sum_{0 \le i \le j \le e-1} t^{2^i + 2^j} \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}.$$

Hence, by (9.15) and (9.19), we have

$$\sigma(\theta'_\zeta) - \theta'_\zeta \equiv \sum_{i=0}^{2^{e-1}} d_i \eta_\zeta^i \mod \mathfrak{p}_{N'^{\mathrm{u}}_\zeta}$$

with some $d_i \in k^{\mathrm{ac}}$. By (9.19), we have

$$\sum_{i=0}^{e-1} (t(\sigma(\gamma'_\zeta) - \gamma'_\zeta))^{2^i} = \sigma(\gamma'_\zeta) - \gamma'_\zeta + \sum_{1 \le i \le j \le e-1} t^{2^i + 2^j} + \sum_{0 \le i < j \le e-1} t^{2^j} (\delta_\zeta - s)^{2^i}.$$

Therefore, again by (9.15) and (9.19), we have

$$d_0 = b_1 + s^2 + \sum_{0 \le i \le j \le e-1} t^{2^i + 2^j} + \sum_{1 \le i \le j \le e-1} t^{2^i + 2^j} + b_1^2 = s + s^2 + t = 0.$$

This implies $\sigma(\theta'_\zeta) - \theta'_\zeta = \eta_\zeta^{2^{e-1}}$, since we know that $\sigma(\theta'_\zeta) - \theta'_\zeta - \eta_\zeta^{2^{e-1}} \in \mathbb{F}_2$ by Lemma 9.8 and $\sigma(\eta_\zeta) - \eta_\zeta = 1$. $\square$

Lemma 11.15. *We have*

$$\varepsilon(\xi'_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta}) = \begin{cases} \frac{1+\sqrt{-1}}{\sqrt{2}} & \text{if } e = 1, \\ \frac{1-\sqrt{-1}}{\sqrt{2}} & \text{if } e \ge 2. \end{cases}$$

Proof. By Proposition 5.3, (10.4), Lemma 11.3, Lemma 11.12 and Lemma 11.13, we have

$$\begin{aligned} \varepsilon(\xi'_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta}) &= 2^{-\frac{1}{2}} \sum_{x \in \mathbb{F}_2} \xi'_{n,\zeta}\big(\delta_\zeta^{2^e+1}(1 + x\delta_\zeta^{-1})\big)^{-1} \psi_{M'^{\mathrm{u}}_\zeta}\big(\delta_\zeta^{2^e+1}(1 + x\delta_\zeta^{-1})\big) \\ &= \begin{cases} 2^{-\frac{1}{2}}\big(1 - \xi'_{n,\zeta}(1 + \delta_\zeta^{-1})^{-1}\big) & \text{if } e = 1, \\ 2^{-\frac{1}{2}}\big(1 + \xi'_{n,\zeta}(1 + \delta_\zeta^{-1})^{-1}\big) & \text{if } e \ge 2. \end{cases} \end{aligned} \tag{11.7}$$

First assume that $e = 1$. Then we know the equality in the claim modulo $\mu_2(\mathbb{C})$ by Lemma 11.5. Hence it suffices to show the equality of the real parts. This follows from (11.7). In particular, we have $\xi'_{2,\zeta}(1 + \delta_\zeta^{-1}) = \sqrt{-1}$.

Next, we consider the general case. We put $\alpha'_1 = 1/(\delta_\zeta^2 - \delta_\zeta + 1)$ and $\varpi' = \alpha_1'^3$. Let $\xi'_{2,1,\zeta}$ denote $\xi'_{2,1}$ in the case where K and ϖ are replaced by $\mathbb{F}_2((\varpi'))$ and ϖ'. By applying Lemma 11.14 to $\xi'_{n,\zeta}$ and $\xi'_{2,1,\zeta}$, we have $\xi'_{n,\zeta} = \xi'_{2,1,\zeta}$. We know that $\xi'_{2,1,\zeta}(1 + \delta_\zeta^{-1}) = \sqrt{-1}$ by the result in the case $e = 1$. Hence, we have $\xi'_{n,\zeta}(1 + \delta_\zeta^{-1}) = \sqrt{-1}$, which shows the claim. $\square$

Lemma 11.16. *We have*

$$\varepsilon(\xi_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta}) = (-1)^{1+\epsilon_0}.$$

Proof. The epsilon factor $\varepsilon(\xi_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta})$ equals $\varepsilon(\xi'_{n,\zeta}, \psi_{M'^{\mathrm{u}}_\zeta})$ times

$$\begin{cases} \left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{-3(2^e+1)} & \text{if } e \ne 2, \\ -\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^{-3(2^e+1)} & \text{if } e = 2 \end{cases}$$

by Lemma 4.2, (9.12), Lemma 10.3. Hence, the claim follows from Lemma 11.15. $\square$

A Realization in cohomology of Artin–Schreier variety

We realize τ_n in the cohomology of an Artin–Schreier variety. Let ν_{n-2} be the quadratic form on $\mathbb{A}^{n-2}_{k^{\mathrm{ac}}}$ defined by

$$\nu_{n-2}\big((y_i)_{1\le i\le n-2}\big)=-\frac{1}{n'}\sum_{1\le i\le j\le n-2} y_i y_j.$$

Let X be the smooth affine variety over k^{ac} defined by

$$x^p-x=y^{p^e+1}+\nu_{n-2}\big((y_i)_{1\le i\le n-2}\big)\ \text{ in } \mathbb{A}^n_{k^{\mathrm{ac}}}.$$

We define a right action of $Q\rtimes\mathbb{Z}$ on X by

$$\begin{gathered}(x,y,(y_i)_{1\le i\le n-2})((a,b,c),0)=\left(x+\sum_{i=0}^{e-1}(by)^{p^i}+c,a(y+b^{p^e}),(a^{\frac{p^e+1}{2}}y_i)_{1\le i\le n-2}\right),\\ (x,y,(y_i)_{1\le i\le n-2})\,\mathrm{Fr}(1)=(x^p,y^p,(y_i^p)_{1\le i\le n-2}).\end{gathered}$$

We consider the morphism

$$\pi_{n-2}\colon \mathbb{A}^{n-1}_{k^{\mathrm{ac}}}\to\mathbb{A}^1_{k^{\mathrm{ac}}};\ (y,(y_i)_{1\le i\le n-2})\mapsto y^{p^e+1}+\nu_{n-2}\big((y_i)_{1\le i\le n-2}\big).$$

Then we have a decomposition

$$H^{n-1}_{\mathrm{c}}(X,\overline{\mathbb{Q}}_\ell)\cong\bigoplus_{\psi\in\mathbb{F}_p^\vee\setminus\{1\}}H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}},\pi^*_{n-2}\mathcal{L}_\psi)\tag{A.1}$$

as $Q\rtimes\mathbb{Z}$ representations. Let ρ_n be the representation over $\mathbb{C}$ of $Q\rtimes\mathbb{Z}$ defined by

$$H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}},\pi^*_{n-2}\mathcal{L}_{\psi_0})\Big(\frac{n-1}{2}\Big)$$

and ι, where $\left(\frac{n-1}{2}\right)$ means the twist by the character $((a,b,c),m)\mapsto p^{m(n-1)/2}$.

Lemma A.1. *If $p\neq 2$, then we have* $\det\nu_{n-2}=-(-2n')^n\in\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$.

Proof. This is an easy calculation. □

Proposition A.2. *We have $\tau_n\simeq\rho_n$.*

Proof. Let Y be the smooth affine variety over k^{ac} defined by

$$x^p-x=\nu_{n-2}\big((y_i)_{1\le i\le n-2}\big)\ \text{ in } \mathbb{A}^{n-1}_{k^{\mathrm{ac}}}.$$

We define a right action of $Q\rtimes\mathbb{Z}$ on Y by

$$\begin{gathered}\big(x,(y_i)_{1\le i\le n-2}\big)\big((a,b,c),0\big)=\big(x,(a^{\frac{p^e+1}{2}}y_i)_{1\le i\le n-2}\big),\\ \big(x,(y_i)_{1\le i\le n-2}\big)\,\mathrm{Fr}(1)=\big(x^p,(y_i^p)_{1\le i\le n-2}\big).\end{gathered}$$

Using the action of $Q\rtimes\mathbb{Z}$ on Y, we can define an action of $Q\rtimes\mathbb{Z}$ on $H^{n-2}_{\mathrm{c}}(\mathbb{A}^{n-2}_{k^{\mathrm{ac}}},\nu^*_{n-2}\mathcal{L}_{\psi_0})$. Then we have

$$H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}},\pi^*_{n-2}\mathcal{L}_{\psi_0})\cong H^1_{\mathrm{c}}(\mathbb{A}^1_{k^{\mathrm{ac}}},\pi^*\mathcal{L}_{\psi_0})\otimes H^{n-2}_{\mathrm{c}}(\mathbb{A}^{n-2}_{k^{\mathrm{ac}}},\nu^*_{n-2}\mathcal{L}_{\psi_0})\tag{A.2}$$

by the Künneth formula, where the isomorphism is compatible with the actions of $Q \rtimes \mathbb{Z}$. By (A.2), it suffices to show the action of $Q \rtimes \mathbb{Z}$ on

$$H_{\mathrm{c}}^{n-2}(\mathbb{A}_{k^{\mathrm{ac}}}^{n-2}, \nu_{n-2}^{*}\mathcal{L}_{\psi_0})\Big(\frac{n-1}{2}\Big) \tag{A.3}$$

is equal to the character (2.9) via ι.

First, consider the case where $p \neq 2$. The equality of the actions of Q follows from [DL98, Lemma 2.2.3]. We have

$$\begin{aligned}(-1)^{n-2} \sum_{\mathbf{y} \in \mathbb{F}_p^{n-2}} \psi_0\big(\nu_{n-2}(\mathbf{y})\big) &= \Big(\frac{-1}{p}\Big)\Big(-\Big(\frac{-2n'}{p}\Big)\Big)^{n} (\epsilon(p)\sqrt{p})^{n-2} \\ &= \left(-\epsilon(p)\left(\frac{-2n'}{p}\right)\right)^{n} \sqrt{p}^{n-2}\end{aligned} \tag{A.4}$$

by Lemma A.1. The equality of the actions of $\mathrm{Fr}(1) \in Q \rtimes \mathbb{Z}$ follows from [Del77, Sommes trig. Scholie 1.9] and (A.4).

If $p = 2$, the equality follows from [IT20, Proposition 4.5] and $\big(\frac{2}{n-1}\big) = (-1)^{n(n-2)/8}$. □

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