Affinoids in the Lubin-Tate perfectoid space and simple epipelagic representations II: wild case

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Abstract

We construct a family of affinoids in the Lubin-Tate perfectoid space and their formal models such that the middle cohomology of their reductions realizes the local Langlands correspondence and the local Jacquet-Langlands correspondence for the simple epipelagic representations. The reductions of the formal models are isomorphic to the perfections of some Artin-Schreier varieties, whose cohomology realizes primitive Galois representations.

Introduction

Let K be a non-archimedean local field with a residue field k. Let p be the characteristic of k. We write $\mathcal{O}_K$ for the ring of integers of K, and $\mathfrak{p}$ for the maximal ideal of $\mathcal{O}_K$. We fix an algebraic closure k^{ac} of k. The Lubin-Tate spaces are deformation spaces of the one-dimensional formal $\mathcal{O}_K$-module over k^{ac} of height n with level structures. We take a prime number ℓ that is different from p. The local Langlands correspondence (LLC) and the local Jacquet-Langlands correspondence (LJLC) for cuspidal representations of GL_n are realized in the ℓ-adic cohomology of Lubin-Tate spaces. This is proved in [Boy99] and [HT01] by global automorphic arguments. On the other hand, the relation between these correspondences and the geometry of Lubin-Tate spaces is not well understood.

In this direction, Yoshida constructs a semi-stable model of the Lubin-Tate space with a full level $\mathfrak{p}$-structure, and studies its relation with the LLC in [Yos10]. In this case, the Deligne-Lusztig varieties appear as open subschemes in the reductions of the semi-stable models, and their cohomology realizes the LLC for depth zero supercuspidal representations. In [BW16], Boyarchenko-Weinstein construct a family of affinoids in the Lubin-Tate perfectoid space and their formal models so that the cohomology of the reductions realizes the LLC and the LJLC for some representations which are related to unramified extensions of K (*cf.* [Wei10a] for some special case at a finite level). It generalizes a part of the result in [Yos10] to higher conductor cases. In the Lubin-Tate perfectoid setting, the authors study the case for the essentially tame simple epipelagic representations in [IT13], where simple epipelagic means that the exponential Swan conductor is equal to one. See [BH05] for the notion of essentially tame representations. The result in [IT13] is generalized to some higher conductor essentially tame cases by Tokimoto in [Tok16] (*cf.* [IT15a] for some special case at a finite level).

In all the above cases, Langlands parameters are of the form $\mathrm{Ind}_{W_L}^{W_K} \chi$ for a finite separable extension L over K and a character χ of W_L, where W_K and W_L denote the Weil groups of K and L respectively. Further, the construction of affinoids directly involves CM points which have multiplication by L. In this paper, we study the case for simple epipelagic representations which are not essentially tame. In this case, the Langlands parameters can not be written as inductions of characters. Hence, we have no canonical candidate of CM points which may be used for constructions of affinoids.

We will explain our main result. All the representation are essentially tame if n is prime to p. Hence, we assume that p divides n. We say that a representation is essentially simple epipelagic if it is a character twist of a simple epipelagic representation. Let q be the number of the elements of k and D be the central division algebra over K of invariant $1/n$. We write $q = p^f$ and $n = p^e n'$, where n' is prime to p. We put $m = \gcd(e, f)$. The main theorem is the following:

Theorem. *For $r \in \mu_{q-1}(K)$, there is an affinoid $\mathcal{X}_r$ in the Lubin-Tate perfectoid space and its formal model $\mathfrak{X}_r$ such that*

- *the special fiber $\overline{\mathfrak{X}_r}$ of $\mathfrak{X}_r$ is isomorphic to the perfection of the affine smooth variety defined by*
$$z^{p^m} - z = y^{p^e+1} - \frac{1}{n'} \sum_{1 \leq i \leq j \leq n-2} y_i y_j \quad \text{in } \mathbb{A}^n_{k^{\mathrm{ac}}},$$
- *the stabilizer $H_r \subset GL_n(K) \times D^\times \times W_K$ of $\mathcal{X}_r$ naturally acts on $\overline{\mathfrak{X}_r}$, and*
- $\text{c-Ind}_{H_r}^{GL_n(K) \times D^\times \times W_K} H_c^{n-1}(\overline{\mathfrak{X}_r}, \overline{\mathbb{Q}}_\ell)$ *realizes the LLC and the LJLC for essentially simple epipelagic representations.*

See Theorem 2.5 and Theorem 6.4 for precise statements. As we mentioned, we have no candidate of CM points for the construction of affinoids. First, we consider a CM point ξ which has multiplication by a field extension of K obtained by adding an n-th root of a uniformizer of K. If we imitate the construction of affinoids in [IT13] using the CM point ξ, we can get a non-trivial affinoid and its model, but the reduction degenerates in some sense, and the cohomology of the reduction does not give a supercuspidal representation. What we will do in this paper is to modify the CM point ξ using information of field extensions which appear in the study of our simple epipelagic Langlands parameter. The modified point is no longer CM point, but we can use this point for a construction of a desired affinoid.

In the above mentioned preceding researches, the Langlands parameters are inductions of characters, and realized from commutative group actions on varieties. In the case for Deligne-Lusztig varieties, they come from the natural action of tori. In our simple epipelagic case, they come from non-commutative group actions. For example, the restriction to the inerita subgroup of a simple epipelagic Langlands parameter factors through a semi-direct product of a cyclic group with a Heisenberg type group, which acts on our Artin-Schreier variety in a very non-trivial way.

In the following, we briefly explain the content of each section. In Section 1, we collect known results on the Lubin-Tate perfectoid space, its formal model and group action on it.

In Section 2, we construct a family of affinoids and their formal models. Further we determine the reductions of them. The reduction is isomorphic to the perfection of some Artin-Schreier variety.

In Section 3, we describe the group action on the reductions. To determine the action of some special element on the cohomology of the reduction, we need to study half-dimensional cycle classes on some Artin-Schreier variety. This is done in Section 4.

In Section 6, we give an explicit description of the LLC and the LJLC for essentially simple epipelagic representations, which follows from results in [IT14] and [IT15b]. In Section 6, we give a geometric realization of the LLC and the LJLC in cohomology of our reduction.

Notation

For a non-archimedean valuation field F, its valuation ring is denoted by $\mathcal{O}_F$. For $a \in \mathbb{Q}$ and elements f, g with valuation v that takes values in $\mathbb{Q}$, we write $f \equiv g \mod a$ if $v(f-g) \geq a$, and $f \equiv g \mod_{>} a$ if $v(f-g) > a$. For a topological field extension E over F, let $\mathrm{Gal}(E/F)$ denote the group of the continuous automorphisms of E over F.

1 Lubin-Tate perfectoid space

1.1 Lubin-Tate perfectoid space and its formal model

Let K be a non-archimedean local field with a residue field k of characteristic p. Let q the number of the elements of k. We write $\mathfrak{p}$ for the maximal ideal of $\mathcal{O}_K$. We fix an algebraic closure K^{ac} of K. Let k^{ac} be the residue field of K^{ac}.

Let n be a positive integer. We take a one-dimensional formal $\mathcal{O}_K$-module $\mathcal{G}_0$ over k^{ac} of height n, which is unique up to isomorphism. Let K^{ur} be the maximal unramified extension of K in K^{ac}. We write $\widehat{K}^{\mathrm{ur}}$ for the completion of K^{ur}. Let $\mathcal{C}$ be the category of complete Noetherian local $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$-algebras with residue field k^{ac}.

Let $\mathcal{G}$ be a formal $\mathcal{O}_K$-module over $R \in \mathcal{C}$. For $a \in \mathcal{O}_K$, let $[a]_{\mathcal{G}} \colon \mathcal{G} \to \mathcal{G}$ be the multiplication by a, and $\mathcal{G}[a]$ be the kernel of $[a]_{\mathcal{G}}$. For an integer $m \geq 0$, we define $\mathcal{G}[\mathfrak{p}^m]$ to be $\mathcal{G}[a]$ for some $a \in \mathfrak{p}^m \setminus \mathfrak{p}^{m+1}$.

We consider the functor $\mathcal{C} \to \mathbf{Sets}$ which associates to an object $R \in \mathcal{C}$ the set of isomorphism classes of triples $(\mathcal{G}, \phi, \iota)$, where $(\mathcal{G}, \iota)$ is a deformation of $\mathcal{G}_0$ to R and ϕ is a Drinfeld level $\mathfrak{p}^m$-structure on $\mathcal{G}$. This functor is represented by a regular local ring A_m by [Dri74, Proposition 4.3]. Then, $\{A_m\}_{m \geq 0}$ makes an inductive system. Let I the ideal of $\varinjlim A_m$ generated by the maximal ideal of A_0. Let A be the I-adic completion of $\varinjlim A_m$. We put $\mathbf{M}_{\mathcal{G}_0,\infty} = \mathrm{Spf}\, A$.

Let K^{ab} be the maximal abelian extension of K in K^{ac}. We write $\widehat{K}^{\mathrm{ab}}$ for the completion of K^{ab}. Let $\wedge\mathcal{G}_0$ denote the one-dimensional formal $\mathcal{O}_K$-module over k^{ac} of height one. Then we have $\mathbf{M}_{\wedge\mathcal{G}_0,\infty} \simeq \mathrm{Spf}\, \mathcal{O}_{\widehat{K}^{\mathrm{ab}}}$ by the Lubin-Tate theory. We have a determinant morphism

$$\mathbf{M}_{\mathcal{G}_0,\infty} \to \mathbf{M}_{\wedge\mathcal{G}_0,\infty} \tag{1.1}$$

by [Wei10b, 2.5 and 2.7] (*cf.* [Hed10]). Then, we have the ring homomorphism $\mathcal{O}_{\widehat{K}^{\mathrm{ab}}} \to A$ determined by (1.1).

We fix a uniformizer ϖ of K. Let $\mathcal{M}_\infty$ be the open adic subspace of $\mathrm{Spa}(A, A)$ defined by $|\varpi(x)| \neq 0$ (*cf.* [Hub94, 2]). We regard $\mathcal{M}_\infty$ as an adic space over $\widehat{K}^{\mathrm{ur}}$. For a deformation $\mathcal{G}$ of $\mathcal{G}_0$ over $\mathcal{O}_{\mathbf{C}}$, we put

$$V_{\mathfrak{p}}(\mathcal{G}) = \big(\varprojlim \mathcal{G}[\mathfrak{p}^m](\mathcal{O}_{\mathbf{C}})\big) \otimes_{\mathcal{O}_K} K,$$

where the transition maps are multiplications by ϖ. By the construction, each point of $\mathcal{M}_\infty(\mathbf{C})$ corresponds to a triple $(\mathcal{G}, \phi, \iota)$ that consists of a formal $\mathcal{O}_K$-module over $\mathcal{O}_{\mathbf{C}}$, an isomorphism $\phi \colon K^n \to V_{\mathfrak{p}}(\mathcal{G})$ and an isomorphism $\iota \colon \mathcal{G}_0 \to \mathcal{G} \otimes_{\mathcal{O}_{\mathbf{C}}} k^{\mathrm{ac}}$ (*cf.* [BW16, Definition 2.10.1]).

We put $\eta = \mathrm{Spa}(\widehat{K}^{\mathrm{ab}}, \mathcal{O}_{\widehat{K}^{\mathrm{ab}}})$. By the ring homomorphism $\mathcal{O}_{\widehat{K}^{\mathrm{ab}}} \to A$, we can regard $\mathcal{M}_\infty$ as an adic space over η, for which we write $\mathcal{M}_{\infty,\eta}$. Let $\mathbf{C}$ be the completion of K^{ac}. We put $\bar{\eta} = \mathrm{Spa}(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. We have a natural embedding $\widehat{K}^{\mathrm{ab}} \hookrightarrow \mathbf{C}$. We put

$$\mathcal{M}_{\infty,\overline{\eta}} = \mathcal{M}_{\infty,\eta} \times_\eta \overline{\eta}.$$

Then, $\mathcal{M}_{\infty,\overline{\eta}}$ is a perfectoid space over $\mathbf{C}$ in the sense of [Sch12, Definition 6.15] by [Wei10b, Lemma 2.10.1]. We call $\mathcal{M}_{\infty,\overline{\eta}}$ the Lubin-Tate perfectoid space.

In the following, we recall an explicit description of $A^\circ = A\widehat{\otimes}_{\mathcal{O}_{\widehat{K}^{\mathrm{ab}}}}\mathcal{O}_{\mathbf{C}}$ given in [Wei10b, (2.9.2)]. Let $\widehat{\mathcal{G}}_0$ be the formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ whose logarithm is

$$\sum_{i=0}^{\infty} \frac{X^{q^{in}}}{\varpi^i}$$

(*cf.* [BW16, 2.3]). Let $\mathcal{G}_0$ be the formal $\mathcal{O}_K$-module over k^{ac} obtained as the reduction of $\widehat{\mathcal{G}}_0$. We put $\mathcal{O}_D = \mathrm{End}\, \mathcal{G}_0$ and $D = \mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Q}$, which is the central division algebra over K of invariant $1/n$. Let $[\,\cdot\,]$ denote the action of $\mathcal{O}_D$ on $\mathcal{G}_0$. Let φ be the element of D such that $[\varphi](X) = X^q$. Let K_n be the unramified extension of K of degree n. For an element $a \in \mathcal{O}_{\mathbf{C}}$, its image in the residue field is denoted by $\bar{a}$. We consider the K-algebra embedding of K_n into D determined by

$$[\zeta](X) = \bar{\zeta}X \quad \text{for} \quad \zeta \in \mu_{q^n-1}(K_n).$$

Then we have $\varphi^n = \varpi$ and $\varphi\zeta = \zeta^q\varphi$ for $\zeta \in \mu_{q^n-1}(K_n)$. Let $\widehat{\wedge\mathcal{G}_0}$ be the one-dimensional formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ whose logarithm is

$$\sum_{i=0}^{\infty} (-1)^{(n-1)i} \frac{X^{q^i}}{\varpi^i}.$$

We choose a compatible system $\{t_m\}_{m\geq 1}$ such that

$$t_m \in K^{\mathrm{ac}} \quad (m \geq 1), \quad t_1 \neq 0, \quad [\varpi]_{\widehat{\wedge\mathcal{G}_0}}(t_1) = 0, \quad [\varpi]_{\widehat{\wedge\mathcal{G}_0}}(t_m) = t_{m-1} \quad (m \geq 2). \tag{1.2}$$

We put

$$t = \lim_{m\to\infty} (-1)^{q(n-1)(m-1)} t_m^{q^{m-1}} \in \mathcal{O}_{\mathbf{C}}.$$

Let v be the normalized valuation of K such that $v(\varpi) = 1$. The valuation v naturally extends to a valuation on $\mathbf{C}$, for which we again write v. Note that $v(t) = 1/(q-1)$. For an integer $i \geq 0$, we put $t^{q^{-i}} = \lim_{m\to\infty} (-1)^{q(n-1)(m-1)} t_m^{q^{m-i-1}}$.

Let W_K be the Weil group of K. Let $\mathrm{Art}_K \colon K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ be the Artin reciprocity map normalized such that a uniformizer is sent to a lift of the geometric Frobenius element. We use similar normalizations also for the Artin reciprocity maps for other non-archimedean local fields. Let $\sigma \in W_K$. Let n_σ be the image of σ under the composite

$$W_K \twoheadrightarrow W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} K^\times \xrightarrow{v} \mathbb{Z}.$$

Let $a_K \colon W_K \to \mathcal{O}_K^\times$ be the homomorphism given by the action of W_K on $\{t_m\}_{m\geq 1}$. It induces an isomorphism $a_K \colon \mathrm{Gal}(\widehat{K}^{\mathrm{ab}}/\widehat{K}^{\mathrm{ur}}) \simeq \mathcal{O}_K^\times$.

For $m \geq 0$, we put

$$\delta_m(X_1, \ldots, X_n) = {}^{\widehat{\wedge\mathcal{G}_0}}\!\!\sum_{(m_1,\ldots,m_n)} \mathrm{sgn}(m_1, \ldots, m_n) X_1^{q^{m_1-m}} \cdots X_n^{q^{m_n-m}} \tag{1.3}$$

in $\mathcal{O}_K[[X_1^{1/q^\infty}, \ldots, X_n^{1/q^\infty}]]$, where

- the symbol ${}^{\widehat{\wedge\mathcal{G}_0}}\sum$ denotes the sum under the additive operation of $\widehat{\wedge\mathcal{G}_0}$,
- we take the sum over n-tuples $(m_1, \ldots, m_n)$ of integers such that $m_1+\cdots+m_n = n(n-1)/2$ and $m_i \not\equiv m_j \mod n$ for $i \neq j$,

- $\mathrm{sgn}(m_1,\ldots,m_n)$ is the sign of the permutation on $\mathbb{Z}/n\mathbb{Z}$ defined by $i \mapsto m_i$.

We put

$$\delta = \lim_{m\to\infty} \delta_m^{q^m} \in \mathcal{O}_{\mathbf{C}}[[X_1^{1/q^\infty},\ldots,X_n^{1/q^\infty}]].$$

For $l \geq 1$, we put $\delta^{q^{-l}} = \lim_{m\to\infty} \delta_m^{q^{m-l}}$. The following theorem follows from [Wei10b, (2.9.2)] and the proof of [BW16, Theorem 2.10.3] (*cf.* [SW13, Theorem 6.4.1]).

Theorem 1.1. *Let* $\sigma \in \mathrm{Gal}(\widehat{K}^{\mathrm{ab}}/\widehat{K}^{\mathrm{ur}})$. *We put* $A^\sigma = A\widehat{\otimes}_{\mathcal{O}_{\widehat{K}^{\mathrm{ab}}},\sigma}\mathcal{O}_{\mathbf{C}}$. *Then, we have an isomorphism*

$$A^\sigma \simeq \mathcal{O}_{\mathbf{C}}[[X_1^{1/q^\infty},\ldots,X_n^{1/q^\infty}]]/(\delta(X_1,\ldots,X_n)^{q^{-m}} - \sigma(t^{q^{-m}}))_{m\geq 0}. \tag{1.4}$$

For $\sigma \in \mathrm{Gal}(\widehat{K}^{\mathrm{ab}}/\widehat{K}^{\mathrm{ur}})$, let $\mathcal{M}_{\infty,\bar{\eta},\sigma}$ be the base change of $\mathcal{M}_{\infty,\eta}$ by $\bar{\eta} \to \eta \xrightarrow{\sigma} \eta$. For $\sigma \in \mathrm{Gal}(\widehat{K}^{\mathrm{ab}}/\widehat{K}^{\mathrm{ur}})$ and $\alpha = a_K(\sigma) \in \mathcal{O}_K^\times$, we write A^α for A^σ and $\mathcal{M}_{\infty,\bar{\eta},\alpha}^{(0)}$ for $\mathcal{M}_{\infty,\bar{\eta},\sigma}^{(0)}$. We put

$$\mathbf{M}_{\infty,\mathcal{O}_{\mathbf{C}}}^{(0)} = \coprod_{\alpha\in\mathcal{O}_K^\times} \mathrm{Spf}\, A^\alpha, \quad \mathcal{M}_{\infty,\bar{\eta}}^{(0)} = \coprod_{\alpha\in\mathcal{O}_K^\times} \mathcal{M}_{\infty,\bar{\eta},\alpha}. \tag{1.5}$$

Then $\mathcal{M}_{\infty,\bar{\eta}}^{(0)}$ is the generic fiber of $\mathbf{M}_{\infty,\mathcal{O}_{\mathbf{C}}}^{(0)}$, and $\mathcal{M}_{\infty,\bar{\eta}}^{(0)}(\mathbf{C}) = \mathcal{M}_\infty(\mathbf{C})$.

Let $+_{\widehat{\mathcal{G}_0}}$ and $+_{\widehat{\wedge\mathcal{G}_0}}$ be the additive operations for $\widehat{\mathcal{G}}_0$ and $\widehat{\wedge\mathcal{G}_0}$ respectively.

Lemma 1.2. 1. *We have* $X_1 +_{\widehat{\mathcal{G}_0}} X_2 \equiv X_1 + X_2$ *modulo terms of total degree* q^n.
2. *We have* $X_1 +_{\widehat{\wedge\mathcal{G}_0}} X_2 \equiv X_1 + X_2$ *modulo terms of total degree* q.

Proof. This follows from the descriptions of the logarithms of $\widehat{\mathcal{G}}_0$ and $\widehat{\wedge\mathcal{G}_0}$ (*cf.* [Wei10b, Lemma 5.2.1]). □

Let $\boldsymbol{X}_i$ be $(X_i^{q^{-j}})_{j\geq 0}$ for $1 \leq i \leq n$. We write $\delta(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n)$ for the q-th power compatible system $(\delta(X_1,\ldots,X_n)^{q^{-j}})_{j\geq 0}$.

For q-th power compatible systems $\boldsymbol{X} = (X^{q^{-j}})_{j\geq 0}$ and $\boldsymbol{Y} = (Y^{q^{-j}})_{j\geq 0}$ that take values in $\mathcal{O}_{\mathbf{C}}$, we define q-th power compatible systems $\boldsymbol{X}+\boldsymbol{Y}$, $\boldsymbol{X}-\boldsymbol{Y}$ and $\boldsymbol{X}\boldsymbol{Y}$ by the requirement that their j-th components for $j \geq 0$ are

$$\lim_{m\to\infty}(X^{q^{-m}} + Y^{q^{-m}})^{q^{m-j}}, \quad \lim_{m\to\infty}(X^{q^{-m}} - Y^{q^{-m}})^{q^{m-j}}, \quad \text{and} \quad X^{q^{-j}}Y^{q^{-j}}$$

respectively. For such $\boldsymbol{X} = (X^{q^{-j}})_{j\geq 0}$, we put $v(\boldsymbol{X}) = v(X)$. We put

$$\delta_0'(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n) = \sum_{(m_1,\ldots,m_n)} \mathrm{sgn}(m_1,\ldots,m_n)\boldsymbol{X}_1^{q^{m_1}}\cdots\boldsymbol{X}_n^{q^{m_n}},$$

where we take the sum in the above sense and the index set is the same as (1.3). We recall the following lemma from [IT13, Lemma 1.6].

Lemma 1.3. *Assume that* $n \geq 2$ *and* $v(\boldsymbol{X}_i) \geq (nq^{i-1}(q-1))^{-1}$ *for* $1 \leq i \leq n$. *Then, we have*

$$\delta(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n) \equiv \delta_0'(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n) \mod_{>} \frac{1}{n} + \frac{1}{q-1}.$$

1.2 Group action on the formal model

We define a group action on the formal scheme $\mathbf{M}^{(0)}_{\infty,\mathcal{O}_{\mathbf{C}}}$, which is compatible with usual group actions on Lubin-Tate spaces with finite level (*cf.* [BW16, 2.11]). We put

$$G = GL_n(K) \times D^{\times} \times W_K.$$

Let G^0 denote the kernel of the following homomorphism:

$$G \to \mathbb{Z};\ (g,d,\sigma) \mapsto v\big(\det(g)^{-1}\mathrm{Nrd}_{D/K}(d)\mathrm{Art}_K^{-1}(\sigma)\big).$$

Then, the formal scheme $\mathbf{M}^{(0)}_{\infty,\mathcal{O}_{\mathbf{C}}}$ admits a right action of G^0. We write down the action. In the sequel, we use the following notation:

> For $a \in \mu_{q^n-1}(K_n) \cup \{0\}$, let $a^{q^{-m}}$ denote the q^m-th root of a in $\mu_{q^n-1}(K_n) \cup \{0\}$ for a positive integer m, and we simply write a also for the q-th power compatible system $(a^{q^{-m}})_{m \geq 0}$.

For q-th power compatible systems $\boldsymbol{X} = (X^{q^{-j}})_{j \geq 0}$ and $\boldsymbol{Y} = (Y^{q^{-j}})_{j \geq 0}$ that take values in $\mathcal{O}_{\mathbf{C}}$, we define a q-th power compatible system $\boldsymbol{X} +_{\widehat{\mathcal{G}}_0} \boldsymbol{Y}$ by the requirement that their j-th components for $j \geq 0$ are $\lim_{m \to \infty}(X^{q^{-m}} +_{\widehat{\mathcal{G}}_0} Y^{q^{-m}})^{q^{m-j}}$. The symbol $\widehat{\mathcal{G}}_0 \sum$ denotes this summation for q-th power compatible systems.

First, we define a left action of $GL_n(K) \times D^{\times}$ on the ring $B_n = \mathcal{O}_{\mathbf{C}}[[X_1^{1/q^{\infty}}, \ldots, X_n^{1/q^{\infty}}]]$. For $a = \sum_{j=l}^{\infty} a_j \varpi^j \in K$ with $l \in \mathbb{Z}$ and $a_j \in \mu_{q-1}(K) \cup \{0\}$, we set

$$[a] \cdot \boldsymbol{X}_i = \widehat{\mathcal{G}}_0 \sum_{j=l}^{\infty} a_j \boldsymbol{X}_i^{q^{jn}}.$$

for $1 \leq i \leq n$. Let $g \in GL_n(K)$. We write $g = (a_{i,j})_{1 \leq i,j \leq n}$. Then, let g act on the ring B_n by

$$g^* \colon B_n \to B_n;\ \boldsymbol{X}_i \mapsto \widehat{\mathcal{G}}_0 \sum_{j=1}^{n} [a_{j,i}] \cdot \boldsymbol{X}_j \quad \text{for } 1 \leq i \leq n. \tag{1.6}$$

Let $d \in D^{\times}$. We write $d^{-1} = \sum_{j=l}^{\infty} d_j \varphi^j \in D^{\times}$ with $l \in \mathbb{Z}$ and $d_j \in \mu_{q^n-1}(K_n) \cup \{0\}$. Then, let d act on B_n by

$$d^* \colon B_n \to B_n;\ \boldsymbol{X}_i \mapsto \widehat{\mathcal{G}}_0 \sum_{j=l}^{\infty} d_j \boldsymbol{X}_i^{q^j} \quad \text{for } 1 \leq i \leq n. \tag{1.7}$$

Now, we give a right action of G^0 on $\mathbf{M}^{(0)}_{\infty,\mathcal{O}_{\mathbf{C}}}$ using (1.6) and (1.7). Let $(g,d,1) \in G^0$. We set $\gamma(g,d) = \det(g)\mathrm{Nrd}_{D/K}(d)^{-1} \in \mathcal{O}_K^{\times}$. We put $\boldsymbol{t} = (t^{q^{-m}})_{m \geq 0}$. Let $(g,d,1)$ act on $\mathbf{M}^{(0)}_{\infty,\mathcal{O}_{\mathbf{C}}}$ by

$$A^{\alpha} \to A^{\gamma(g,d)^{-1}\alpha};\ \boldsymbol{X}_i \mapsto (g,d) \cdot \boldsymbol{X}_i \quad \text{for } 1 \leq i \leq n,$$

where $\alpha \in \mathcal{O}_K^{\times}$. This is well-defined, because the equation

$$\delta((g,d) \cdot \boldsymbol{X}_1, \ldots, (g,d) \cdot \boldsymbol{X}_n) = \mathrm{Art}_K(\alpha)(\boldsymbol{t})$$

is equivalent to $\delta(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = \mathrm{Art}_K(\gamma(g,d)^{-1}\alpha)(\boldsymbol{t})$. Let $(1, \varphi^{-n_\sigma}, \sigma) \in G^0$ act on $\mathbf{M}^{(0)}_{\infty,\mathcal{O}_{\mathbf{C}}}$ by

$$A^{\alpha} \to A^{a_K(\sigma)\alpha};\ \boldsymbol{X}_i \mapsto \boldsymbol{X}_i, \quad x \mapsto \sigma(x) \quad \text{for } 1 \leq i \leq n \text{ and } x \in \mathcal{O}_{\mathbf{C}},$$

where $\alpha \in \mathcal{O}_K^{\times}$. Thus, we have a right action of G^0 on $\mathbf{M}_{\infty,\mathcal{O}_{\mathbf{C}}}$, which induces a right action on $\mathcal{M}^{(0)}_{\infty,\bar{\eta}}(\mathbf{C}) = \mathcal{M}_{\infty}(\mathbf{C})$.

Remark 1.4. *For $a \in K^{\times}$, the action of $(a,a,1) \in G^0$ on $\mathbf{M}_{\infty,\mathcal{O}_{\mathbf{C}}}$ is trivial by the definition.*

1.3 CM points

We recall the notion of CM points from [BW16, 3.1]. Let L be a finite extension of K of degree n inside $\mathbf{C}$.

Definition 1.5. *A deformation $\mathcal{G}$ of $\mathcal{G}_0$ over $\mathcal{O}_{\mathbf{C}}$ has CM by L if there is an isomorphism $L \xrightarrow{\sim} \mathrm{End}(\mathcal{G}) \otimes_{\mathcal{O}_K} K$ as K-algebras such that the induced map $L \to \mathrm{End}(\mathrm{Lie}\,\mathcal{G}) \otimes_{\mathcal{O}_K} K \simeq \mathbf{C}$ coincides with the natural embedding $L \subset \mathbf{C}$.*

We say that a point of $\mathcal{M}_\infty(\mathbf{C})$ has CM by L if the corresponding deformation over $\mathcal{O}_{\mathbf{C}}$ has CM by L.

Let $\xi \in \mathcal{M}_\infty(\mathbf{C})$ be a point that has CM by L. Let $(\mathcal{G}, \phi, \iota)$ be the triple corresponding to ξ. Then we have embeddings $i_{M,\xi}\colon L \to M_n(K)$ and $i_{D,\xi}\colon L \to D$ characterized by the commutative diagrams

$$\begin{tikzcd} K^n \arrow[r, "\phi"] \arrow[d, "i_{M,\xi}(a)"'] & V_{\mathfrak{p}}\mathcal{G} \arrow[d, "V_{\mathfrak{p}}(a)"] \\ K^n \arrow[r, "\phi"] & V_{\mathfrak{p}}\mathcal{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{G}_0 \arrow[r, "\iota"] \arrow[d, "i_{D,\xi}(a)"'] & \mathcal{G} \otimes_{\mathcal{O}_{\mathbf{C}}} k^{\mathrm{ac}} \arrow[d, "a\otimes\mathrm{id}"] \\ \mathcal{G}_0 \arrow[r, "\iota"] & \mathcal{G} \otimes_{\mathcal{O}_{\mathbf{C}}} k^{\mathrm{ac}} \end{tikzcd}$$

in the isogeny category for $a \in L$. We put $i_\xi = (i_{M,\xi}, i_{D,\xi})\colon L \to M_n(K) \times D$. We put

$$(GL_n(K) \times D^\times)^0 = \{(g,d) \in GL_n(K) \times D^\times \mid (g,d,1) \in G^0\}.$$

Lemma 1.6. *[BW16, Lemma 3.1.2] The group $(GL_n(K) \times D^\times)^0$ acts transitively on the set of the points of $\mathcal{M}_\infty(\mathbf{C})$ that have CM by L. For $\xi \in \mathcal{M}_\infty(\mathbf{C})$ that has CM by L, the stabilizer of ξ in $(GL_n(K) \times D^\times)^0$ is $i_\xi(L^\times)$.*

2 Good reduction of affinoids

2.1 Construction of affinoids

We take a uniformizer ϖ of K. Let $r \in \mu_{q-1}(K)$. We put $\varpi_r = r\varpi$. We take $\varphi_r \in \mathbf{C}$ such that $\varphi_r^n = \varpi_r$. We put $L_r = K(\varphi_r)$. Let $\mathcal{G}_r$ be the one-dimensional formal $\mathcal{O}_{L_r}$-module over $\mathcal{O}_{\widehat{L}_r^{\mathrm{ur}}}$ defined by

$$[\varphi_r]_{\mathcal{G}_r}(X) = \varphi_r X + X^q, \quad [\zeta]_{\mathcal{G}_r}(X) = \zeta X \quad \text{for} \quad \zeta \in \mu_{q-1}(L) \cup \{0\}. \tag{2.1}$$

We take a compatible system $\{t_{r,m}\}_{m\geq 1}$ in $\mathbf{C}$ such that

$$t_{r,1} \neq 0, \quad [\varphi_r]_{\mathcal{G}_r}(t_{r,1}) = 0, \quad [\varphi_r]_{\mathcal{G}_r}(t_{r,m}) = t_{r,m-1}$$

for $m \geq 2$. We apply results in Section 1 replacing ϖ by ϖ_r and taking a model of $\mathcal{G}_0$ given by the reduction of (2.1). We put

$$\varphi_{M,r} = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ \varpi_r & \mathbf{0} \end{pmatrix} \in M_n(K).$$

Let $\varphi_{D,r} \in D$ be the image of φ_r under the composite

$$\mathcal{O}_{L_r} \to \mathrm{End}\,\mathcal{G}_r \to \mathrm{End}\,\mathcal{G}_0 = \mathcal{O}_D.$$

For $\xi \in \mathcal{M}_{\infty,\overline{\eta}}^{(0)}(\mathbf{C})$, we write $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n)$ for the coordinate of ξ with respect to $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$, where $\boldsymbol{\xi}_i = (\xi_i^{q^{-j}})_{j\geq 0}$ for $1 \leq i \leq n$.

Lemma 2.1. *There exists $\xi_r \in \mathcal{M}^{(0)}_{\infty,\overline{\eta}}(\mathbf{C})$ such that*

$$\xi_{r,i}^{q^{-j}} = \lim_{m\to\infty} t_{r,m}^{q^{m-i-j}} \in \mathcal{O}_{\mathbf{C}} \tag{2.2}$$

for $1 \le i \le n$ and $j \ge 0$. Further, we have the following:

(i) ξ_r has CM by L_r.

(ii) We have $i_{\xi_r}(\varphi_r) = (\varphi_{M,r}, \varphi_{D,r}) \in M_n(K) \times D$.

(iii) $\boldsymbol{\xi}_{r,i} = \boldsymbol{\xi}_{r,i+1}^q$ for $1 \le i \le n-1$.

(iv) $v(\xi_{r,i}) = 1/(nq^{i-1}(q-1))$ for $1 \le i \le n$.

Proof. This is proved as in the same way as [IT13, Lemma 2.2]. $\square$

We take ξ_r as in Lemma 2.1. We can replace the choice of (1.2) so that $\delta(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n) = \boldsymbol{t}$. Then we have $\xi_r \in \mathcal{M}^{(0)}_{\infty,\overline{\eta},1}$. Let $\mathcal{D}^{n,\mathrm{perf}}_{\mathbf{C}}$ be the generic fiber of $\mathrm{Spf}\, \mathcal{O}_{\mathbf{C}}[[X_1^{1/q^\infty}, \cdots, X_n^{1/q^\infty}]]$. We consider $\mathcal{M}^{(0)}_{\infty,\overline{\eta},1}$ as a subspace of $\mathcal{D}^{n,\mathrm{perf}}_{\mathbf{C}}$ by (1.4). We put $\boldsymbol{\eta}_r = \boldsymbol{\xi}_{r,1}^{q-1}$. Note that $v(\boldsymbol{\eta}_r) = 1/n$. We write $n = p^e n'$ with $(p, n') = 1$. We put

$$\varepsilon_0 = \begin{cases} (n'+1)/2 & \text{if } p^e = 2, \\ 0 & \text{if } p^e \neq 2. \end{cases}$$

We take q-th power compatible systems $\boldsymbol{\theta}_r$ and $\boldsymbol{\lambda}_r$ in $\mathbf{C}$ satisfying

$$\boldsymbol{\theta}_r^{p^{2e}} + \boldsymbol{\eta}_r^{p^e-1}(\boldsymbol{\theta}_r + 1) = 0, \quad \boldsymbol{\lambda}_r^q - \boldsymbol{\eta}_r^{q-1}(\boldsymbol{\lambda}_r - \boldsymbol{\theta}_r^{p^e}(\boldsymbol{\theta}_r + 1) + \varepsilon_0 \boldsymbol{\eta}_r) = 0 \tag{2.3}$$

Note that

$$v(\boldsymbol{\theta}_r) = \frac{p^e - 1}{np^{2e}}, \quad v(\boldsymbol{\lambda}_r) = \frac{1}{n}\left(1 - \frac{1}{qp^e}\right). \tag{2.4}$$

We define $\xi'_r \in \mathcal{D}^{n,\mathrm{perf}}_{\mathbf{C}}$ by

$$\begin{aligned} &\boldsymbol{\xi}'_{r,1} = \boldsymbol{\xi}_{r,1}(1 + \boldsymbol{\theta}_r), \quad \boldsymbol{\xi}'_{r,i+1} = \boldsymbol{\xi}'^{\frac{1}{q}}_{r,i} \quad \text{for } 1 \le i \le n-2, \\ &\boldsymbol{\xi}'_{r,n} = \boldsymbol{\xi}'^{\frac{1}{q}}_{r,n-1}\big((1 + \boldsymbol{\theta}_r)^{-n}(1 + n'\boldsymbol{\lambda}_r)\big)^{\frac{1}{q^{n-1}}}. \end{aligned}$$

Proposition 2.2. *There uniquely exists $\xi_r^0 \in \mathcal{M}^{(0)}_{\infty,\overline{\eta},1}$ satisfying*

$$\boldsymbol{\xi}^0_{r,i} = \boldsymbol{\xi}'_{r,i} \quad \text{for } 1 \le i \le n-1, \quad \boldsymbol{\xi}^0_{r,n} \equiv \boldsymbol{\xi}'_{r,n} \mod_{>} \frac{q^2 - q + 1}{nq^{n-1}(q-1)}.$$

Proof. We have

$$\delta(\boldsymbol{\xi}'_r) \equiv \boldsymbol{t} \mod_{>} \frac{1}{q-1} + \frac{1}{n}.$$

Hence, we see the claim by Newton's method. $\square$

We take ξ_r^0 as in Proposition 2.2. We put $\boldsymbol{x}_i = \boldsymbol{X}_i / \boldsymbol{\xi}^0_{r,i}$ for $1 \le i \le n$. We define $\mathcal{X}_r \subset \mathcal{M}^{(0)}_{\infty,\overline{\eta},1}$ by

$$\begin{aligned} &v\left(\frac{\boldsymbol{x}_i}{\boldsymbol{x}_{i+1}} - \left(\frac{\boldsymbol{x}_{n-1}}{\boldsymbol{x}_n}\right)^{q^{n-1-i}}\right) \ge \frac{1}{2nq^i} \quad \text{for } 1 \le i \le n-2, \\ &v(\boldsymbol{x}_i - 1) \ge \frac{1}{nq^{n-1}(p^e+1)} \quad \text{for } n-1 \le i \le n. \end{aligned} \tag{2.5}$$

The definition of $\mathcal{X}_r$ is independent of the choice of $\boldsymbol{\theta}_r$ and $\boldsymbol{\lambda}_r$. We define $\mathcal{B}_r \subset \mathcal{D}^{n,\mathrm{perf}}_{\mathbf{C}}$ by the same condition (2.5).

2.2 Formal models of affinoids

Let $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ be the coordinate of $\mathcal{B}_r$. We put $h(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = \prod_{i=1}^n \boldsymbol{X}_i^{q^{i-1}}$. Further, we put

$$f(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = 1 - \frac{\delta(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)}{h(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)}, \tag{2.6}$$

$$f_0(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = \sum_{i=1}^{n-1} \left(\frac{\boldsymbol{X}_i}{\boldsymbol{X}_{i+1}} \right)^{q^{i-1}(q-1)} + \left(\frac{\boldsymbol{X}_n^{q^n}}{\boldsymbol{X}_1} \right)^{\frac{q-1}{q}}. \tag{2.7}$$

We simply write $f(\boldsymbol{X})$ for $f(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$, and $f(\boldsymbol{\xi}_r)$ for $f(\boldsymbol{\xi}_{r,1}, \ldots, \boldsymbol{\xi}_{r,n})$. We will use the similar notations also for other functions. We put

$$\boldsymbol{S} = f_0(\boldsymbol{X}) - f_0(\boldsymbol{\xi}_r^0). \tag{2.8}$$

Lemma 2.3. *We have*

$$f(\boldsymbol{X}) \equiv f_0(\boldsymbol{X}) \mod_{>} \frac{q-1}{nq} \quad \textit{and} \quad \boldsymbol{S} \equiv f(\boldsymbol{X}) - f(\boldsymbol{\xi}_r^0) \mod_{>} \frac{1}{n}.$$

Proof. We put

$$f_1(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = \sum_{i=1}^{n-3} \sum_{j=i+2}^{n-1} \left(\frac{\boldsymbol{X}_i}{\boldsymbol{X}_{i+1}} \right)^{q^{i-1}(q-1)} \left(\frac{\boldsymbol{X}_j}{\boldsymbol{X}_{j+1}} \right)^{q^{j-1}(q-1)} + \left(\frac{\boldsymbol{X}_n^{q^n}}{\boldsymbol{X}_1} \right)^{\frac{q-1}{q}} \sum_{i=1}^{n-3} \left(\frac{\boldsymbol{X}_{i+1}}{\boldsymbol{X}_{i+2}} \right)^{q^i(q-1)}.$$

Then we see that

$$f(\boldsymbol{X}) \equiv 1 - \frac{\delta_0(\boldsymbol{X})}{h(\boldsymbol{X})} \equiv f_0(\boldsymbol{X}) - f_1(\boldsymbol{X}) \mod_{>} \frac{1}{n}$$

using Lemma 1.3 and the definition of δ_0. The claims follow from this, because

$$v(f_1(\boldsymbol{X})) \geq \frac{2(q-1)}{nq} \quad \text{and} \quad v\big(f_1(\boldsymbol{X}) - f_1(\boldsymbol{\xi}_r^0)\big) > \frac{2(q-1)}{nq}.$$

$\square$

We put $\boldsymbol{s}_i = (\boldsymbol{x}_i/\boldsymbol{x}_{i+1})^{q^i(q-1)}$ for $1 \leq i \leq n-1$, and

$$\boldsymbol{s}_i \boldsymbol{s}_{n-1}^{-1} = 1 + \boldsymbol{Y}_i \quad \text{for } 1 \leq i \leq n-2, \quad \boldsymbol{s}_{n-1} = 1 + \boldsymbol{Y}_{n-1}. \tag{2.9}$$

We put $m = \gcd(e, f)$ and

$$\boldsymbol{z} = \sum_{i=0}^{\frac{e}{m}-1} \left(\frac{\boldsymbol{\theta}_r^{p^e} \boldsymbol{Y}_{n-1}}{\boldsymbol{\eta}_r} \right)^{p^{im}} - \frac{1}{n'} \sum_{i=0}^{\frac{f}{m}-1} \left(\frac{\boldsymbol{S}}{\boldsymbol{\eta}_r} \right)^{p^{im}}. \tag{2.10}$$

We put $f = m_0$ and $e = m_1$. We define $m_2, \ldots, m_{N+1}$ by the Euclidean algorithm as follows: We have

$$m_{i-1} = n_i m_i + m_{i+1} \quad \text{with} \;\; n_i \geq 0 \;\; \text{and} \;\; 0 \leq m_{i+1} < m_i \quad \text{for} \;\; 1 \leq i \leq N,$$
$$m_N = m, \quad m_{N+1} = 0.$$

We put

$$\boldsymbol{T}_0 = \frac{\boldsymbol{\theta}_r^{p^e} \boldsymbol{Y}_{n-1}}{\boldsymbol{\eta}_r}, \quad \boldsymbol{T}_1 = \frac{-\boldsymbol{S}}{n' \boldsymbol{\eta}_r} \tag{2.11}$$

and define $\boldsymbol{T}_2, \dots, \boldsymbol{T}_N$ by

$$\boldsymbol{T}_{i+1} = \boldsymbol{T}_{i-1} + \sum_{j=0}^{n_i-1} \boldsymbol{T}_i^{p^{jm_i+m_{i+1}}} \quad \text{for } 1 \leq i \leq N-1.$$

Then we see that

$$\boldsymbol{z} = \sum_{j=0}^{\frac{m_{i+1}}{m}-1} \boldsymbol{T}_i^{p^{jm}} + \sum_{j=0}^{\frac{m_i}{m}-1} \boldsymbol{T}_{i+1}^{p^{jm}} \quad \text{for } 1 \leq i \leq N-1$$

inductively by (2.10). We see also that

$$(-1)^{N-i}\boldsymbol{T}_i = \sum_{j=0}^{\frac{m_i}{m}-1} \boldsymbol{T}_N^{p^{jm}} + P_i(\boldsymbol{z}) \tag{2.12}$$

with some $P_i(x) \in \mathbb{Z}[x]$ for $0 \leq i \leq N-1$. We put

$$\boldsymbol{Y} = \frac{(-1)^N \boldsymbol{\eta}_r}{\boldsymbol{\theta}_r^{p^e}} \boldsymbol{T}_N^{p^{f-m}}. \tag{2.13}$$

Then we have

$$\boldsymbol{Y} \equiv \boldsymbol{Y}_{n-1} \mod_{>} 1/(n(p^e+1)) \tag{2.14}$$

by (2.12) and (2.13). We define a subaffinoid $\mathcal{B}'_r \subset \mathcal{B}_r$ by $v(\boldsymbol{z}) \geq 0$. We choose a square root $\boldsymbol{\eta}_r^{1/2}$ and a (p^e+1)-root $\boldsymbol{\eta}_r^{1/(p^e+1)}$ of $\boldsymbol{\eta}_r$ compatibly.

We set

$$\begin{aligned} \boldsymbol{Y}_i &= \boldsymbol{\eta}_r^{1/2} \boldsymbol{y}_i \quad \text{with} \quad \boldsymbol{y}_i = (y_i^{q^{-j}})_{j \geq 0} \quad \text{for } 1 \leq i \leq n-2, \\ \boldsymbol{Y} &= \boldsymbol{\eta}_r^{1/(p^e+1)} \boldsymbol{y} \quad \text{with} \quad \boldsymbol{y} = (y^{q^{-j}})_{j \geq 0} \end{aligned} \tag{2.15}$$

on $\mathcal{B}'_r$. Let $\mathcal{B}$ be the generic fiber of $\operatorname{Spf} \mathcal{O}_{\mathbf{C}}\langle y^{1/q^\infty}, y_1^{1/q^\infty}, \dots, y_{n-2}^{1/q^\infty}, z^{1/q^\infty} \rangle$. The parameters $\boldsymbol{y}, \boldsymbol{y}_1, \dots, \boldsymbol{y}_{n-1}, \boldsymbol{z}$ give the morphism $\Theta \colon \mathcal{B}'_r \to \mathcal{B}$. We simply say an analytic function on $\mathcal{B}$ for a q-th power compatible system of analytic functions on $\mathcal{B}$.

We put

$$1 + \boldsymbol{\theta}'_r = (1+\boldsymbol{\theta}_r)^{-n}(1+n'\boldsymbol{\lambda}_r)\left(\frac{\boldsymbol{\xi}_n^0}{\boldsymbol{\xi}'_n}\right)^{q^{n-1}}.$$

Lemma 2.4. Θ *is an isomorphism.*

Proof. We will construct the inverse morphism of Θ. We can write $\boldsymbol{Y}_{n-1}$ and $\boldsymbol{S}$ as analytic functions on $\mathcal{B}$ by (2.11), (2.12), (2.13) and (2.15). Then we can write $\boldsymbol{x}_i/\boldsymbol{x}_{i+1}$ as an analytic function on $\mathcal{B}$ by (2.9). By (2.7) and (2.8), we have

$$\frac{\boldsymbol{\eta}_r^{-(q-1)}\boldsymbol{S}^q}{(1+\boldsymbol{\theta}_r)^{(q-1)^2}} = \sum_{i=1}^{n-2}(\boldsymbol{s}_i - 1) + \frac{\boldsymbol{s}_{n-1}-1}{(1+\boldsymbol{\theta}'_r)^{q-1}} + (1+\boldsymbol{\theta}'_r)^{q(q-1)}\Big(\boldsymbol{x}_n^{(q-1)(q^n-1)} \prod_{i=1}^{n-1}(\boldsymbol{x}_i^{-1}\boldsymbol{x}_{i+1}) - 1\Big).$$

By this equation, we can write $\boldsymbol{x}_n$ as an analytic functions on $\mathcal{B}$. Hence, we have the inverse morphism of Θ. □

We put

$$\delta_{\mathcal{B}}(y, y_1, \dots, y_{n-1}, z) = (\delta|_{\mathcal{B}'_r}) \circ \Theta^{-1}$$

equipped with its q^j-th root $\delta_A^{q^{-j}}$ for $j \geq 0$. We put

$$\mathfrak{X}_r = \operatorname{Spf} \mathcal{O}_{\mathbf{C}}\langle y^{1/q^\infty}, y_1^{1/q^\infty}, \dots, y_{n-1}^{1/q^\infty}, z^{1/q^\infty} \rangle / (\delta_{\mathcal{B}}^{q^{-j}})_{j \geq 0}.$$

Theorem 2.5. *The formal scheme $\mathfrak{X}_r$ is a formal model of $\mathcal{X}_r$, and the special fiber of $\mathfrak{X}_r$ is isomorphic to the perfection of the affine smooth variety defined by*

$$z^{p^m} - z = y^{p^e+1} - \frac{1}{n'} \sum_{1 \le i \le j \le n-2} y_i y_j \quad \textit{in } \mathbb{A}^n_{k^{\mathrm{ac}}}. \tag{2.16}$$

Proof. Let $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ be the coordinate of $\mathcal{B}_r$. By Lemma 2.3, we have

$$v(f(\boldsymbol{X})) \ge \frac{q-1}{nq} \quad \text{and} \quad v(\boldsymbol{S}) > \frac{q-1}{nq}. \tag{2.17}$$

We have

$$h(\boldsymbol{X})^{q-1} = \left(\frac{\boldsymbol{X}_n^{q^n}}{\boldsymbol{X}_1} \right) \prod_{i=1}^{n-1} \left(\frac{\boldsymbol{X}_i}{\boldsymbol{X}_{i+1}} \right)^{q^i}. \tag{2.18}$$

We have

$$\left(\frac{\boldsymbol{X}_n^{q^n}}{\boldsymbol{X}_1} \right)^{q-1} = \Big(\boldsymbol{\eta}_r (1+\boldsymbol{\theta}_r)^{q-1} (1+\boldsymbol{\theta}'_r)^q \Big)^{q-1} \left(\frac{h(\boldsymbol{X})}{h(\boldsymbol{\xi}_r^0)} \right)^{(q-1)^2} \prod_{i=1}^{n-1} \boldsymbol{s}_i^{-1} \tag{2.19}$$

by (2.18). We put

$$R(\boldsymbol{X}) = \frac{1-f(\boldsymbol{\xi}_r^0)}{1-f(\boldsymbol{X})} - (1+\boldsymbol{S}). \tag{2.20}$$

Then we have $v(R(\boldsymbol{X})) > 1/n$ by Lemma 2.3 and (2.17). The equation $\delta(\boldsymbol{X}) = \delta(\boldsymbol{\xi}_r^0)$ is equivalent to

$$\left(\frac{\boldsymbol{X}_n^{q^n}}{\boldsymbol{X}_1} \right)^{q-1} = \Big(\boldsymbol{\eta}_r (1+\boldsymbol{\theta}_r)^{q-1} (1+\boldsymbol{\theta}'_r)^q \Big)^{q-1} \big(1+\boldsymbol{S}+R(\boldsymbol{X})\big)^{(q-1)^2} \prod_{i=1}^{n-1} \boldsymbol{s}_i^{-1} \tag{2.21}$$

by (2.6), (2.19) and (2.20). We put

$$F(\boldsymbol{X}) = (1+\boldsymbol{\theta}'_r)^{q(q-1)} \big(1+\boldsymbol{S}+R(\boldsymbol{X})\big)^{(q-1)^2} \prod_{i=1}^{n-1} \boldsymbol{s}_i^{-1}.$$

The equation (2.21) is equivalent to

$$f_0(\boldsymbol{X})^q = \boldsymbol{\eta}_r^{q-1} (1+\boldsymbol{\theta}_r)^{(q-1)^2} \left(\sum_{i=1}^{n-2} \boldsymbol{s}_i + \frac{\boldsymbol{s}_{n-1}}{(1+\boldsymbol{\theta}'_r)^{q-1}} + F(\boldsymbol{X}) \right). \tag{2.22}$$

The equation (2.22) is equivalent to

$$\boldsymbol{S}^q = \boldsymbol{\eta}_r^{q-1} (1+\boldsymbol{\theta}_r)^{(q-1)^2} \left(\sum_{i=1}^{n-2} (\boldsymbol{s}_i - 1) + \frac{\boldsymbol{s}_{n-1}-1}{(1+\boldsymbol{\theta}'_r)^{q-1}} + F(\boldsymbol{X}) - F(\boldsymbol{\xi}_r^0) \right). \tag{2.23}$$

We put

$$\begin{aligned} R_1(\boldsymbol{X}) =& (1+\boldsymbol{\theta}_r)^{(q-1)^2} \left(\sum_{i=1}^{n-2} (\boldsymbol{s}_i - 1) + \frac{\boldsymbol{s}_{n-1}-1}{(1+\boldsymbol{\theta}'_r)^{q-1}} + F(\boldsymbol{X}) - F(\boldsymbol{\xi}_r^0) \right) \\ & - \left(\boldsymbol{S} + \sum_{1 \le i \le j \le n-2} \boldsymbol{Y}_i \boldsymbol{Y}_j - n' \Big(\boldsymbol{Y}_{n-1}^{p^e+1} + (1+\boldsymbol{\theta}_r) \boldsymbol{Y}_{n-1}^{p^e} + \boldsymbol{\theta}_r^{p^e} \boldsymbol{Y}_{n-1} \Big) \right). \end{aligned}$$

Then we have $v(R_1(\boldsymbol{X})) > 1/n$. The equation (2.23) is equivalent to

$$\boldsymbol{S}^q = \boldsymbol{\eta}_r^{q-1}\Bigg(\boldsymbol{S} + \sum_{1\le i\le j\le n-2} \boldsymbol{Y}_i\boldsymbol{Y}_j - n'\Big(\boldsymbol{Y}_{n-1}^{p^e+1} + (1+\boldsymbol{\theta}_r)\boldsymbol{Y}_{n-1}^{p^e} + \boldsymbol{\theta}_r^{p^e}\boldsymbol{Y}_{n-1}\Big) + R_1(\boldsymbol{X})\Bigg). \quad (2.24)$$

The equation (2.24) is equivalent to

$$\boldsymbol{z}^{p^m} - \boldsymbol{z} = \boldsymbol{\eta}_r^{-1}\Bigg(\boldsymbol{Y}_{n-1}^{p^e+1} - \frac{1}{n'}\sum_{1\le i\le j\le n-2}\boldsymbol{Y}_i\boldsymbol{Y}_j - \frac{R_1(\boldsymbol{X})}{n'}\Bigg). \quad (2.25)$$

As a result, $\delta(\boldsymbol{X}) = \delta(\boldsymbol{\xi}_L)$ is equivalent to (2.25) on $\mathcal{B}_r$. By Lemma 2.3 and (2.25), we have $v(\boldsymbol{z}) \geq 0$ on $\mathcal{X}_r$. This implies $\mathcal{X}_r \subset \mathcal{B}'_r$. We have the first claim by Lemma 2.4 and the construction of $\mathfrak{X}_r$. The second claim follows from (2.14) and (2.25). □

Remark 2.6. *If $n = p = 2$, then the curve over k defined by* (2.16) *is the supersingular elliptic curve, which appears in a semi-stable reduction of a one-dimensional Lubin-Tate space in [IT11] and [IT12].*

3 Group action on the reductions

Action of GL_n and $D^\times$ Let $\mathfrak{I} \subset M_n(\mathcal{O}_K)$ be the inverse image under the reduction map $M_n(\mathcal{O}_K) \to M_n(k)$ of the ring consisting of upper triangular matrices in $M_n(k)$.

Lemma 3.1. *Let $(g,d,1) \in G^0$. We take the integer l such that $d\varphi_{D,r}^{-l} \in \mathcal{O}_D^\times$. Let $(\boldsymbol{X}_1,\dots,\boldsymbol{X}_n)$ be the coordinate of $\mathcal{X}_r$. Assume $v((g,d)\cdot\boldsymbol{X}_i) = v(\boldsymbol{X}_i)$ for $1 \le i \le n$ at some point of $\mathcal{X}_r$. Then we have $(g,d) \in (\varphi_{M,r},\varphi_{D,r})^l(\mathfrak{I}^\times \times \mathcal{O}_D^\times)$.*

Proof. This is proved as in the same way as [IT13, Lemma 3.1]. □

We put

$$\mathbf{g}_r = (\varphi_{M,r},\varphi_{D,r},1) \in G. \quad (3.1)$$

We put

$$\varepsilon_1 = \begin{cases} 1 & \text{if } p^e = 2, \\ 0 & \text{if } p^e \neq 2. \end{cases}$$

Proposition 3.2. 1. *The action of $\mathbf{g}_r$ stabilizes $\mathcal{X}_r$, and induces the automorphism of $\overline{\mathfrak{X}_r}$ defined by*

$$\begin{aligned}&(\boldsymbol{z},\boldsymbol{y},(\boldsymbol{y}_i)_{1\le i\le n-2}) \\ &\mapsto \Big(\boldsymbol{z}+\varepsilon_1(\boldsymbol{y}_{n-2}+1), \boldsymbol{y}, -\sum_{i=1}^{n-3}\boldsymbol{y}_i - 2\boldsymbol{y}_{n-2} + \varepsilon_1, (\boldsymbol{y}_{i-1}-\boldsymbol{y}_{n-2}+\varepsilon_1)_{2\le i\le n-2}\Big).\end{aligned} \quad (3.2)$$

2. *Assume $p^e \neq 2$. Let $g_r \in GL_{n-1}(k)$ be the matrix corresponding to the action of $\mathbf{g}_r$ on $(\boldsymbol{y},(\boldsymbol{y}_i)_{1\le i\le n-2})$ in* (3.2)*. Then,* $\det(g_r) = (-1)^{n+1}$.

Proof. By (1.6) and (1.7), we have

$$\mathbf{g}_r^*\boldsymbol{X}_1 = \boldsymbol{X}_n^{q^{n-1}}, \quad \mathbf{g}_r^*\boldsymbol{X}_i = \boldsymbol{X}_{i-1}^{\frac{1}{q}} \quad \text{for } 2 \le i \le n. \quad (3.3)$$

By (3.3), we have $\mathbf{g}_r^*(h(\boldsymbol{X})) = h(\boldsymbol{X})$. Hence, we have

$$\mathbf{g}_r^* \boldsymbol{S} \equiv \boldsymbol{S} \mod_{>} \frac{1}{n} \tag{3.4}$$

by (2.6), (2.8) and Lemma 2.3. By (2.21) and (3.3), we have

$$\mathbf{g}_r^* \boldsymbol{s}_1 \equiv \prod_{i=1}^{n-1} \boldsymbol{s}_i^{-1} \mod_{>} \frac{1}{2n}. \tag{3.5}$$

We have also

$$\mathbf{g}_r^* \boldsymbol{s}_i = \boldsymbol{s}_{i-1} \quad \text{for } 2 \leq i \leq n-2, \quad \mathbf{g}_r^* \boldsymbol{s}_{n-1} = \boldsymbol{s}_{n-2}(1+\boldsymbol{\theta}_r')^{1-q} \tag{3.6}$$

by (3.3). We have

$$\mathbf{g}_r^* \boldsymbol{Y}_1 \equiv (1+\boldsymbol{\theta}_r)^n (1+\boldsymbol{Y}_{n-2})^{-2} \prod_{i=1}^{n-3} (1+\boldsymbol{Y}_i)^{-1} - 1 \mod_{>} \frac{1}{2n} \tag{3.7}$$

by (3.5) and (3.6). We have also

$$\begin{aligned} \mathbf{g}_r^* \boldsymbol{Y}_i &\equiv (1+\boldsymbol{\theta}_r)^n (1+\boldsymbol{Y}_{i-1})(1+\boldsymbol{Y}_{n-2})^{-1} - 1 \mod_{>} \frac{1}{2n} \quad \text{for } 2 \leq i \leq n-2, \\ \mathbf{g}_r^* \boldsymbol{Y}_{n-1} &\equiv (1+\boldsymbol{\theta}_r)^{-n} (1+\boldsymbol{Y}_{n-2})(1+\boldsymbol{Y}_{n-1}) - 1 \mod_{>} \frac{1}{p^e n} \end{aligned} \tag{3.8}$$

by (3.6). The claim follows from (3.4), (3.7) and (3.8). □

Let $\mathfrak{P}$ be the Jacobson radical of the order $\mathfrak{I}$, and $\mathfrak{p}_D$ be the maximal ideal of $\mathcal{O}_D$. We put

$$U_{\mathfrak{I}}^1 = 1 + \mathfrak{P}, \quad U_D^1 = 1 + \mathfrak{p}_D$$

and

$$(U_{\mathfrak{I}}^1 \times U_D^1)^1 = \{(g,d) \in U_{\mathfrak{I}}^1 \times U_D^1 \mid \det(g)^{-1} \operatorname{Nrd}_{D/K}(d) = 1\}.$$

Let $\mathrm{pr}_{\mathcal{O}_K/k} \colon \mathcal{O}_K \to k$ be the reduction map. We put

$$h_r(g,d) = \frac{1}{n'} (\mathrm{Tr}_{k/\mathbb{F}_{p^m}} \circ \mathrm{pr}_{\mathcal{O}_K/k}) \Big(\mathrm{Trd}_{D/K}(\varphi_{D,r}^{-1}(d-1)) - \mathrm{tr}(\varphi_{M,r}^{-1}(g-1)) \Big)$$

for $(g,d) \in U_{\mathfrak{I}}^1 \times U_D^1$.

Proposition 3.3. *The stabilizer of $\mathcal{X}_r$ in $GL_n(K) \times D^\times$ is $i_{\xi_r}(L_r^\times) \cdot (U_{\mathfrak{I}}^1 \times U_D^1)^1$. Further, $(g,d) \in (U_{\mathfrak{I}}^1 \times U_D^1)^1$ induces the automorphism of $\overline{\mathfrak{X}_r}$ defined by*

$$(\boldsymbol{z}, \boldsymbol{y}, (\boldsymbol{y}_i)_{1 \leq i \leq n-2}) \mapsto (\boldsymbol{z} + h_r(g,d), \boldsymbol{y}, (\boldsymbol{y}_i)_{1 \leq i \leq n-2}).$$

Proof. Assume that $(g,d) \in GL_n(K) \times D^\times$ stabilizes $\mathcal{X}_r$. Then we have $\det(g) = \operatorname{Nrd}_{D/K}(d)$. We will show that $(g,d) \in i_{\xi_r}(L_r^\times) \cdot (U_{\mathfrak{I}}^1 \times U_D^1)^1$. By Lemma 3.1 and Proposition 3.2, we may assume that $(g,d) \in \mathfrak{I}^\times \times \mathcal{O}_D^\times$.

We write $g = (a_{i,j})_{1 \leq i,j \leq n} \in \mathfrak{I}$ and $a_{i,j} = \sum_{l=0}^\infty a_{i,j}^{(l)} \varpi_r^l$ with $a_{i,j}^{(l)} \in \mu_{q-1}(K) \cup \{0\}$. By (1.6), we have

$$\begin{aligned} g^* \boldsymbol{X}_1 &\equiv a_{1,1}^{(0)} \boldsymbol{X}_1 + a_{n,1}^{(1)} \boldsymbol{X}_n^{q^n} \mod_{>} q/(n(q-1)), \\ g^* \boldsymbol{X}_i &\equiv a_{i,i}^{(0)} \boldsymbol{X}_i + a_{i-1,i}^{(0)} \boldsymbol{X}_{i-1} \mod_{>} (nq^{i-2}(q-1))^{-1} \quad \text{for } 2 \leq i \leq n. \end{aligned} \tag{3.9}$$

We write $d^{-1} = \sum_{i=0}^{\infty} d_i \varphi_{D,r}^i$ with $d_i \in \mu_{q^n-1}(K_n) \cup \{0\}$. We set $\kappa(d) = d_1/d_0$. By (1.7), we have

$$d^*\boldsymbol{X}_i \equiv d_0 \boldsymbol{X}_i \left(1 + \kappa(d)\boldsymbol{X}_i^{q-1}\right) \quad \mathrm{mod}_{>} (nq^{i-2}(q-1))^{-1} \quad \text{for } 1 \le i \le n. \tag{3.10}$$

By (2.5), (3.9) and (3.10), we have $(g,d) \in i_{\xi_r}(\mathcal{O}_K^\times) \cdot (U_{\mathfrak{J}}^1 \times U_D^1)^1$. Conversely, any element of $i_{\xi_r}(L_r^\times) \cdot (U_{\mathfrak{J}}^1 \times U_D^1)^1$ stabilizes $\mathcal{X}_r$ by Remark 1.4 and Proposition 3.2 and the above arguments.

Let $(g,d) \in (U_{\mathfrak{J}}^1 \times U_D^1)^1$. We put

$$\Delta_g(\boldsymbol{X}) = \sum_{i=1}^{n-1} a_{i,i+1}^{(0)} \left(\frac{\boldsymbol{X}_i}{\boldsymbol{X}_{i+1}}\right)^{q^i} + a_{n,1}^{(1)} \frac{\boldsymbol{X}_n^{q^n}}{\boldsymbol{X}_1}, \quad \Delta_d(\boldsymbol{X}) = \sum_{i=1}^{n} \kappa(d)^{q^{i-1}} \boldsymbol{X}_i^{q^{i-1}(q-1)}.$$

Then, we acquire

$$f_0\big((g,d)^*\boldsymbol{X}\big) \equiv f_0(\boldsymbol{X}) + \Delta_g(\boldsymbol{X}) + \Delta_d(\boldsymbol{X}) \quad \mathrm{mod}_{>} 1/n. \tag{3.11}$$

We have

$$(g,d)^*\boldsymbol{S} \equiv \boldsymbol{S} + \Delta_g(\boldsymbol{X}) + \Delta_d(\boldsymbol{X}) \quad \mathrm{mod}_{>} 1/n \tag{3.12}$$

by (2.8) and (3.11). We have

$$(g,d)^*\boldsymbol{s}_i \equiv \boldsymbol{s}_i \quad \mathrm{mod}\ 1/n \tag{3.13}$$

for $1 \le i \le n-2$. We obtain

$$(g,d)^*\boldsymbol{z} = \boldsymbol{z} + h_r(g,d)$$

by (2.10), (3.12) and (3.13). We can compute the action of (g,d) on $\boldsymbol{y}$ and $\{\boldsymbol{y}_i\}_{1\le i\le n-2}$ by (2.9), (2.14), (2.15) and (3.13). □

Action of Weil group We put $\varphi_r' = \varphi_r^{p^e}$ and $E_r = K(\varphi_r')$. Let $\sigma \in W_{E_r}$ in this paragraph. We put $a_\sigma = \mathrm{Art}_{E_r}^{-1}(\sigma)$, and $u_\sigma = a_\sigma \varphi_r'^{-n_\sigma} \in \mathcal{O}_{E_r}^\times$. We take $b_\sigma \in \mu_{q-1}(K)$ such that $\bar{b}_\sigma^{p^e} = \bar{u}_\sigma \in k$. We put $c_\sigma = b_\sigma^{-n} \mathrm{Nr}_{E_\zeta/K}(u_\sigma) \in U_K^1$. Let $g_\sigma = (a_{i,j})_{1\le i,j\le n} \in \mathcal{O}_K^\times U_{\mathfrak{J}}^1$ be the element defined by $a_{i,i} = b_\sigma$ for $1 \le i \le n-1$, $a_{n,n} = b_\sigma c_\sigma$ and $a_{i,j} = 0$ if $i \ne j$. We put

$$\mathbf{g}_\sigma = (g_\sigma, \varphi_{D,r}^{-n_\sigma}, \sigma) \in G. \tag{3.14}$$

We choose elements α_r, β_r and γ_r such that

$$\begin{gathered} \alpha_r^{p^e+1} = -\varphi_r', \quad \beta_r^{p^{2e}} + \beta_r = -\alpha_r^{-1}, \quad \gamma_r^{p^m} - \gamma_r = \beta_r^{p^e+1} + \varepsilon_0, \\ \alpha_r^{-1}\eta_r^{\frac{p^e}{p^e+1}} \equiv 1, \quad \beta_r^{-1}\theta_r^{p^e}\eta_r^{-\frac{p^e}{p^e+1}} \equiv 1, \quad \gamma_r^{-1}\sum_{i=0}^{\frac{f}{m}-1} (\lambda_r\eta_r^{-1})^{p^{im}} \equiv 1 \quad \mathrm{mod}_{>} 0. \end{gathered} \tag{3.15}$$

For $\sigma \in W_{E_r}$, we set

$$\begin{gathered} a_{r,\sigma} = \sigma(\alpha_r)/(\alpha_r), \quad b_{r,\sigma} = a_{r,\sigma}\sigma(\beta_r) - \beta_r, \\ c_{r,\sigma} = \sigma(\gamma_r) - \gamma_r + \sum_{i=0}^{\frac{e}{m}-1} (b_{r,\sigma}^{p^e}(\beta_r + b_{r,\sigma}))^{p^{im}}. \end{gathered} \tag{3.16}$$

Then we have $a_{r,\sigma}, b_{r,\sigma}, c_{r,\sigma} \in \mathcal{O}_{\mathbf{C}}$.

Let

$$Q = \left\{ g(a,b,c) \;\middle|\; a,b,c \in k^{\mathrm{ac}},\ a^{p^e+1} = 1,\ b^{p^{2e}} + b = 0,\ c^{p^m} - c + b^{p^e+1} = 0 \right\}$$

be the group whose multiplication is given by

$$g(a_1,b_1,c_1)\cdot g(a_2,b_2,c_2)=g\Biggl(a_1a_2,a_1b_2+b_1,c_1+c_2+\sum_{i=0}^{\frac{e}{m}-1}(a_1b_1^{p^e}b_2)^{p^{im}}\Biggr).$$

Let $Q\rtimes\mathbb{Z}$ be the semidirect product, where $l\in\mathbb{Z}$ acts on Q by $g(a,b,c)\mapsto g(a^{q^{-l}},b^{q^{-l}},c^{q^{-l}})$. Let $(g(a,b,c),l)\in Q\rtimes\mathbb{Z}$ act on $\overline{\mathfrak{X}_r}$ by

$$(\boldsymbol{z},\boldsymbol{y},(\boldsymbol{y}_i)_{1\le i\le n-2})\mapsto\Biggl(\Biggl(\boldsymbol{z}+\sum_{i=0}^{\frac{e}{m}-1}(b\boldsymbol{y})^{p^{im}}+c\Biggr)^{q^l},(a(\boldsymbol{y}+b^{p^e}))^{q^l},(a^{\frac{p^e+1}{2}}\boldsymbol{y}_i^{q^l})_{1\le i\le n-2}\Biggr).\tag{3.17}$$

We have the surjective homomorphism

$$\Theta_r\colon W_{E_r}\to Q\rtimes\mathbb{Z};\ \sigma\mapsto\bigl(g(\bar{a}_{r,\sigma},\bar{b}_{r,\sigma},\bar{c}_{r,\sigma}),n_\sigma\bigr)\,.\tag{3.18}$$

Proposition 3.4. *Let $\sigma\in W_{E_r}$. Then, $\mathbf{g}_\sigma\in G$ stabilizes $\mathcal{X}_r$, and induces the automorphism of $\overline{\mathfrak{X}_r}$ given by $\Theta_r(\sigma)$.*

Proof. Let $P\in\mathcal{X}_r(\mathbf{C})$. We have

$$\begin{aligned}\boldsymbol{S}(P\mathbf{g}_\sigma)&=f_0\bigl(\boldsymbol{X}(P\mathbf{g}_\sigma)\bigr)-f_0(\boldsymbol{\xi}_r^0)\\&=f_0\bigl(\boldsymbol{X}(P\mathbf{g}_\sigma)\bigr)-f_0\bigl(\boldsymbol{X}(P(1,\varphi_r^{-n_\sigma},\sigma))\bigr)+\sigma^{-1}\bigl(f_0(\boldsymbol{X}(P))\bigr)-f_0(\boldsymbol{\xi}_r^0)\\&\equiv\Delta_{g_\sigma}\bigl(\boldsymbol{X}(P(1,\varphi_r^{-n_\sigma},\sigma))\bigr)+\sigma^{-1}\bigl(\boldsymbol{S}(P)+f_0(\boldsymbol{\xi}_r^0)\bigr)-f_0(\boldsymbol{\xi}_r^0)\\&\equiv\sigma^{-1}\bigl(\boldsymbol{S}(P)\bigr)+f_0(\sigma^{-1}(\boldsymbol{\xi}_r^0))-f_0(\boldsymbol{\xi}_r^0)\quad\bmod_{>}1/n\end{aligned}\tag{3.19}$$

by (2.10) and (3.11). We have

$$f_0(\sigma^{-1}(\boldsymbol{\xi}_r^0))-f_0(\boldsymbol{\xi}_r^0)\equiv n'(\sigma^{-1}(\boldsymbol{\lambda}_r)-\boldsymbol{\lambda}_r)\quad\bmod_{>}1/n.\tag{3.20}$$

We put $s_i(\boldsymbol{X})=(\boldsymbol{X}_i/\boldsymbol{X}_{i+1})^{q^i(q-1)}$ for $1\le i\le n-1$. We have

$$\begin{aligned}&s_{n-1}(\boldsymbol{\xi}_r^0)\boldsymbol{Y}_{n-1}(P\mathbf{g}_\sigma)=s_{n-1}\bigl(\boldsymbol{X}(P\mathbf{g}_\sigma)\bigr)-s_{n-1}(\boldsymbol{\xi}_r^0)\\&=s_{n-1}\bigl(\boldsymbol{X}(P\mathbf{g}_\sigma)\bigr)-s_{n-1}\bigl(\boldsymbol{X}(P(1,\varphi_r^{-n_\sigma},\sigma))\bigr)+\sigma^{-1}\bigl(s_{n-1}(\boldsymbol{X}(P))\bigr)-s_{n-1}(\boldsymbol{\xi}_r^0)\\&\equiv\sigma^{-1}\bigl(s_{n-1}(\boldsymbol{\xi}_r^0)\boldsymbol{Y}_{n-1}(P)\bigr)+\sigma^{-1}\bigl(s_{n-1}(\boldsymbol{\xi}_r^0)\bigr)-s_{n-1}(\boldsymbol{\xi}_r^0)\quad\bmod_{>}\frac{q-1}{n}+\frac{1}{np^e}\end{aligned}\tag{3.21}$$

by (2.10) and (3.11). Hence, we have

$$\boldsymbol{Y}_{n-1}(P\mathbf{g}_\sigma)\equiv\sigma^{-1}\bigl(\boldsymbol{Y}_{n-1}(P)\bigr)+\sigma^{-1}(\boldsymbol{\theta}_r)-\boldsymbol{\theta}_r\quad\bmod_{>}\frac{1}{np^e}.\tag{3.22}$$

We put $\boldsymbol{\theta}_{r,\sigma}=\sigma(\boldsymbol{\theta}_r)-\boldsymbol{\theta}_r$ and $\boldsymbol{\lambda}_{r,\sigma}=\sigma(\boldsymbol{\lambda}_r)-\boldsymbol{\lambda}_r$. We have

$$\begin{aligned}\sigma\bigl(\boldsymbol{z}(P\mathbf{g}_\sigma)\bigr)&=\sigma\Biggl(\sum_{i=0}^{\frac{e}{m}-1}\biggl(\frac{\boldsymbol{\theta}_r^{p^e}\boldsymbol{Y}_{n-1}(P\mathbf{g}_\sigma)}{\boldsymbol{\eta}_r}\biggr)^{p^{im}}-\frac{1}{n'}\sum_{i=0}^{\frac{f}{m}-1}\biggl(\frac{\boldsymbol{S}(P\mathbf{g}_\sigma)}{\boldsymbol{\eta}_r}\biggr)^{p^{im}}\Biggr)\\&\equiv\boldsymbol{z}(P)+\sum_{i=0}^{\frac{e}{m}-1}\biggl(\frac{\boldsymbol{\theta}_{r,\sigma}^{p^e}\boldsymbol{Y}_{n-1}(P)-\sigma(\boldsymbol{\theta}_r^{p^e})\boldsymbol{\theta}_{r,\sigma}}{\boldsymbol{\eta}_r}\biggr)^{p^{im}}+\sum_{i=0}^{\frac{f}{m}-1}\biggl(\frac{\boldsymbol{\lambda}_{r,\sigma}}{\boldsymbol{\eta}_r}\biggr)^{p^{im}}\\&\equiv\boldsymbol{z}(P)+\sum_{i=0}^{\frac{e}{m}-1}\Biggl(b_{r,\sigma}\frac{\boldsymbol{Y}_{n-1}(P)}{\boldsymbol{\eta}_r^{1/(p^e+1)}}-\sigma(\beta_r)a_{r,\sigma}b_{r,\sigma}^{\frac{1}{p^e}}\Biggr)^{p^{im}}+\sigma(\gamma_r)-\gamma_r\\&\equiv\boldsymbol{z}(P)+\sum_{i=0}^{\frac{e}{m}-1}(b_{r,\sigma}\boldsymbol{y}(P))^{p^{im}}+c_{r,\sigma}\quad\bmod_{>}0\end{aligned}$$

by (3.19), (3.20), (3.21) and (3.22). We see also that

$$\sigma\left(\frac{\boldsymbol{Y}_{n-1}(P\mathbf{g}_\sigma)}{\boldsymbol{\eta}_r^{1/(p^e+1)}}\right) \equiv a_{r,\sigma}^{-p^e}(\boldsymbol{y} - b_{r,\sigma}^{\frac{1}{p^e}}) \equiv a_{r,\sigma}(\boldsymbol{y} + b_{r,\sigma}^{p^e}) \mod_{>} 0$$

by (3.22). By the same argument using (3.10), we have $\boldsymbol{Y}_i(P\mathbf{g}_\sigma) \equiv \sigma^{-1}(\boldsymbol{Y}_i(P)) \mod_{>} 1/(2n)$ for $1 \leq i \leq n-1$. This implies

$$\boldsymbol{y}_i(P\mathbf{g}_\sigma) \equiv \frac{\sigma^{-1}(\boldsymbol{\eta}_r^{1/2})}{\boldsymbol{\eta}_r^{1/2}}\sigma^{-1}(\boldsymbol{y}_i(P)) \equiv a_{r,\sigma}^{(p^e+1)/2}\boldsymbol{y}_i(P)^{q^{n_\sigma}} \mod_{>} 0$$

for $1 \leq i \leq n-1$. □

Stabilizer We put $n_1 = (n, p^m - 1)$. We put $\varphi_r'' = \varphi_r'^{n_1}$ and $F_r = K(\varphi_r'')$. Let $\sigma \in W_{F_r}$. We put $\zeta_\sigma = \sigma^{-1}(\varphi_r')/\varphi_r'$. Let ζ_σ^{1/p^e} be the p^e-th root of ζ_σ in $\mu_{p^m-1}(K)$. We put $\varphi_{r,\sigma} = \zeta_\sigma^{1/p^e}\varphi_r$. Let $\mathcal{G}_{r,\sigma}$ be the one-dimensional formal $\mathcal{O}_{L_r}$-module over $\mathcal{O}_{\widehat{L}_r^{\mathrm{ur}}}$ defined by (2.1) changing φ_r by $\varphi_{r,\sigma}$. We take a compatible system $\{t_{r,j,\sigma}\}_{j\geq 1}$ in $\mathbf{C}$ such that

$$\frac{\sigma^{-1}(t_{r,1})}{t_{r,1,\sigma}} \equiv 1 \mod_{>} 0, \quad [\varphi_{r,\sigma}]_{\mathcal{G}_{r,\sigma}}(t_{r,1,\sigma}) = 0, \quad [\varphi_{r,\sigma}]_{\mathcal{G}_{r,\sigma}}(t_{r,j,\sigma}) = t_{r,j-1,\sigma}$$

for $j \geq 2$. We construct $\xi_{r,\sigma}$ as in Lemma 2.1 using $\{t_{r,j,\sigma}\}_{j\geq 1}$. Then $\xi_{r,\sigma}$ has CM by L_r.

Lemma 3.5. *For $\sigma \in W_{F_r}$, we have*

$$\frac{\sigma^{-1}(\xi_{r,i})}{\xi_{r,\sigma,i}} \equiv 1 \mod \frac{1}{q^{i-1}p^{e-1}(p-1)} \quad \text{for } 1 \leq i \leq n,$$
$$\sigma^{-1}(\boldsymbol{\theta}_r) \equiv \boldsymbol{\theta}_r \mod \frac{1}{n(p^e+1)}.$$

Proof. We have

$$\frac{\sigma^{-1}(\varphi_r)}{\varphi_r} \equiv \zeta_\sigma^{1/p^e} \mod \frac{1}{p^{e-1}(p-1)}. \tag{3.23}$$

We obtain the claims by (3.23) and

$$\bigl(\sigma^{-1}(\boldsymbol{\theta}_r) - \boldsymbol{\theta}_r\bigr)^{p^{2e}} + \boldsymbol{\eta}_r^{p^e-1}\bigl(\sigma^{-1}(\boldsymbol{\theta}_r) - \boldsymbol{\theta}_r\bigr) + \bigl(\mathbf{1} + \sigma^{-1}(\boldsymbol{\theta}_r)\bigr)\bigl(\sigma^{-1}(\boldsymbol{\eta}_r)^{p^e-1} - \boldsymbol{\eta}_r^{p^e-1}\bigr) = \mathbf{0},$$

which follows from (2.3). □

We define $j_r \colon W_{F_r} \to L_r^\times \backslash (GL_n(K) \times D^\times)$ as follows:

> Let $\sigma \in W_{F_r}$. Since $\xi_{r,\sigma}$ has CM by L_r, there exists $(g,d) \in GL_n(K) \times D^\times$ uniquely up to left multiplication by $L_r^\times$ such that $(g,d,1) \in G^0$ and $\xi_{r,\sigma}(g,d,1) = \xi_r$ by Lemma 1.6. We put $j_r(\sigma) = L_r^\times(g, \varphi_{D,r}^{-n_\sigma}d)$.

For $\sigma \in W_{L_r}$, we put $a_\sigma = \mathrm{Art}_{L_r}^{-1}(\sigma) \in L_r^\times$ and $u_\sigma = a_\sigma \varphi_r^{-n_\sigma} \in \mathcal{O}_{L_r}^\times$.

Lemma 3.6. *For $\sigma \in W_{L_r}$, we have $j_r(\sigma) = L_r^\times(1, a_\sigma^{-1})$.*

Proof. This follows from [BW16, Lemma 3.1.3]. Note that our action of W_K is inverse to that in [BW16]. □

We put

$$\mathcal{S}_r = \{(g,d,\sigma) \in G \mid \sigma \in W_{F_r},\ j_r(\sigma) = L_r^\times(g,d)\}.$$

Lemma 3.7. *The action of $\mathcal{S}_r$ on $\mathcal{M}_{\infty,\overline{\eta}}^{(0)}$ stabilizes $\mathcal{X}_r$, and induces the action on $\overline{\mathfrak{X}_r}$.*

Proof. We take an element of $\mathcal{S}_r$, and write it as $(g, \varphi_{D,r}^{-n_\sigma} d, \sigma)$, where $(g,d,1) \in G^0$ and $\sigma \in W_{F_r}$. Since $\xi_{r,\sigma}(g,d,1) = \xi_r$, we have $(g,d) \in (\varphi_{M,r}, \varphi_{D,r})^l(\mathfrak{I}^\times \times \mathcal{O}_D^\times)$ by Lemma 3.1 and Lemma 3.5.

To show the claims, we may assume that $(g,d) \in \mathfrak{I}^\times \times \mathcal{O}_D^\times$ by Proposition 3.2.1. We write $g = (a_{i,j})_{1 \leq i,j \leq n} \in \mathfrak{I}^\times$ and $a_{i,j} = \sum_{l=0}^\infty a_{i,j}^{(l)} \varpi_r^l$ with $a_{i,j}^{(l)} \in \mu_{q-1}(K) \cup \{0\}$, and $d^{-1} = \sum_{i=0}^\infty d_i \varphi_{D,r}^i$ with $d_i \in \mu_{q^n-1}(K_n) \cup \{0\}$. For $1 \leq i \leq n-1$, we have

$$\frac{a_{i,i}^{(0)}}{a_{i+1,i+1}^{(0)}} = d_0^{q-1} \tag{3.24}$$

by $\xi_{r,\sigma}(g,d,1) = \xi_r$ using (3.9), (3.10), $\xi_{r,\sigma,i} = \xi_{r,\sigma,i+1}^q$ and $\xi_{r,i} = \xi_{r,i+1}^q$. The condition on the first line in (2.5) is equivalent to

$$v\left(\frac{\boldsymbol{X}_i}{\boldsymbol{X}_{i+1}} - \left(\frac{\boldsymbol{X}_{n-1}}{\boldsymbol{X}_n}\right)^{q^{n-1-i}}\right) \geq \frac{3}{2nq^i} \quad \text{for } 1 \leq i \leq n-2. \tag{3.25}$$

We see that the condition (3.25) is stable under the action of $(g, \varphi_{D,r}^{-n_\sigma} d, \sigma)$ using (3.9) and (3.10), because $a_{i,i}^{(0)}/a_{i+1,i+1}^{(0)}$ is independent of i by (3.24). We see that the condition on the second line in (2.5) is stable under the action of $(g, \varphi_{D,r}^{-n_\sigma} d, \sigma)$ by Lemma 3.5 using (3.9) and (3.10). □

The group $\mathcal{S}_r$ normalizes $i_{\xi_r}(L_r^\times) \cdot (U_{\mathfrak{I}}^1 \times U_D^1)^1$ by Proposition 3.3. We put

$$H_r = (U_{\mathfrak{I}}^1 \times U_D^1)^1 \cdot \mathcal{S}_r \subset G.$$

Then H_r acts on $\overline{\mathfrak{X}_r}$ by Lemma 3.7 and the proof of Proposition 3.3.

Proposition 3.8. *The subgroup $H_r \subset G^0$ is the stabilizer of $\mathcal{X}_r$ in $\mathcal{M}_{\infty,\overline{\eta}}^{(0)}$.*

Proof. Assume that $(g, \varphi_{D,r}^{-n_\sigma} d, \sigma) \in G^0$ stabilizes $\mathcal{X}_r$. It suffices to show that $(g, \varphi_{D,r}^{-n_\sigma} d, \sigma) \in H_r$. By Lemma 3.1, we have $(g,d) \in (\varphi_{M,r}, \varphi_{D,r})^l(\mathfrak{I}^\times \times \mathcal{O}_D^\times)$. Hence, we may assume that $(g,d) \in \mathfrak{I}^\times \times \mathcal{O}_D^\times$ by Proposition 3.2.1.

First, we show that $\sigma \in W_{F_r}$. We write $g = (a_{i,j})_{1 \leq i,j \leq n} \in \mathfrak{I}^\times$, $a_{i,j} = \sum_{l=0}^\infty a_{i,j}^{(l)} \varpi_r^l$ and $d^{-1} = \sum_{i=0}^\infty d_i \varphi_{D,r}^i$ as in the proof of Lemma 3.7. Since $(g, \varphi_{D,r}^{-n_\sigma} d, \sigma)$ stabilizes $\mathcal{X}_r$, we have

$$\frac{a_{i,i}^{(0)}}{a_{i+1,i+1}^{(0)}} = d_0^{q-1} \quad \text{for } 1 \leq i \leq n-1, \tag{3.26}$$

$$\frac{a_{n,n}^{(0)} d_0 \sigma^{-1}(\xi_{r,n}^0)}{\xi_{r,n}^0} \equiv 1 \mod \frac{1}{nq^{n-1}(p^e+1)} \tag{3.27}$$

by (2.5), (3.9), (3.10) and $\xi_{r,i} = \xi_{r,i+1}^q$. By taking $p^e q^{n-1}(q-1)$-th power of (3.27), we see that

$$d_0^{p^e q^{n-1}(q-1)} \frac{\sigma^{-1}(\varphi_r')}{\varphi_r'} \equiv \left(\frac{1+\theta_r}{1+\sigma^{-1}(\theta_r)}\right)^{p^e(q-1)} \mod \frac{p^e}{n(p^e+1)}. \tag{3.28}$$

This implies that the left hand side of (3.28) is equal to 1. Hence we have $\sigma^{-1}(\varphi_r')/\varphi_r' \in \mu_{q-1}(K)$ and $\sigma^{-1}(\theta_r) \equiv \theta_r \mod 1/(n(p^e+1))$, since $d_0^{q-1} \in \mu_{q-1}(K)$ by (3.26). These happen only if $\sigma \in W_{F_r}$ by the proof of Lemma 3.5 and $\mu_{p^e-1}(K^{\mathrm{ur}}) \cap \mu_{q-1}(K) = \mu_{p^m-1}(K)$. Since $\sigma \in W_{F_r}$, we may assume that $\sigma = 1$ by Lemma 3.7. Then $(g,d,1) \in H_r$ by Proposition 3.3. □

4 Artin-Schreier variety in characteristic two

In this section, we assume that $p = 2$. For an integer $i \geq 0$, we simply write $\mathbb{A}^i$ for an affine space $\mathbb{A}^i_{k^{\mathrm{ac}}}$. Let $n \geq 4$ be an even integer. We consider the affine smooth variety Y of dimension $n-2$ defined by

$$z^{2^m} - z = \sum_{1 \leq i \leq j \leq n-2} y_i y_j \quad \text{in } \mathbb{A}^{n-1}.$$

Then, by the isomorphism

$$u_i = \sum_{j=i}^{n-2} y_j \quad \text{for } 1 \leq i \leq n-2,$$

the variety Y is isomorphic to the affine variety defined by

$$z^{2^m} - z = \sum_{i=1}^{n-2} u_i^2 + \sum_{i=1}^{n-3} u_i u_{i+1} \quad \text{in } \mathbb{A}^{n-1}. \tag{4.1}$$

For each $\zeta \in \mathbb{F}_{2^m}^{\times}$, we consider the homomorphism

$$p_\zeta \colon \mathbb{F}_{2^m} \to \mathbb{F}_2;\ x \mapsto \sum_{i=0}^{m-1} (\zeta^{-2} x)^{2^i}.$$

Then, we consider the quotient $Y_\zeta = Y / \ker p_\zeta$. This variety has the defining equation

$$\zeta^2 (z_\zeta^2 - z_\zeta) = \sum_{i=1}^{n-2} u_i^2 + \sum_{i=1}^{n-3} u_i u_{i+1} \quad \text{in } \mathbb{A}^{n-1},$$

where the relation between z and z_ζ is given by $z_\zeta = \sum_{i=0}^{m-1} (\zeta^{-2} z)^{2^i}$. We set $w_\zeta = \zeta z_\zeta + \sum_{i=1}^{n-2} u_i$. Then, Y_ζ is defined by

$$w_\zeta^2 + \zeta w_\zeta = \zeta \sum_{i=1}^{n-2} u_i + \sum_{i=1}^{n-3} u_i u_{i+1} \quad \text{in } \mathbb{A}^{n-1}. \tag{4.2}$$

Lemma 4.1. *Let $\ell \neq p$ be a prime number. Then we have an isomorphism*

$$H^{n-2}(Y, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\zeta \in \mathbb{F}_{2^m}^{\times}} H^{n-2}(Y_\zeta, \overline{\mathbb{Q}}_\ell)$$

and $\dim H^{n-2}(Y_\zeta, \overline{\mathbb{Q}}_\ell) = 1$.

Proof. By [IT13, Proposition 4.5.1], we know that there exists an isomorphism

$$H^{n-2}(Y, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\psi \in \mathbb{F}_{2^m}^{\vee} \setminus \{1\}} \psi$$

as $\mathbb{F}_{2^m}$-representations. Hence, for each $\psi \in \mathbb{F}_{2^m}^{\vee} \setminus \{1\}$, we acquire

$$H^{n-2}(Y, \overline{\mathbb{Q}}_\ell)[\psi] \simeq \psi.$$

Let $\iota_0 \colon \mathbb{F}_2 \hookrightarrow \overline{\mathbb{Q}}_\ell^{\times}$ be the non-trivial character. Then, for each $\psi \in \mathbb{F}_{2^m}^{\vee} \setminus \{1\}$, there exists a unique element $\zeta \in \mathbb{F}_{2^m}^{\times}$ such that $\psi = \iota_0 \circ p_\zeta$. Hence, we know that $H^{n-2}(Y_\zeta, \overline{\mathbb{Q}}_\ell) = H^{n-2}(Y, \overline{\mathbb{Q}}_\ell)[\psi] \simeq \psi$ as $\mathbb{F}_{2^m}$-representations. Hence, the required assertion follows. $\square$

We put $n_0 = (n-2)/2$. We can write (4.2) as

$$w_\zeta^2 + \zeta w_\zeta = \zeta \sum_{i=1}^{n_0} u_{2i} + \sum_{i=1}^{n_0} u_{2i-1}(u_{2i-2} + u_{2i} + \zeta), \tag{4.3}$$

where we use notation that $u_0 = 0$. Consider the fibration

$$\pi_\zeta \colon Y_\zeta \to \mathbb{A}^{n_0};\ (w_\zeta, (u_i)_{1 \le i \le n-2}) \mapsto ((u_{2i})_{1 \le i \le n_0}).$$

We consider the closed point P in $\mathbb{A}^{n_0}$ defined by

$$u_{2i} = i\zeta \text{ for } 1 \le i \le n_0.$$

We put $N_0 = \binom{n_0+1}{2}$. Then, we have

$$w_\zeta^2 + \zeta w_\zeta = N_0 \zeta^2 \tag{4.4}$$

on $\pi_\zeta^{-1}(P)$ by (4.3). By (4.4), the inverse image $\pi_\zeta^{-1}(P)$ has two connected components. Let $\varrho \in k^{\mathrm{ac}}$ be an element such that $\varrho^2 + \varrho = N_0$. We put

$$\varrho^+ = \varrho, \quad \varrho^- = \varrho + 1.$$

For $\iota \in \{\pm\}$, we define Z_ζ^ι to be the connected component of $\pi_\zeta^{-1}(P)$ defined by $w_\zeta = \zeta\varrho^\iota$. By (4.4), we know that Z_ζ^+ and Z_ζ^- are isomorphic to affine spaces of dimension n_0.

Let $\ell \neq p$ be a prime number. Let

$$\mathrm{cl} \colon CH_{n_0}(Y_\zeta) \to H^{n-2}(Y_\zeta, \overline{\mathbb{Q}}_\ell)$$

be the cycle class map.

Lemma 4.2. 1. *The fibration $\pi_\zeta \colon Y_\zeta \to \mathbb{A}^{n_0}$ is an affine bundle over $\mathbb{A}^{n_0} \setminus \{P\}$.*
2. *The cohomology group $H^{n-2}(Y_\zeta, \overline{\mathbb{Q}}_\ell)$ is generated by the cycle class $\mathrm{cl}([Z_\zeta^+])$, and we have $\mathrm{cl}([Z_\zeta^+]) = -\mathrm{cl}([Z_\zeta^-])$.*

Proof. The first claim follows from (4.3) easily.

We set $U = \pi_\zeta^{-1}(\mathbb{A}^{n_0} \setminus \{P\})$. We have the long exact sequence

$$H^{n-3}(U, \overline{\mathbb{Q}}_\ell) \to H^{n-2}_{\pi_\zeta^{-1}(P)}(Y_\zeta, \overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell(-n_0)^{\oplus 2} \to H^{n-2}(Y_\zeta, \overline{\mathbb{Q}}_\ell) \to H^{n-2}(U, \overline{\mathbb{Q}}_\ell)$$

and $H^{n-2}(U, \overline{\mathbb{Q}}_\ell) \simeq H^{n-2}(\mathbb{A}^{n_0} \setminus \{P\}, \overline{\mathbb{Q}}_\ell) = 0$, which follows from the first claim. Hence, $H^{n-2}(Y_\zeta, \overline{\mathbb{Q}}_\ell)$ is generated by the cycle classes $\mathrm{cl}([Z_\zeta^+])$ and $\mathrm{cl}([Z_\zeta^-])$. On the other hand, we have $\mathrm{cl}([Z_\zeta^+]) = -\mathrm{cl}([Z_\zeta^-])$, since $[Z_\zeta^+] + [Z_\zeta^-] = 0$ in $CH_{n_0}(Y_\zeta)$. Therefore, we obtain the claim. □

Remark 4.3. *Using Lemma 4.1, Lemma 4.2 and [IT13, Proposition 4.5], we can verify that a generalization of the Tate conjecture in [Jan90, 7.13] holds for the variety Y.*

For $\iota \in \{\pm\}$, we consider the other n_0-dimensional cycle $Z_\zeta'^\iota$ defined by

$$u_{2i-1} = (n_0 + 1 - i)\zeta \quad \text{for } 1 \le i \le n_0, \quad w_\zeta = \zeta\varrho^\iota.$$

Proposition 4.4. *For $\zeta \in \mathbb{F}_{2^m}^\times$ and $\iota \in \{\pm\}$, we have*

$$\left[Z_\zeta^\iota\right] = (-1)^{n_0} \left[Z_\zeta'^\iota\right] \quad \textit{in } CH_{n_0}(Y_\zeta).$$

Proof. We show that $[Z_\zeta^+] - (-1)^{n_0}[Z_\zeta'^+]$ is rationally equivalent to zero. For each $1 \le j \le n_0$, let $Y_{\zeta,j}$ be the (n_0+1)-dimensional closed subvariety of Y_ζ defined by

$$\begin{aligned} u_{2i-1} &= (n_0+1-i)\zeta && \text{for } 1 \le i \le j-1, \\ u_{2i-2} + u_{2i} &= \zeta && \text{for } j+1 \le i \le n_0. \end{aligned}$$

Note that

$$\sum_{i=1}^{j-1} u_{2i-1} = \left(N_0 + \binom{n_0+2-j}{2} \right) \zeta \tag{4.5}$$

on $Y_{\zeta,j}$. We see that the equality (4.3) becomes

$$\begin{aligned} w_\zeta^2 + \zeta w_\zeta &= \zeta \left(\sum_{i=1}^{j-1} u_{2i-1} + \sum_{i=j-1}^{n_0} u_{2i} \right) + u_{2j-3}u_{2j-2} + u_{2j-1}(u_{2j-2} + u_{2j} + \zeta) \\ &= N_0\zeta^2 + \big(u_{2j-1} + (n_0+1-j)\zeta\big)(u_{2j-2} + u_{2j} + \zeta) \end{aligned}$$

on $Y_{\zeta,j}$, using (4.5). Therefore, we acquire

$$\operatorname{div}(w_\zeta - \zeta\varrho) = [u_{2j-1} + (n_0+1-j)\zeta] + [u_{2j-2} + u_{2j} + \zeta] \tag{4.6}$$

on $Y_{\zeta,j}$. For $0 \le j \le n_0$, let $Z_{\zeta,j}^+$ be the n_0-dimensional cycle on Y_ζ defined by

$$\begin{aligned} u_{2i-1} &= (n_0+1-i)\zeta && \text{for } 1 \le i \le j, \\ u_{2i-2} + u_{2i} &= \zeta && \text{for } j+1 \le i \le n_0 \end{aligned}$$

and $w_\zeta = \zeta\varrho$. Note that $Z_{\zeta,0}^+ = Z_\zeta^+$ and $Z_{\zeta,n_0}^+ = Z_\zeta'^+$. By (4.6), we have

$$[Z_{\zeta,j}^+] + [Z_{\zeta,j+1}^+] = 0$$

in $CH_{n_0}(Y_\zeta)$ for $0 \le j \le n_0 - 1$. Hence, we have the claim for Z_ζ^+. We can prove the claim for Z_ζ^- replacing the condition $w_\zeta = \zeta\varrho$ by $w_\zeta = \zeta(\varrho+1)$ in the above argument. □

Corollary 4.5. *Assume that $n \ge 4$. Let g be the automorphism of Y defined by*

$$(z, (y_i)_{1\le i\le n-2}) \mapsto \Big(z + \varepsilon_1(y_{n-2}+1), \sum_{i=1}^{n-3} y_i + \varepsilon_1, (y_{i-1} + y_{n-2} + \varepsilon_1)_{2\le i\le n-2}\Big).$$

Then, g^* acts on $H^{n-2}(Y)(n_0)$ by -1.

Proof. First note that g induces the automorphism

$$\big(z, (u_i)_{1\le i\le n-2}\big) \mapsto \big(z + \varepsilon_1(u_{n-2}+1), u_{n-2}, (u_{i-1} + iu_{n-2} + (i+1)\varepsilon_1)_{2\le i\le n-2}\big).$$

We can check that $g^*w_\zeta = w_\zeta + \varepsilon_1$. Hence, we have

$$g^{-1}(Z_\zeta'^+) = \begin{cases} Z_\zeta^- & \text{if } e = 1, \\ Z_\zeta^+ & \text{otherwise.} \end{cases}$$

Therefore, we obtain

$$g^*\big(\mathrm{cl}([Z_\zeta^+])\big) = (-1)^{n_0} g^*\big(\mathrm{cl}([Z_\zeta'^+])\big) = -\mathrm{cl}([Z_\zeta^+])$$

in $H^{n-2}(Y)(n_0)$ using Lemma 4.2 and Proposition 4.4. Hence, the claim follows from Lemma 4.1 and Lemma 4.2. □

5 Explicit LLC and LJLC

5.1 Galois representations

Let X be the affine smooth variety over k^{ac} defined by (2.16). We define an action of $Q \rtimes \mathbb{Z}$ on X similarly as (3.17).

We choose an isomorphism $\iota \colon \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. Let $q^{1/2} \in \overline{\mathbb{Q}}_\ell$ be the 2-nd root of q such that $\iota(q^{1/2}) > 0$. For a rational number $r \in 2^{-1}\mathbb{Z}$, let $\overline{\mathbb{Q}}_\ell(r)$ be the unramified representation of $\mathrm{Gal}(k^{\mathrm{ac}}/k)$ of degree 1, on which the geometric Frobenius Frob_q acts as scalar multiplication by q^{-r}. We simply write Q for the subgroup $Q \times \{0\} \subset Q \rtimes \mathbb{Z}$. We consider the morphism

$$\Phi \colon \mathbb{A}^{n-1}_{k^{\mathrm{ac}}} \to \mathbb{A}^1_{k^{\mathrm{ac}}};\ (y, (y_i)_{1 \leq i \leq n-2}) \mapsto y^{p^e+1} - \frac{1}{n'} \sum_{1 \leq i \leq j \leq n-2} y_i y_j.$$

Let $\mathcal{L}_\psi$ be the Artin-Schreier $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{A}^1_{k^{\mathrm{ac}}}$ associated to ψ, which is $\mathfrak{F}(\psi)$ in the notation of [Del77, Sommes trig. 1.8 (i)]. Then we have a decomposition

$$H^{n-1}_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\psi \in \mathbb{F}^\vee_{p^m} \backslash \{1\}} H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}}, \Phi^* \mathcal{L}_\psi) \tag{5.1}$$

as $Q \rtimes \mathbb{Z}$-representations. We put

$$\tau_{\psi,n} = H^{n-1}_{\mathrm{c}}(\mathbb{A}^{n-1}_{k^{\mathrm{ac}}}, \Phi^* \mathcal{L}_\psi)\left(\frac{n-1}{2}\right)$$

as a $Q \rtimes \mathbb{Z}$-representation for each $\psi \in \mathbb{F}^\vee_{p^m} \backslash \{1\}$. We write $\tau^0_{r,\psi}$ for the inflation of $\tau_{\psi,n}$ by Θ_r. We put $\tau_{r,\psi} = \mathrm{Ind}_{E_r/K} \tau^0_{r,\psi}$.

5.2 Correspondence

Definition 5.1. *We say that an irreducible finite dimensional continuous ℓ-adic representation of W_K is simple epipelagic if its exponential Swan conductor is one.*

We apply the same definition to a smooth irreducible supercuspidal representation of $GL_n(K)$ and a smooth irreducible representation of $D^\times$.

Remark 5.2. *The words "simple" and "epipelagic" come from [GR10] and [RY14]. Our "simple epipelagic" representations are called "epipelagic" in [BH14].*

We define $\psi_0 \in \mathbb{F}_p^\vee$ by $\iota(\psi_0(1)) = e^{2\pi\sqrt{-1}/p}$. We put $\psi^0 = \mathrm{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_p} \circ \psi_0$. We take an additive character $\psi_K \colon K \to \overline{\mathbb{Q}}_\ell^\times$ such that $\psi_K(x) = \psi^0(\bar{x})$ for $x \in \mathcal{O}_K$. In the following, for each triple $(\zeta, \chi, c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \overline{\mathbb{Q}}_\ell^\times$, we define a $GL_n(K)$-representation $\pi_{\zeta,\chi,c}$, a $D^\times$-representation $\rho_{\zeta,\chi,c}$ and a W_K-representation $\tau_{\zeta,\chi,c}$.

We use notations in Subsection 2.1, replacing $r \in \mu_{q-1}(K)$ by $\zeta \in \mu_{q-1}(K)$. We have the K-algebra embeddings

$$L_\zeta \to M_n(K);\ \varphi_\zeta \mapsto \varphi_{M,\zeta}, \quad L_\zeta \to D;\ \varphi_\zeta \mapsto \varphi_{D,\zeta}.$$

Set $\varphi_{\zeta,n} = n'\varphi_\zeta$. Let $\Lambda_{\zeta,\chi,c} \colon L_\zeta^\times U^1_{\mathfrak{J}} \to \overline{\mathbb{Q}}_\ell^\times$ be the character defined by

$$\Lambda_{\zeta,\chi,c}(\varphi_\zeta) = (-1)^{n-1}c, \quad \Lambda_{\zeta,\chi,c}(x) = \chi(\bar{x}) \quad \text{for } x \in \mathcal{O}_K^\times,$$
$$\Lambda_{\zeta,\chi,c}(x) = (\psi_K \circ \mathrm{tr})(\varphi_{\zeta,n}^{-1}(x-1)) \quad \text{for } x \in U^1_{\mathfrak{J}}.$$

We put $\pi_{\zeta,\chi,c} = \text{c-Ind}_{L_\zeta^\times U_{\mathfrak{J}}^1}^{GL_n(K)} \Lambda_{\zeta,\chi,c}$. Then, $\pi_{\zeta,\chi,c}$ is a simple epipelagic representation of $GL_n(K)$, and every simple epipelagic representation is isomorphic to $\pi_{\zeta,\chi,c}$ for a uniquely determined $(\zeta,\chi,c) \in \mu_{q-1}(K) \times (k^\times)^\vee \times \overline{\mathbb{Q}}_\ell^\times$ (*cf.* [BH14, 2.1, 2.2]).

Let $\theta_{\zeta,\chi,c} \colon L_\zeta^\times U_D^1 \to \overline{\mathbb{Q}}_\ell^\times$ be the character defined by

$$\theta_{\zeta,\chi,c}(\varphi_\zeta) = c, \quad \theta_{\zeta,\chi,c}(x) = \chi(\bar{x}) \quad \text{for } x \in \mathcal{O}_K^\times,$$
$$\theta_{\zeta,\chi,c}(d) = \left(\psi_K \circ \mathrm{Trd}_{D/K}\right)\left(\varphi_{\zeta,n}^{-1}(d-1)\right) \quad \text{for } d \in U_D^1.$$

We put $\rho_{\zeta,\chi,c} = \mathrm{Ind}_{L_\zeta^\times U_D^1}^{D^\times} \theta_{\zeta,\chi,c}$. The isomorphism class of this representation does not depend on the choice of the embedding $L_\zeta \hookrightarrow D$.

Let $\phi_c \colon W_{E_\zeta} \to \overline{\mathbb{Q}}_\ell^\times$ be the character defined by $\phi_c(\sigma) = c^{n_\sigma}$. Let $\mathrm{Frob}_p \colon k^\times \to k^\times$ be the map defined by $x \mapsto x^{p^{-1}}$ for $x \in k^\times$. We consider the composite

$$\nu_\zeta \colon W_{E_\zeta}^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_{E_\zeta}^{-1}} E_\zeta^\times \to \mathcal{O}_{E_\zeta}^\times \xrightarrow{\text{can.}} k^\times \xrightarrow{\mathrm{Frob}_p^e} k^\times,$$

where the second homomorphism is given by $E_\zeta^\times \to \mathcal{O}_{E_\zeta}^\times;\ x \mapsto x\varphi_\zeta'^{-v_{E_\zeta}(x)}$. We simply write τ_ζ^0 for τ_{ζ,ψ^0}^0. We set $\tau_{\zeta,\chi,c}^0 = \tau_\zeta^0 \otimes (\chi \circ \nu_\zeta) \otimes \phi_c$ and $\tau_{\zeta,\chi,c} = \mathrm{Ind}_{E_\zeta/K} \tau_{\zeta,\chi,c}^0$.

The following theorem follows from [IT14] and [IT15b].

Theorem 5.3. *Let* JL *and* LL *denote the local Jacquet-Langlands correspondence and the local Langlands correspondence for $GL_n(K)$ respectively. For $\zeta \in \mu_{q-1}(K)$, $\chi \in (k^\times)^\vee$ and $c \in \overline{\mathbb{Q}}_\ell^\times$, we have* $\mathrm{JL}(\rho_{\zeta,\chi,c}) = \pi_{\zeta,\chi,c}$ *and* $\mathrm{LL}(\pi_{\zeta,\chi,c}) = \tau_{\zeta,\chi,c}$.

Definition 5.4. *We say that an irreducible finite dimensional continuous ℓ-adic representation of W_K is essentially simple epipelagic if it is a character twist of a simple epipelagic representation.*

We apply the same definition to a smooth irreducible representation of $GL_n(K)$ and a smooth irreducible representation of $D^\times$.

Let $\omega \colon K^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character. We put

$$\pi_{\zeta,\chi,c,\omega} = \pi_{\zeta,\chi,c} \otimes (\omega \circ \det), \quad \rho_{\zeta,\chi,c,\omega} = \rho_{\zeta,\chi,c} \otimes (\omega \circ \mathrm{Nrd}_{D/K}), \quad \tau_{\zeta,\chi,c,\omega} = \tau_{\zeta,\chi,c} \otimes (\omega \circ \mathrm{Art}_K^{-1}),$$

and

$$\Lambda_{\zeta,\chi,c,\omega} = \Lambda_{\zeta,\chi,c} \otimes (\omega \circ \det|_{L_\zeta^\times U_{\mathfrak{J}}^1}), \quad \theta_{\zeta,\chi,c,\omega} = \theta_{\zeta,\chi,c} \otimes (\omega \circ \mathrm{Nrd}_{D/K}|_{L_\zeta^\times U_D^1}),$$
$$\tau_{\zeta,\chi,c,\omega}^0 = \tau_{\zeta,\chi,c}^0 \otimes (\omega \circ \mathrm{Nr}_{L_\zeta'/K} \circ \mathrm{Art}_{E_\zeta}^{-1}).$$

Then we have

$$\pi_{\zeta,\chi,c,\omega} = \text{c-Ind}_{L_\zeta^\times U_{\mathfrak{J}}^1}^{GL_n(K)} \Lambda_{\zeta,\chi,c,\omega}, \quad \rho_{\zeta,\chi,c,\omega} = \mathrm{Ind}_{L_\zeta^\times U_D^1}^{D^\times} \theta_{\zeta,\chi,c,\omega}, \quad \tau_{\zeta,\chi,c,\omega} = \mathrm{Ind}_{L_\zeta'/K} \tau_{\zeta,\chi,c,\omega}^0.$$

Corollary 5.5. *We have* $\mathrm{LL}(\pi_{\zeta,\chi,c,\omega}) = \tau_{\zeta,\chi,c,\omega}$ *and* $\mathrm{JL}(\rho_{\zeta,\chi,c,\omega}) = \pi_{\zeta,\chi,c,\omega}$.

Proof. This follows from Theorem 5.3, because LL and JL are compatible with character twists. □

6 Geometric realization

We fix $s \in \mu_{\frac{n_1(q-1)}{p^m-1}}(K)$. We take an element $r \in \mu_{q-1}(K)$ such that $r^{\frac{p^m-1}{n_1}} = s$. We put

$$H_{\mathfrak{X}_r} = H_{\mathrm{c}}^{n-1}(\overline{\mathfrak{X}}_r)\left(\frac{n-1}{2}\right)$$

as H_r-representations. Further, we put

$$\Pi_s = \text{c-Ind}_{H_r}^G H_{\mathfrak{X}_r},$$

whose isomorphism class as a G-representation depends only on s. For simplicity, we write G_1 and G_2 for $GL_n(K)$ and $D^\times \times W_K$ respectively, and consider them as subgroups of G. We put

$$H = \{g \in U_{\mathfrak{J}}^1 \mid \det(g) = 1\}.$$

We have $H = H_r \cap G_1$ by Proposition 3.3. Let $\overline{H}_r$ be the image of H_r in $G/G_1 \simeq G_2$.

Let $a \in \mu_{q-1}(K)$. We define a character $\Lambda_r^a \colon U_{\mathfrak{J}}^1 \to \overline{\mathbb{Q}}_\ell^\times$ by $x \mapsto (\psi_K \circ \mathrm{tr})((a\varphi_{r,n})^{-1}(x-1))$. Let π be a smooth irreducible representation of $GL_n(K)$.

Lemma 6.1. *If π is not essentially simple epipelagic, then we have $\mathrm{Hom}_H(\Lambda_r^a, \pi) = 0$. Further, we have*

$$\dim \mathrm{Hom}_H(\Lambda_r^a, \pi_{\zeta,\chi,c,\omega}) = \begin{cases} 1 & \text{if } a^n r = \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We assume that $\mathrm{Hom}_H(\Lambda_r^a, \pi) \neq 0$, and show that π is essentially simple epipelagic. Let ω_π be the centaral character of π. Then ω_π is trivial on $K^\times \cap H$ by $\mathrm{Hom}_H(\Lambda_r^a, \pi) \neq 0$. Hence, we may assume that Then ω_π is trivial on $K^\times \cap U_{\mathfrak{J}}^1$, changing π by a character twist. Then, there is a character $\Lambda_{r,\omega_\pi}^a \colon K^\times U_{\mathfrak{J}}^1 \to \overline{\mathbb{Q}}_\ell^\times$ such that

$$\Lambda_{r,\omega_\pi}^a|_{U_{\mathfrak{J}}^1} = \Lambda_\zeta^a, \quad \Lambda_{r,\omega_\pi}^a|_{K^\times} = \omega_\pi.$$

Then we have

$$\mathrm{Hom}_H(\Lambda_r^a, \pi) \simeq \mathrm{Hom}_{K^\times H}(\Lambda_{r,\omega_\pi}^a, \pi) \simeq \mathrm{Hom}_{K^\times U_{\mathfrak{J}}^1}\left(\mathrm{Ind}_{K^\times H}^{K^\times U_{\mathfrak{J}}^1}(\Lambda_{r,\omega_\pi}^a|_{K^\times H}), \pi\right) \tag{6.1}$$

by Frobenius reciprocity. We have the natural isomorphism

$$K^\times U_{\mathfrak{J}}^1/(K^\times H) \xrightarrow{\det} (K^\times)^n U_K^1/(K^\times)^n \simeq U_K^1/(U_K^1)^n. \tag{6.2}$$

For a smooth character ϕ of $U_K^1/(U_K^1)^n$, let ϕ' denote the character of $K^\times U_{\mathfrak{J}}^1$ obtained by ϕ and the isomorphism (6.2). We have a natural isomorphism

$$\mathrm{Ind}_{K^\times H}^{K^\times U_{\mathfrak{J}}^1}(\Lambda_{r,\omega_\pi}^a|_{K^\times H}) \simeq \bigoplus_{\phi \in (U_K^1/(U_K^1)^n)^\vee} \Lambda_{r,\omega_\pi}^a \otimes \phi'. \tag{6.3}$$

Let ϕ be a smooth character of $U_K^1/(U_K^1)^n$, and regard it as a character of U_K^1. We extend ϕ to a character $\tilde{\phi}$ of $K^\times$ such that $\tilde{\phi}(\varpi) = 1$ and $\tilde{\phi}$ is trivial on $\mu_{q-1}(K)$. We have

$$\mathrm{Hom}_{K^\times U_{\mathfrak{J}}^1}(\Lambda_{r,\omega_\pi}^a \otimes \phi', \pi) \simeq \mathrm{Hom}_{G_1}\Big(\big(\text{c-Ind}_{K^\times U_{\mathfrak{J}}^1}^{G_1} \Lambda_{r,\omega_\pi}^a\big) \otimes \tilde{\phi}, \pi\Big). \tag{6.4}$$

We take $\chi' \in (k^\times)^\vee$ such that $\chi'(\bar{x}) = \omega_\pi(x)$ for $x \in \mu_{q-1}(K)$. For $c' \in \overline{\mathbb{Q}}_\ell^\times$, we define the character $\Lambda^a_{r,\chi',c'} \colon L_r^\times U^1_{\mathfrak{J}} \to \overline{\mathbb{Q}}_\ell^\times$ by

$$\Lambda^a_{r,\chi',c'}|_{U^1_{\mathfrak{J}}} = \Lambda^a_r, \quad \Lambda^a_{r,\chi',c'}(\varphi_{M,r}) = c', \quad \Lambda^a_{r,\chi',c'}(x) = \chi'(\bar{x}) \quad \text{for } x \in \mu_{q-1}(K).$$

We put $\pi^a_{r,\chi',c'} = \text{c-Ind}^{G_1}_{L_r^\times U^1_{\mathfrak{J}}} \Lambda^a_{r,\chi',c'}$. Then we have

$$\text{c-Ind}^{G_1}_{K^\times U^1_{\mathfrak{J}}} \Lambda^a_{r,\omega_\pi} \simeq \bigoplus_{c' \in \overline{\mathbb{Q}}_\ell^\times} \pi^a_{r,\chi',c'}. \tag{6.5}$$

Note that

$$\pi^a_{r,\chi',c'} \simeq \pi_{a^n r,\chi',\chi'(a)c'} \tag{6.6}$$

by the constructions. Then we see that π is simple epipelagic by (6.1), (6.3), (6.4), (6.5), (6.6) and the assumption $\mathrm{Hom}_H(\Lambda^a_r, \pi) \neq 0$.

For an irreducible admissible representation π of G_1, we write $\mathrm{a}(\pi)$ for its Artin conductor exponent. Then, if $\phi \neq 1$, we have $\mathrm{a}(\tilde{\phi}) \geq 2n$. Hence, by $\mathrm{a}(\pi^a_{r,\chi,c'}) = n+1$, we obtain $\mathrm{a}(\pi^a_{r,\chi,c'} \otimes \tilde{\phi}) = \mathrm{a}(\tilde{\phi}) \geq 2n$. Therefore, we acquire

$$\dim \mathrm{Hom}_{G_1}(\pi^a_{r,\chi',c'} \otimes \tilde{\phi}, \pi_{\zeta,\chi,c}) = \begin{cases} 1 & \text{if } \phi = 1,\ a^n r = \zeta \text{ and } \chi'(a)c' = c, \\ 0 & \text{otherwise} \end{cases}$$

by (6.6) and [BH14, 2.2]. To show the second claim, we may assume that $\omega = 1$. Hence, we obtain the second claim by the above discussion, using that $\omega_{\pi_{\zeta,\chi,c}}$ is trivial on U^1_K. □

Proposition 6.2. 1. *If π is not essentially simple epipelagic, then we have $\mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi) = 0$. Further, we have*

$$\dim \mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{\zeta,\chi,c,\omega}) = \begin{cases} p^e n_1 & \text{if } \zeta^{\frac{p^m-1}{n_1}} = s, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

2. *We have $L_r^\times U^1_D \times W_{E_r} \subset \overline{H}_r$ and an injective homomorphism*

$$\theta_{r,\chi,c,\omega} \otimes \tau^0_{r,\chi,c,\omega} \hookrightarrow \mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{r,\chi,c,\omega})$$

as $L_r^\times U^1_D \times W_{E_r}$-representations.

Proof. By (5.1), we have a decomposition

$$H_{\mathfrak{X}_r} \simeq \bigoplus_{\psi \in \mathbb{F}_{p^m}^\vee \setminus \{1\}} \tau_{\psi,n} \tag{6.8}$$

as representations of $Q \rtimes \mathbb{Z}$. By Proposition 3.3 and (6.8), we have

$$H_{\mathfrak{X}_r} \simeq \bigoplus_{a \in \mu_{p^m-1}(K)} (\Lambda_r^{-a})^{\oplus p^e} \tag{6.9}$$

as H-representations. We prove the first claim. If $\zeta^{\frac{p^m-1}{n_1}} \neq s$, the claim follows from Lemma 6.1 and (6.9). Assume that $\zeta^{\frac{p^m-1}{n_1}} = s$. By Lemma 6.1 and (6.9), we have

$$\mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{\zeta,\chi,c,\omega}) \simeq \bigoplus_{a \in \mu_{p^m-1}(K),\, a^n r = \zeta} \mathrm{Hom}_H(\Lambda_r^{-a}, \pi_{\zeta,\chi,c,\omega})^{\oplus p^e}, \tag{6.10}$$

and the dimension of this space is $p^e n_1$.

We prove the second claim. We consider the element $(\varphi_{D,r}, 1) \in L_r^\times U_D^1 \times W_{E_r} \subset G_2$ and its lifting $\mathbf{g}_r \in G$ in (3.1) with respect to $G \to G_2$. We have $\mathbf{g}_r \in H_r$ by Proposition 3.2.1. The element $(\varphi_{D,r}, 1)$ acts on $\theta_{r,\chi,c,\omega} \otimes \tau^0_{r,\chi,c,\omega}$ as scalar multiplication by $c\omega((-1)^{n-1}r\varpi)$. By Proposition 3.2.2, Corollary 4.5, [IT13, Proposition 4.2.3], the element $\mathbf{g}_r$ acts on $\mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{r,\chi,c,\omega})$ as scalar multiplication by $c\omega((-1)^{n-1}r\varpi)$.

Let $zd \in \mathcal{O}_K^\times U_D^1$ with $z \in \mu_{q-1}(K)$ and $d \in U_D^1$. Let $g = (a_{i,j})_{1 \le i,j \le n} \in U_{\mathfrak{J}}^1$ be the element defined by $a_{1,1} = \mathrm{Nrd}_{D/K}(d)$, $a_{i,i} = 1$ for $2 \le i \le n$ and $a_{i,j} = 0$ if $i \neq j$. We have $\det(g) = \mathrm{Nrd}_{D/K}(d)$ and $(zg, zd, 1) \in H_r$. The element $(zd, 1) \in L_r^\times U_D^1 \times W_{E_r}$ acts on $\theta_{r,\chi,c,\omega} \otimes \tau^0_{r,\chi,c,\omega}$ as scalar multiplication by $\chi(\bar{z})\theta_{r,\chi,c}(d)\omega(\mathrm{Nrd}_{D/K}(zd))$. We have the subspace

$$\mathrm{Hom}_H(\tau_{-\psi^0,n}, \pi_{r,\chi,c,\omega}) \subset \mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{r,\chi,c,\omega}) \tag{6.11}$$

by the decomposition (6.8). By Remark 1.4, Proposition 3.3 and [IT13, Propositions 4.2.1 and 4.5.1], the element $(zg, zd, 1)$ acts on the subspace (6.11) as scalar multiplication by $\chi(\bar{z})\theta_{r,\chi,c}(d)\omega(\det(zg))$.

Let $\sigma \in W_{E_r}$ such that $n_\sigma = 1$. We take $\mathbf{g}_\sigma$ as in (3.14). By Proposition 3.4, the element $\mathbf{g}_\sigma$ acts on the subspace (6.11) by

$$\chi(\bar{b}_\sigma)\tau^0_{r,\psi^0}(\sigma)\omega(\det(g_\sigma)).$$

On the other hand, the element $(\varphi_{D,r}^{-1}, \sigma) \in L_r^\times U_D^1 \times W_{E_r}$ acts on $\theta_{r,\chi,c,\omega} \otimes \tau^0_{r,\chi,c,\omega}$ by

$$(\chi \circ \nu_r)(\sigma)\tau^0_{r,\psi^0}(\sigma)\omega(\mathrm{Nr}_{E_r/K}(u_\sigma)).$$

Hence, the required assertion follows. $\square$

Proposition 6.3. *If π is not essentially simple epipelagic, then we have* $\mathrm{Hom}_{GL_n(K)}(\Pi_s, \pi) = 0$. *Further, we have*

$$\mathrm{Hom}_{GL_n(K)}(\Pi_s, \pi_{\zeta,\chi,c,\omega}) \simeq \begin{cases} \rho_{\zeta,\chi,c,\omega} \otimes \tau_{\zeta,\chi,c,\omega} & \text{if } \zeta^{\frac{p^m-1}{n_1}} = s, \\ 0 & \text{otherwise} \end{cases}$$

as $D^\times \times W_K$-representations.

Proof. For $g \in H_r \backslash G/G_1$, we choose an element $\tilde{g} \in G_2$ whose image in $\overline{H}_r \backslash G_2$ equals g under the natural isomorphism $H_r \backslash G/G_1 \simeq \overline{H}_r \backslash G_2$. We put $H^{\tilde{g}} = \tilde{g}^{-1} H \tilde{g}$. Let $H^{\tilde{g}}_{\mathfrak{X}_r}$ denote the representation of $H^{\tilde{g}}$ which is the conjugate of $H_{\mathfrak{X}_r}$ by $\tilde{g}$. Then, we have

$$\Pi_s|_{G_1} \simeq \bigoplus_{g \in H_r \backslash G/G_1} \text{c-Ind}^{G_1}_{H^{\tilde{g}}} H^{\tilde{g}}_{\mathfrak{X}_r} \simeq \Pi_s|_{G_1} \simeq \bigoplus_{\overline{H}_r \backslash G_2} \text{c-Ind}^{G_1}_{H} H_{\mathfrak{X}_r} \tag{6.12}$$

as G_1-representations, since we have $H^{\tilde{g}} = H$ and $H_{\mathfrak{X}_r} \simeq H^{\tilde{g}}_{\mathfrak{X}_r}$ as H-representations. By (6.12) and Frobenius reciprocity, we acquire

$$\mathrm{Hom}_{G_1}(\Pi_s, \pi_{\zeta,\chi,c,\omega}) \simeq \bigoplus_{\overline{H}_r \backslash G_2} \mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{\zeta,\chi,c,\omega}). \tag{6.13}$$

If $\zeta^{\frac{p^m-1}{n_1}} \neq s$, the required assertion follows from (6.13) and Proposition 6.2.1. Now, assume that $\zeta^{\frac{p^m-1}{n_1}} = s$. Without loss of generality, we may assume that ζ equals r, because Π_s depends only on s. By Proposition 6.2 and Frobenius reciprocity, we obtain a non-zero map

$$\mathrm{Ind}^{\overline{H}_r}_{L_r^\times U_D^1 \times W_{E_r}}(\theta_{r,\chi,c,\omega} \otimes \tau^0_{r,\chi,c,\omega}) \to \mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{r,\chi,c,\omega}). \tag{6.14}$$

By applying $\mathrm{Ind}_{\overline{H}_r}^{G_2}$ to the map (6.14), we acquire a non-zero map

$$\rho_{r,\chi,c,\omega} \otimes \tau_{r,\chi,c,\omega} \to \mathrm{Ind}_{\overline{H}_r}^{G_2} \mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{r,\chi,c,\omega}). \tag{6.15}$$

We have $\dim \rho_{r,\chi,c,\omega} = (q^n-1)/(q-1)$ and $\dim \tau_{r,\chi,c,\omega} = n$. Moreover, we have $[G_2 : \overline{H}_r] = n'(q^n-1)/n_1(q-1)$. Hence, the both sides of (6.15) are $n(q^n-1)/(q-1)$-dimensional by Proposition 6.2.1. Since $\rho_{r,\chi,c,\omega} \otimes \tau_{r,\chi,c,\omega}$ is an irreducible representation of G_2, we know that (6.15) is an isomorphism as G_2-representations. On the other hand, we have a non-zero map

$$\mathrm{Ind}_{\overline{H}_r}^{G_2} \mathrm{Hom}_H(H_{\mathfrak{X}_r}, \pi_{r,\chi,c,\omega}) \to \mathrm{Hom}_{G_1}(\Pi_s, \pi_{r,\chi,c,\omega}) \tag{6.16}$$

by Frobenius reciprocity. Since the left hand side is an irreducible representation of G_2 and the both sides have the same dimension by (6.13), we know that (6.16) is an isomorphism. Hence, the required assertion follows from the isomorphisms (6.15) and (6.16). $\square$

Theorem 6.4. *Let* LJ *be the inverse of* JL *in Proposition 5.3. We put*

$$\Pi = \bigoplus_{s \in \mu_{\frac{n_1(q-1)}{p^m-1}}(K)} \Pi_s.$$

Let π be a smooth irreducible representation of $GL_n(K)$. Then, we have

$$\mathrm{Hom}_{GL_n(K)}(\Pi, \pi) \simeq \begin{cases} \mathrm{LJ}(\pi) \otimes \mathrm{LL}(\pi) & \text{if } \pi \text{ is essentially simple epipelagic,} \\ 0 & \text{otherwise} \end{cases}$$

as $D^\times \times W_K$-representations.

Proof. This follows from Proposition 5.3 and Lemma 6.3, because every essentially simple epipelagic representation is isomorphic to $\pi_{\zeta,\chi,c,\omega}$ for some $\zeta \in \mu_{q-1}(K)$, $\chi \in (k^\times)^\vee$, $c \in \overline{\mathbb{Q}}_\ell^\times$ and a smooth character $\omega \colon K^\times \to \overline{\mathbb{Q}}_\ell^\times$. $\square$

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